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EXISTENCE AND UNIQUENESS FOR NON-LINEAR SINGULAR INTEGRAL EQUATIONS USED IN FLUID MECHANICS

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Abstract. Non-linear singular integral equations are investigated in connection with some basic applications in two-dimensional fluid mechanics. A general existence and uniqueness analysis is proposed for non-linear singular integral equations defined on a Banach space. Therefore, the non-linear equations are defined over a finite set of contours and the existence of solutions is investigated for two different kinds of equations, the first and the second kind. Moreover, the existence of solutions is further studied for non-linear singular integral equations over a finite number of arbitrarily ordered arcs. An application to fluid mechanics theory is finally given for the determination of the form of the profiles of a turbomachine in two-dimensional flow of an incompressible fluid.

Keywords: non-linear singular integral equations, existence and uniqueness theorems, Banach spaces, Hölder conditions, fluid mechanics

MSC 2000: 65L10, 65R20

1. Introduction

Many problems of mathematical physics, like elasticity, plasticity, viscoelasticity, thermoelasticity and fluid mechanics, reduce to the solution of a non-linear singular integral equation. Hence, there is interest in the solution of such non-linear integral equations, since these are connected with a wide range of problems of an applied character. The theory of non-linear singular integral equations seems to be particularly complicated if closely linked with applied mechanics problems.

Having in mind the implications for different problems of mathematical physics, E. G. Ladopoulos [1]–[4] introduced linear finite-part singular integral equations. This type of linear equations has been applied to many problems of elasticity, plasticity, fracture mechanics and aerodynamics.

In addition, E. G. Ladopoulos [5]–[9] investigated linear multidimensional singular integral equations, by introducing the Singular Integral Operators Method (S.I.O.M.), which has been applied to many problems of structural and fluid mechanics theory.

Therefore, in the present report the theories obtained for linear singular integral equations and linear multidimensional singular integral equations shall be extended to non-linear ones.

On the other hand, some studies have been published, investigating non-linear integral equations of simpler form, without any singularities. Among those who studied nonlinear theories used in applied mechanics, we shall mention the following: J. Andrews and J. M. Ball [10], S. S. Antman [11], [12], S. S. Antman and E. R. Carbone [13], J. M. Ball [14]–[16], H. Brezis [17], P. G. Ciarlet and P. Destuynder [18], P. G. Ciarlet and J. Necas [19], [20], C. M. Dafermos [21], [22], C. M. Dafermos and L. Hsiao [23], J. E. Dendy [24], Guo Zhong-Heng [25], H. Hattori [26], D. Hoff and J. Smoller [27], W. J. Hrusa [28], R. C. MacCamy [29]–[31], B. Neta [32], [33], R. W. Ogden [34], R. L. Pego [35], M. Slemrod [36] and O. J. Staffans [37].

In the present communication some existence and uniqueness theorems are established for non-linear singular integral equations defined on Banach spaces. These non-linear equations are first defined over finite sets of contours and the existence of solutions is studied for two different kinds of non-linear singular integral equations, the first and the second kind.

In addition, the existence of solutions is further established for non-linear singular integral equations over a finite number of arbitrarily ordered arcs, by introducing a Banach space and a special set defined in the space.

An application to fluid mechanics theory is finally given, consisting in the determination of the form of the lattice of profiles of turbomachines in two-dimensional steady flow of an incompressible fluid. This two-dimensional fluid mechanics problem reduces to the solution of a non-linear singular integral equation, by obtaining the form of the profile of the turbomachine, supposing that the velocity on the surface of the profile, the incident velocity, the velocity downstream of the lattice and the lattice spacing are known.

2. Existence and uniqueness theorems for non-linear singular integral equations over finite sets of contours

Definition 2.1. Consider a non-linear singular integral equation of the first kind

$$\int_L \frac{M(t,x)}{x-t} u(x)\,\mathrm{d}x + \xi \int_L \frac{K\big[t,x,u(x)\big]}{x-t}\,\mathrm{d}x = f(t) \tag{2.1}$$

where L is a finite set of smooth disconnected contours, x, $t \in L$, $u(x)$ the unknown, $f(t)$, $M(t,x)$ known functions Hölder-continuous in L, $K[t,x,u(x)]$ a non-linear kernel Hölder-continuous in L and ξ a real parameter.

Definition 2.2. We also consider a non-linear singular integral equation of the second kind

$$\xi \int_L \frac{K[t,x,u(x)]}{x-t}\,\mathrm{d}x = u(t) \tag{2.2}$$

in which L is a finite set of smooth disconnected contours, $x, t \in L$, $K[t,x,u(x)]$ a non-linear kernel Hölder-continuous in L, $u(x)$ the unknown and ξ a real parameter.

Definition 2.3. The singular kernel

$$\frac{M(t,x)}{t-x} \tag{2.3}$$

is said to be closed when the integral equation

$$\int_L \frac{M(t,x)}{t-x} u(t)\,\mathrm{d}t = 0 \tag{2.4}$$

has only the zero solution $u = 0$ in the set of functions satisfying Hölder's conditions.

Theorem 2.1 (Schauder's Fixed Point Theorem [38]). *Let X be a closed, compact and convex subset of a Banach space B. For an arbitrary continuous transformation $T\colon X \to X$ there exists a point $x \in X$ such that $T(x) = x$.*

Theorem 2.2 (Privalov's Theorem [39]. *Let a complex function $K(t,u)$ of two variables be defined in a domain $t \in L$, $u \in P$, where L denotes a smooth contour and P is a certain region or a line in the complex plane. If the function $K(t,u)$ satisfies Hölder's condition with respect to both variables:*

$$\left|K(t,u) - K(t_1,u_1)\right| < k\left[|t-t_1|^{\nu} + |u-u_1|^{\mu}\right], \tag{2.5}$$

$0 < \nu < 1$, $0 < \mu \leqslant 1$, *then the Cauchy singular integral*

$$\Phi(t,u) = \int_L \frac{K(t,u)}{t-x}\,\mathrm{d}t \tag{2.6}$$

satisfies the inequality

$$\left|\Phi(t,u)\right| < \pi R + Dk \tag{2.7}$$

and Hölder's condition with respect to both variables:

$$|\Phi(t_1,u_1)-\Phi(t,u)| < Ck\big[|t-t_1|^{\nu}+|u-u_1|^{\mu_1}\big] \tag{2.8}$$

where μ_1 is an arbitrary positive constant, $\mu_1<\mu$, R an upper bound of the function $|K(t,u)|$, C a positive constant depending on the line L and the choice of μ_1 and D is an upper bound of the integral $\int_L |t-x|^{\nu-1}\,\mathrm{d}t$.

Theorem 2.3. *Let us consider a non-linear singular integral equation of the first kind (2.1) defined over a finite set of disconnected contours L. Moreover, consider the kernel function $M(t,x)/(t-x)$ to be closed, and the constants $|\xi|$, $Z_f=\sup\limits_{t\in L}|f(t)|$ and k_f to be sufficiently small, where k_f is the Hölder coefficient of the function f in Hölder's inequality, such that the following inequalities hold:*

$$(1+k_a)h^{-1}\big[|\xi|N_k+|\xi|\pi^{-2}N_\varphi+(c_1Z_f+c_2k_f)\big]\leqslant A \tag{2.9}$$

$$\begin{aligned}&C_MA+|\xi|h^{-1}\big[\lambda+2|L|^{\mu/2}\big]+2|\xi|N_kk_Nh^{-2}|L|^{\mu/2}+|\xi|C_b\\&+Bh^{-1}k_f|L|^{\mu/2}+2(c_1Z_f+c_2k_f)h^{-2}k_M|L|^{\mu/2}\leqslant\lambda,\ 0<\mu\leqslant 1\end{aligned} \tag{2.10}$$

where

$$N_k=\sup\big|K(t,x,u(x))\big|, \tag{2.11}$$

$$N_\varphi=\sup\int_L\big|\Phi(t,x,u(x))\big|\,\mathrm{d}x,\ t\in L,\ |u|\leqslant A, \tag{2.12}$$

A given positive number,

$$h=\inf\big|M(t,t)\big|,\ t\in L, \tag{2.13}$$

$$k_a=\sup\int_L\big|G(\theta,x)\big|\,\mathrm{d}\ell_x, \tag{2.14}$$

$$\Phi(\theta,x,u)=\int_L\frac{K(t,x,u)}{(t-\theta)(x-t)}\,\mathrm{d}t \tag{2.15}$$

with $G(\theta,x)$ a function dependent on the function

$$F(\theta,x)/M(\theta,\theta):\ F(\theta,x)=\int_L\frac{M(t,x)\,\mathrm{d}t}{(t-\theta)(x-t)}, \tag{2.16}$$

and k_M is the Hölder coefficient of the function $M(t,x)$ in Hölder's inequality, c_1 and c_2 are certain positive constants depending on the contour L, c_M is a positive constant depending on the function M, B a positive constant which depends on the *contour L, λ an arbitrary constant, $|L|$ the diameter of the set L and c_b a positive constant depending on the functions M, K and the contour L.*

Then, the non-linear singular integral equation (2.1) *possesses at least one solution $u^*(t)$ satisfying Hölder's condition.*

Proof. By multiplying both sides of the non-linear singular integral equation (2.1) by $1/(t-\theta)$, $\theta \in L$, and integrating with respect to the variable t we reduce (2.1) to its equivalent weakly-singular form

$$\int_L \frac{\mathrm{d}t}{t-\theta}\left[\int_L \frac{M(t,x)}{x-t} u(x)\,\mathrm{d}x\right] + \xi \int_L \frac{\mathrm{d}t}{t-\theta}\left[\int_L \frac{K\big[t,x,u(x)\big]}{x-t}\,\mathrm{d}x\right] = \int_L \frac{f(t)\,\mathrm{d}t}{x-\theta},\ \theta \in L. \tag{2.17}$$

Moreover, by applying the Poincaré-Betrand transformation formula to the integrals on the left hand side of (2.17), we obtain the equation

$$-\pi^2 M(\theta,\theta)u(\theta) - \xi\pi^2 K\big[\theta,\theta,u(\theta)\big] + \int_L F(\theta,x)u(x)\,\mathrm{d}x + \xi \int_L \Phi\big[\theta,x,u(x)\big]\,\mathrm{d}x = f_a(\theta) \tag{2.18}$$

where

$$f_a(\theta = \int_L \frac{f(t)\,\mathrm{d}t}{t-\theta} \tag{2.19}$$

and $F(\theta,x)$, $\Phi\big[\theta,x,u(x)\big]$ are given by (2.16) and (2.15), respectively.

We have

$$\frac{1}{(t-\theta)(x-t)} = \frac{1}{x-\theta}\left[\frac{1}{t-\theta} + \frac{1}{x-t}\right], \tag{2.20}$$

hence eqs (2.15) and (2.16) can be written as

$$F(\theta,x) = \frac{\delta(\theta,x) - \delta(x,x)}{x-\theta} \tag{2.21}$$

and

$$\Phi(\theta,x,u) = \frac{R(\theta,x,u) - R(x,x,u)}{x-\theta} \tag{2.22}$$

where the functions $\delta(\theta,x)$ and $R(\theta,x,u)$ are defined by the Cauchy singular integrals

$$\delta(\theta,x) = \int_L \frac{M(t,x)}{t-\theta}\,\mathrm{d}t \tag{2.23}$$

and

$$R(\theta, x, u) = \int_L \frac{K(t, x, u)}{t - \theta}\, \mathrm{d}t. \tag{2.24}$$

We will apply Schauder's fixed point theorem (Theorem 2.1) to the integral equation (2.18). Assuming that $M(\theta, \theta) \neq 0$, $\theta \in L$, we can write (2.18) in the form

$$u(\theta) - \frac{1}{\pi^2} \int_L \frac{F(\theta, x)}{M(\theta, \theta)} u(x)\, \mathrm{d}x = -\frac{\xi}{M(\theta, \theta)} K\big[\theta, \theta, u(\theta)\big] + \frac{\xi}{\pi^2} \int_L \frac{\Phi\big[\theta, x, u(x)\big]}{M(\theta, \theta)}\, \mathrm{d}x - \frac{f_a(\theta)}{\pi^2 M(\theta, \theta)}. \tag{2.25}$$

The kernel $M(t, x)/(t - x)$ is closed and therefore, the homogeneous equation

$$u(\theta) - \frac{1}{\pi^2} \int_L \frac{F(\theta, x)}{M(\theta, \theta)} u(x)\, \mathrm{d}x = 0 \tag{2.26}$$

possesses only the zero solution.

Moreover, on the basis of the theory of weakly singular integral equations, the only solution of the integral equation

$$u(\theta) - \frac{1}{\pi^2} \int_L \frac{F(\theta, x)}{M(\theta, \theta)} u(x)\, \mathrm{d}x = q(\theta) \tag{2.27}$$

has the form

$$u(\theta) = q(\theta) + \int_L G(\theta, x) q(x)\, \mathrm{d}x \tag{2.28}$$

where $G(\theta, x)$ is a function dependent on the function $F(\theta, x)/M(\theta, \theta)$. Therefore, the solution (2.28) satisfies the inequality

$$\big|u(\theta)\big| \leqslant (1 + k_a) \sup |q| \tag{2.29}$$

where k_a is given by (2.14). Thus, the function $u(\theta)$ satisfies the inequality

$$u(\theta) \leqslant \frac{1 + k_a}{h} \Big[|\xi| N_k + \frac{\xi}{\pi^2} N_\varphi + (c_1 N_f + c_2 k_f) \Big] \tag{2.30}$$

where

$$N_f = \sup \big|f(t)\big|, \ t \in L \tag{2.31}$$

and N_k, N_φ, h are given by (2.11), (2.12) and (2.13), respectively.

Furthermore, the function $u(\theta)$ satisfies Hölder's condition with exponent $\mu/2$, $0 < \mu \leqslant 1$:

$$|u(\theta) - u(\theta_1)| < k_u|\theta - \theta_1|^{\mu/2}. \tag{2.32}$$

Hence the integral equation (2.25) implies

$$\begin{aligned} k_u \leqslant c_M \sup|u| + |\xi|h^{-1}\big[\lambda + 2|L|^{\mu/2}\big] + 2|\xi|N_k k_M h^{-2}|L|^{\mu/2} \\ + |\xi|c_b + Bh^{-1}k_f|L|^{\mu/2} + 2(c_1 z_f + c_2 k_f)h^{-2}k_M|L|^{\mu/2}. \end{aligned} \tag{2.33}$$

Therefore, from (2.30) and (2.33) we obtain the inequalities (2.9) and (2.10). Thus, all conditions of Schauder's Theorem are satisfied as the set of all points $u(t)$ is closed, convex and compact. Therefore, we can assert that there exists at least one fixed point $u^*(t)$ of the transformation (2.25), i.e. there exists a solution $u^*(t)$ of the non-linear singular integral equation (2.1) satisfying Hölder's condition. □

Theorem 2.4. *Consider the non-linear singular integral equation of the second kind* (2.2) *defined over a finite set of disconnected contours L. Furthermore, assume that the non-linear kernel $K(t,x,u)$ satisfies Hölder's condition of the form*

$$\begin{aligned} |K(t,x,u) - K(t_1,x_1,u_1)| < k\big[|t - t_1|^\nu + |x - x_1|^\mu + |u - u_1|\big], \\ 0 < \mu < \nu \leqslant 1,\ k > 0 \end{aligned} \tag{2.34}$$

and the parameter ξ satisfies the condition

$$|\xi| \leqslant \min\left[\frac{A}{k(1+\tau)E + \pi P}, \frac{\tau}{k(1+\tau)c}\right] \tag{2.35}$$

where E denotes the upper bound of the integral

$$\int_L \frac{\mathrm{d}\ell_\nu}{|x - t|^{1-\mu}} \tag{2.36}$$

and A is a given positive number, τ the Hölder coefficient of the function $u(x)$ in Hölder's inequality, P the upper bound of the function $|K(t,x,u)|$ and c a positive constant which depends on the contour L.

Then the non-linear singular integral equation of the second kind (2.2) *possesses at least one solution $u^*(t)$ in the set of functions satisfying Hölder's condition with exponent μ.*

P r o o f. Consider the transformation function $\omega(t)$ which transforms the non-linear singular integral equation (2.2) as follows:

$$\omega(t)=\xi\int_L \frac{K\big[t,x,u(x)\big]}{x-t}\,\mathrm{d}x. \tag{2.37}$$

Furthermore, we write (2.37) as

$$\omega(t)=\xi\int_L \frac{K\big[t,x,u(x)\big]-K\big[t,t,u(t)\big]}{x-t}\,\mathrm{d}x+K\big[t,t,u(t)\big]\int_L \frac{\mathrm{d}x}{x-t}. \tag{2.38}$$

Therefore, we obtain

$$\big|\omega(t)\big|\leqslant|\xi|k(1+\tau)\int_L \frac{\mathrm{d}\ell_x}{|x-t|^{1-\mu}}+\pi|\xi|P \tag{2.39}$$

where P denotes the upper bound of the function $\big|K(t,x,u)\big|$.

By using Privalov's Theorem (Theorem 2.2), we can prove that the function $\omega(t)$ satisfies Hölder's condition of the form

$$\big|\omega(t)-\omega(t_1)\big|\leqslant k|\xi|(1+\tau)c|t-t_1|^{\mu} \tag{2.40}$$

where c is a positive constant which depends on the contours L.

From (2.39) and (2.40) one obtains the following two inequalities:

$$|\xi|k(1+\tau)E+\pi|\xi|P\leqslant A \tag{2.41}$$

and

$$|\xi|k(1+\tau)c\leqslant\tau \tag{2.42}$$

in which A is a given positive number and E the upper bound of the integral (2.36). Thus, by using (2.41) and (2.42) we obtain the inequality (2.35) which has to be satisfied by the parameter ξ.

Hence all conditions of Schauder's Theorem are satisfied as the set of all points $\omega(t)$ is closed, convex and compact and therefore there exists at least one fixed point $u^*(t)$ of the transformation (2.37). Thus, there exists at least one function $u^*(t)$ which is a solution of the non-linear singular integral equation (2.2) for sufficiently small $|\xi|$. □

3. Existence and uniqueness theorems for non-linear singular integral equations over a system of arcs

Consider a set of points $L = \ell_1 + \ell_2 + \ldots + \ell_n$ in the complex plane, consisting of a finite number of arbitrarily ordered arcs $\ell_1, \ell_2, \ldots, \ell_n$. Furthermore, the end-points $c_1, c_2, \ldots, c_q$ of these arcs are also ordered arbitrarily but independently of the order of arcs.

Definition 3.1. Let us define the class H_α^μ as the set of all complex functions $u(t)$ defined at each point t of the set L, or the set $L' = L - \sum\limits_{\nu=1}^{q} c_\nu$ satisfying the inequality

$$|u(t)| < \frac{A}{\prod\limits_{i=1}^{q} |t - c_i|^a}, \qquad t \in L',\ A = \text{constant} \tag{3.1}$$

and the generalized Hölder condition

$$|u(t) - u(t_1)| < \frac{A|t - t_1|^\mu}{\big[|t - c_\nu||t_1 - c_{\nu_1}|\big]^{\alpha+\mu}} \tag{3.2}$$

for each pair of points t, t_1 lying inside the same arbitrary arc $\ell = c_\nu \overset{\frown}{c}_{\nu_1}$, while the point t_1 is on the arc $\overset{\frown}{tc_\nu}$. Moreover, let the real parameters a and μ, fixed for the given class, satisfy the conditions

$$0 \leqslant a < 1,\ 0 < \mu < 1,\ \alpha + \mu < 1 \tag{3.3}$$

where the constant A may take any positive values.

Definition 3.2. Consider the function space F consisting of complex functions u, defined and continuous in the open set L' and satisfying the inequality

$$\sup_{t \in L'} \left[\prod_{i=1}^{q} \left|t - c_i\right|^{\alpha+\mu} \left|u(t)\right| \right] < \infty. \tag{3.4}$$

The norm of the point u is defined as follows:

$$\|u\| = \sup_{t \in L'} \left[\prod_{i=1}^{q} |t - c_i|^{\alpha+\mu} \left|u(t)\right| \right]. \tag{3.5}$$

The space F is linear, metric, normed and complete and thus, a Banach space.

Definition 3.3. In the space F let us define the set S of all functions u for which

$$\prod_{i=1}^{q} |t - c_i|^{\alpha} \big|u(t)\big| \leqslant D_1 \tag{3.6}$$

and

$$\big[|t - c_\nu||t_1 - c_{\nu_1}|\big]^{\alpha+\mu} \big|u(t) - u(t_1)\big| \leqslant D_2 |t - t_1|^{\mu} \tag{3.7}$$

where t_1, t_2 is an arbitrary pair of points lying on the same arbitrary arc $c_\nu \widehat{c}_{\nu_1}$ and $t_1 \in t\widehat{c}_{\nu_1}$, D_1 and D_2 are arbitrary positive numbers. Furthermore, the functions u of the set S belong to the class H_{α}^{μ}.

Theorem 3.1. *Consider the non-linear singular integral equation of the second kind* (2.2) *where* $\xi = 1$*, defined over a system of arcs* L*. Moreover, suppose that the non-linear function* $K(t, x, u)$ *satisfies the condition*

$$\big|K(t, x, u)\big| < B_1 |u| + \frac{B_2}{\prod\limits_{i=1}^{q} (x - c_i)^{a}} \tag{3.8}$$

and the generalized Hölder-Lipschitz condition

$$\big|K(t, x, u) - K(t_1, x_1, u_1)\big| < B_3 |u - u_1| + B_4 \frac{|t - t_1|^{\mu_1} + |x - x_1|^{\mu}}{\big[|x - c_\nu||x_1 - c_{\nu_1}|\big]^{\alpha+\mu}} \tag{3.9}$$

where B_1, B_2, B_3 *and* B_4 *are given positive constants and the positive exponents* α, μ, μ_1 *satisfy the inequalities*

$$\alpha + \mu < 1, \ 0 < \mu < \mu_1 \leqslant 1 \tag{3.10}$$

and $x \in c_\nu \widehat{c}_{\nu_1}$, $x_1 \in x\widehat{c}_{\nu_1}$, $t_1, t_2 \in \ell_e$ $(e = 1, 2, \ldots, n)$.

Furthermore, the constants B_1 *and* B_3 *are sufficiently small so that the following conditions hold:*

$$G_1 B_1 < 1, \ G_2 B_3 < 1 \tag{3.11}$$

and

$$G_1 B_1 + G_2 B_3 + (G_3 G_4 - G_1 G_2) B_1 B_3 < 1 \tag{3.12}$$

where G_1, G_3 *are positive constants depending only on the arcs* $\ell_1, \ell_2, \ldots, \ell_n$ *and* G_2, G_4 *are positive constants.*

Then in the class H_α^μ there exists at least one solution $u^*(t)$ of the non-linear singular integral equation of the second kind (2.2) *for the system of arcs.*

P r o o f. The existence of the solution of the non-linear singular integral equation (2.2) will be proved by Schauder's topological method.

Let us transform the set S defined by Definition 3.3 according to the formula

$$\omega(t) = \int_L \frac{K\big[t, x, u(x)\big]}{x - t}\,\mathrm{d}x \tag{3.13}$$

which assignes to each point u of the set S a certain point ω of the space F, defined by Definition 3.2.

Furthermore, because of the inequalities (3.6) and (3.8), for each point u of the set S one has

$$\Big|K\big[t, x, u(x)\big]\Big| < B_1\big|u(x)\big| + \frac{B_2}{\prod\limits_{i=1}^{q}|t - c_i|^a} < \frac{B_1D_1 + B_2}{\prod\limits_{i=1}^{q}|t - c_i|^a}. \tag{3.14}$$

Moreover, because of the inequalities (3.7) and (3.9) the non-linear kernel $K\big[t, x, u(x)\big]$ satisfies the generalized Hölder condition for the points of the set S

$$\Big|K\big[t, x, u(x)\big] - K\big[t_1, x_1, u(x_1)\big]\Big| < \frac{(B_3D_2 + B_4)\big[|t - t_1|^\mu + |x - x_1|^\mu\big]}{\big[|x - c_\nu||x_1 - c_{\nu_1}|\big]^{\alpha+\mu}}. \tag{3.15}$$

Therefore, by using the inequality (3.15), it is easily proved that the transformed function (3.13), continuous in the open set L', satisfies the formula

$$\big|\omega(t)\big| < \frac{G_1(B_1D_1 + B_2) + G_3(B_3D_2 + B_4)}{\prod\limits_{i=1}^{q}|t - c_i|^\alpha} \tag{3.16}$$

where G_1, G_3 are positive constants depending only on the arcs $\ell_1, \ell_2, \ldots, \ell_n$. Hence, the point ω belongs to the space F.

Furthermore, the transformed function ω satisfies Hölder's condition

$$\big|\omega(t) - \omega(t_1)\big| < \frac{G_4(B_1D_1 + B_2) + G_2(B_3D_2 + B_4)}{\big[|t - c_\nu||t_1 - c_{\nu_1}|\big]^{\alpha+\mu}}|t - t_1|^\mu \tag{3.17}$$

where G_2, G_4 are positive constants, $t \in c_\nu\overset{\frown}{c}_{\nu_1}$, $t_1 \in t\overset{\frown}{c}_{\nu_1}$. The transformed function ω belongs to the class H_α^μ, as the functions u of the set S.

Hence, comparing the inequalities (3.16) and (3.17) with (3.6) and (3.7), we conclude that the set S^* of all the transformed points ω is a subset of the set S, provided that the constants of the problem satisfy the inequalities

$$G_1(B_1D_1 + B_2) + G_3(B_3D_2 + B_4) \leqslant D_1 \tag{3.18}$$

and

$$G_4(B_1D_1 + B_2) + G_2(B_3D_2 + B_4) \leqslant D_2. \tag{3.19}$$

Therefore, if the two constants D_1 and D_2 satisfy the inequalities

$$D_1 > G_1B_2 + G_3B_4 \tag{3.20}$$

and

$$D_2 > G_4B_2 + G_2B_4 \tag{3.21}$$

then the conditions (3.18) and (3.19) are satisfied, provided the positive constants B_1 and B_3 are sufficiently small.

Moreover, in order for the inequalities (3.18) and (3.19) to be satisfied, the conditions (3.11) have to hold. If the constants D_1 and D_2 are further regarded as rectangular coordinates in the plane, then the set of the points (D_1, D_2) satisfying the condition (3.18) is a half-plane lying below the straight line with the equation

$$D_2 = \frac{1 - G_1B_1}{G_3B_3}D_1 - \frac{G_1B_2 + G_3B_4}{G_3B_3}. \tag{3.22}$$

Also, the set of the points (D_1, D_2) satisfying the condition (3.19) is a half-plane lying above the straight line with the equation

$$D_2 = \frac{G_4B_1}{1 - G_2B_3}D_1 + \frac{G_4B_2 + G_2B_4}{1 - G_2B_3}. \tag{3.23}$$

Therefore, a necessary and sufficient condition for the existence of a set points (D_1, D_2) possessing positive coordinates and satisfying the inequalities (3.18) and (3.19) is

$$\frac{1 - G_1B_1}{G_3B_3} > \frac{G_4B_2}{1 - G_2B_3} \tag{3.24}$$

or the equivalent condition (3.12).

Furthermore, the transformation of the set S by the relation (3.13) is continuous in the space F and the set S' transformed from the set S by the relation (3.13) is relatively compact. Hence, in the set S there exists at least one point $u^*(t)$ fixed with respect to the transformation (3.13). Or, in other words, there exists, in the class H_α^μ, at least one solution $u^*(t)$ of the non-linear singular integral equation (2.2), which completes the assertion of Theorem 3.1.

The existence of solutions of the non-linear singular integral equation (2.2) defined over a system of arcs L can be also proved by the method of successive approximations, by using the following theorem of Banach-Cacciopoli, when more restrictive assumptions are imposed on the non-linear kernel $K\big[t, x, u(x)\big]$. □

Theorem 3.2 [Banach-Cacciopoli [40], [41]). *If in a complete metric space an operation P associates with every two points f and g, points $\widehat{P}(f)$ and $\widehat{P}(g)$ of this space, the distances of which satisfy the inequality*

$$\delta\big[\widehat{P}(f), \widehat{P}(g)\big] \leqslant \beta\delta(f, g) \tag{3.25}$$

the positive constant β being smaller than unity and independent of the pair f and g, then there exists a unique point u of the space which satisfies the equation

$$u = \widehat{P}(u), \tag{3.26}$$

i.e. a fixed point with respect to the operation $\widehat{P}$.

Theorem 3.3. *Let a complex valued function $u(\zeta, \theta; t, x)$ of the real variables ζ, θ and complex parameters t, x, be defined in the domain*

$$|\zeta + \mathrm{i}\theta| \leqslant D,\ t \in L,\ x \in L \tag{3.27}$$

where L is a system of smooth arcs, and let the following Hölder-Lipschitz conditions be satisfied for the partial derivatives of u:

$$\begin{aligned} &\big|u_\zeta(\zeta, \theta, t, x) - u_\zeta(\zeta_1, \theta_1, t_1, x_1)\big| \\ &\qquad \leqslant k\big[|\zeta - \zeta_1| + |\theta - \theta_1| + |t - t_1|^\nu + |x - x_1|^\mu\big], \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\big|u_\theta(\zeta, \theta, t, x) - u_\theta(\zeta_1, \theta_1, t_1, x_1)\big| \\ &\qquad \leqslant k\big[|\zeta - \zeta_1| + |\theta - \theta_1| + |t - t_1|^\nu + |x - x_1|^\mu\big] \end{aligned} \tag{3.29}$$

where $0 < \mu < \nu \leqslant 1$. Then the equality

$$\begin{aligned} u(\zeta^*, \theta^*, t, x) - u(\zeta, \theta, t, x) = (\zeta^* - \zeta)&E_1(\zeta, \zeta^*, \theta, \theta^*, t, x) \\ + (\theta^* - \theta)&E_2(\zeta, \zeta^*, \theta, \theta^*, t, x) \end{aligned} \tag{3.30}$$

holds. Moreover, the complex functions E_1 *and* E_2 *determined in the domain*

$$|\zeta+\mathrm{i}\theta|\leqslant D,\ |\zeta^*+\mathrm{i}\theta^*|\leqslant D,\ t,\ x\in L \tag{3.31}$$

satisfy the Hölder-Lipschitz condition

$$\begin{aligned}&\bigl|E_c(\zeta,\zeta^*,\theta,\theta^*,t,x)-E_c(\zeta',\zeta^{*\prime},\theta',\theta^{*\prime},t',x')\bigr|\\&\quad\leqslant k\bigl[|\zeta-\zeta'|+|\zeta^*-\zeta^{*\prime}|+|\theta-\theta'|+|\theta^*-\theta^{*\prime}|+|t-t'|^{\nu}+|x-x'|^{\mu}\bigr].\end{aligned} \tag{3.32}$$

P r o o f. The proof of the present theorem follows from the integral representation of the difference between two arbitrary values of the complex valued function $u(\zeta,\theta,t,x)$:

$$\begin{aligned}u(\zeta^*,\theta^*,t,x)-u(\zeta,\theta,t,x)&=(\zeta^*-\zeta)\int_0^1 u_\zeta\bigl[\zeta+y(\zeta^*-\zeta),\theta^*,t,x\bigr]\,\mathrm{d}y\\&\quad+(\theta^*-\theta)\int_0^1 u_\theta\bigl[\zeta,\theta+y(\theta^*-\theta),t,x\bigr]\,\mathrm{d}y.\end{aligned} \tag{3.33}$$

□

Theorem 3.4. *Consider the non-linear singular integral equation* (2.2) *with* $\xi=1$, *and a system of arcs* L *defined in the class* H_*^μ, *where* H_*^μ *denotes the set of all functions of the class* H_a^μ *for which* a *takes all values such that* $0<a<\frac{1}{2}(1-\mu)$. *The non-linear kernel* $K\bigl[t,x,u(x)\bigr]$ *of* (2.2) *is defined in the open domain*

$$t,x\in L',\ u=\zeta+\mathrm{i}\theta\in\mathbb{C} \tag{3.34}$$

where $\mathbb{C}$ *denotes the plane of the complex variable, and satisfies the inequality*

$$\bigl|K(t,x,u)\bigr|<k_1|u|+k_1' \tag{3.35}$$

and the Hölder-Lipschitz condition

$$\bigl|K(t,x,u)-K(t_1,x_1,u_1)\bigr|<k_2\bigl[|t-t_1|^{\mu_1}+|x-x_1|^{\mu}+|u-u_1|\bigr]. \tag{3.36}$$

Furthermore, the real part $K^{(\mathrm{Re})}$ *and the imaginary part* $K^{(\mathrm{Im})}$ *of the non-linear kernel* K,

$$K(t,x,u)=K^{(\mathrm{Re})}(t,x,\zeta,\theta)+\mathrm{i}K^{(\mathrm{Im})}(t,x,\zeta,\theta) \tag{3.37}$$

have partial derivatives with respect to the variables ζ and θ, which satisfy the Hölder-Lipschitz conditions

$$(3.38)\qquad \left|K_\zeta^{(e)}(t,x,\zeta,\theta)-K_\zeta^{(e)}(t_1,x_1,\zeta_1,\theta_1)\right|<k_3\big[|t-t_1|^{\mu_1}+|x-x_1|^{\mu} + |\zeta-\zeta_1|+|\theta-\theta_1|\big],$$

$$(3.39)\qquad \left|K_\theta^{(e)}(t,x,\zeta,\theta)-K_\theta^{(e)}(t_1,x_1,\zeta_1,\theta_1)\right|<k_3\big[|t-t_1|^{\mu_1}+|x-x_1|^{\mu} + |\zeta-\zeta_1|+|\theta-\theta_1|\big],$$

with $0<\mu<\mu_1\leqslant 1$, and the symbol e must be replaced by either Re *or* Im.

The constants k_1, k_2 and k_3 are sufficiently small and such that the following inequalities hold:

$$(3.40)\qquad A_0k_1+A_1^*k_2+(A_0^*A_1-A_0A_1^*)k_1k_2<1$$

$$(3.41)\qquad A_0k_1<1,\ A_1^*k_2<1$$

$$(3.42)\quad G=\max\Big\{\big[(A_0+A_1)k_2+4(2D_2+1)k_3(A_0^*+A_1^*)\big],4(A_0^*+A_1^*)k_4\Big\}<1$$

where k_4 is the least upper bound of the absolute values of the derivatives of the functions $K^{(\mathrm{Re})}$ and $K^{(\mathrm{Im})}$ with respect to the variables ζ and θ, A_0, A_0^*, A_1, A_1^* and D_2 are positive constants.

Then there exists a unique solution of the non-linear singular integral equation (2.2) in the class H_*^μ, which is at each interior point t of the set L' the limit of the sequence

$$(3.43)\qquad u_0(t),u_1(t),\dots,u_n(t)$$

which is defined by means of the recursive relation

$$(3.44)\qquad u_{n+1}(t)=\int_L\frac{K\big[t,x,u_n(x)\big]}{x-t}\,\mathrm{d}x.$$

P r o o f. By using the method of successive approximations, we prove the existence of the sequence (3.43). Hence, let us assume that $u_0(t)$ belongs to the class H_a^μ, i.e. satisfies the inequalities (3.6) and (3.7). Therefore, all functions of the sequence (3.43) are defined in the class H_a^μ if the constants k_1 and k_2 satisfy the inequalities

$$(3.45)\qquad \begin{aligned} A_0k_1(D_1+A_2)+A_0k_2(D_2+A_3)&\leqslant D_1,\\ A_1k_1(D_1+A_2)+A_1k_2(D_2+A_3)&\leqslant D_2\end{aligned}$$

where

$$A_2 = \sup_{t\in L}\left[\prod_{i=1}^{q}|t-c_i|^a\right], \tag{3.46}$$

$$A_3 = \max \sup_{t,t_1\in L}\left[\prod_{i=1}^{q}|t-c_\nu|\,|t_1-c_{\nu_1}|\right]^{a+\mu}.$$

Thus, by making use of the arbitrary choice of the positive constants D_1 and D_2 we can show that the inequalities (3.40) and (3.41) are necessary and sufficient for the existence of positive values D_1 and D_2 for which the inequalities (3.45) hold. Also, we can easily see from the above that the inequalities (3.35) and (3.36) are sufficient for the existence of the sequence (3.43).

Moreover, in order to prove the convergence of the sequence of successive approximations we shall proceed by the following method. For the proof we use the more restrictive assumptions (3.38) and (3.39), the numerical series $\sum_{n=0}^{\infty} F[u_{n+1}-u_n]$ of the upper bounds of the products

$$\sup_{t\in L'}\left[\prod_{i=1}^{q}|t-c_i|^a\big|u_{n+1}(t)-u_n(t)\big|\right] = F\big[u_{n+1}(t)-u_n(t)\big] \tag{3.47}$$

and the series $\sum_{n=0}^{\infty} H^*\big[u_{n+1}(t)-u_n(t)\big]$ which are determined by the relation

$$H^*\big[u(t)\big] = \max_{\gamma} \sup_{t,t_1\in c_\nu \widehat{c}_{\nu_1}} \left\{\big[|t-c_\nu||t_1-c_{\nu_1}|\big]^{2a+\mu}\frac{\big|u(t)u(t_1)\big|}{|t-t_1|^\mu}\right\} \tag{3.48}$$

where $\ell_\gamma = c_\nu\widehat{c}_{\nu_1}$ and $t_1 \in t\widehat{c}_{\nu_1}$.

Hence, in order to prove the convergence of the above series, we shall use Theorem 3.3. Therefore, let us write

$$u_{n+1}(t)-u_n(t) = \int_L \frac{\lambda_n(t,x)}{x-t}\,\mathrm{d}x \tag{3.49}$$

with

$$\lambda_n(t,x) = K\big[t,x,u_n(t)\big] - K\big[t,x,u_{n-1}(x)\big]. \tag{3.50}$$

Furthermore, let us define the generalized Hölder coefficient of the function of two variables by

$$H^*\big[\lambda_n(t,x)\big] = \max_{\gamma,\gamma^*} \sup_{\substack{t,t_1\in\ell_\gamma\\ x,x_1\in\ell_{\gamma^*}}} \left\{\big[|x-c_\nu||x_1-c_{\nu_1}|\big]^{2a+\mu}\frac{\big|\lambda_n(t,x)-\lambda_n(t_1,x_1)\big|}{|t-t_1|^{\mu_1}+|x-x_1|^\mu}\right\} \tag{3.51}$$

where $\ell_\gamma = c_\nu\widehat{c}_{\nu_1}$ and $x_1 \in x\widehat{c}_{\nu_1}$.

Hence, by using Theorem 3.3, we obtain the inequality

$$H^*\big[\lambda_n(t,x)\big] \leqslant 4k_4 H^*\big[u_n(t)-u_{n-1}(t)\big] + 4(2D_2+1)k_3 F\big[u_n(t)-u_{n-1}(t)\big]. \tag{3.52}$$

Moreover, by using the relation

$$F^*\big[\lambda_n(t,x)\big] = \sup_{t,x\in L}\left\{\prod_{i=1}^{q}|x-c_i|^a\big|\lambda_n(t,x)\big|\right\} \tag{3.53}$$

we obtain

$$F^*\big[\lambda_n(t,x)\big] \leqslant k_2 F[u_n-u_{n-1}] \tag{3.54}$$

and finally we conclude

$$F[u_{n+1}-u_n] \leqslant A_1F^*\big[\lambda_n(t,x)\big] + A_1^*H^*\big[\lambda_n(t,x)\big], \tag{3.55}$$

$$H^*[u_{n+1}-u_n] \leqslant A_2F^*\big[\lambda_n(t,x)\big] + A_2^*H^*\big[\lambda_n(t,x)\big]. \tag{3.56}$$

Therefore, by using (3.51) and (3.52) we obtain

$$\begin{aligned} F\big[u_{n+1}(t)-u_n(t)\big] \leqslant{}& \big[A_0k_2+4(2D_2+1)A_0^*k_3\big]F\big[u_n(t)-u_{n-1}(t)\big] \\ &+4A_0^*k_4H^*\big[u_n(t)-u_{n-1}(t)\big], \end{aligned} \tag{3.57}$$

$$\begin{aligned} H^*\big[u_{n+1}(t)-u_n(t)\big] \leqslant{}& \big[A_1k_2+4(2D_2+1)A^*k_3\big]F\big[u_n(t)-u_{n-1}(t)\big] \\ &+4A^*k_4H^*\big[u_n(t)-u_{n-1}(t)\big]. \end{aligned} \tag{3.58}$$

Now (3.57) and (3.58) imply that the numerical series

$$\sum_{n=0}^{\infty}F\big[u_{n+1}(t)-u_n(t)\big], \quad \sum_{n=0}^{\infty}H^*\big[u_{n+1}(t)-u_n(t)\big] \tag{3.59}$$

converge provided the constants k_2, k_3 and k_4 are sufficiently small.

Hence, by summing both sides of (3.57) and (3.58) we obtain

$$\begin{aligned} &F\big[u_{n+1}(t)-u_n(t)\big]+H^*\big[u_{n+1}(t)-u_n(t)\big] \\ &\qquad\leqslant G\Big\{F\big[u_n(t)-u_{n-1}(t)\big]+H^*\big[u_n(t)-u_{n-1}(t)\big]\Big\} \end{aligned} \tag{3.60}$$

where G is given by (3.42).

Thus, the functional series $\sum\limits_{n=0}^{\infty}\big[u_{n+1}(t)-u_n(t)\big]$ is absolutely and uniformly convergent in L and in accordance with theorem 3.2 (Banach-Cacciopoli) the limit function

$$u(t)=\lim_{n\to\infty}u_n(t) \tag{3.61}$$

exists and $u(t)$ is the required unique solution of the non-linear singular integral equation (2.2). □

4. An application to fluid mechanics theory

As an application of the theory of non-linear singular integral equations we will determine the form of the profiles of a turbomachine. Consider the lattice of profiles of turbomachines in two-dimensional steady flow of an incompressible fluid. Let v denote the velocity on the surface of the profile, $v_1 \mathrm{e}^{\mathrm{i}\omega_1}$ the inlet velocity, $v_2 \mathrm{e}^{\mathrm{i}\omega_2}$ the outlet velocity of the lattice and a the lattice spacing (see Fig. 1).

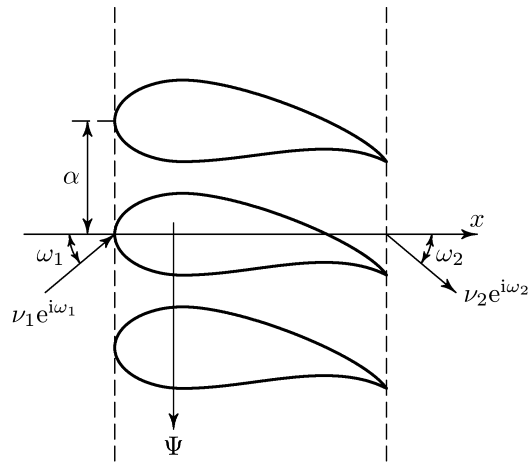

Fig. 1: A lattice of profiles of turbomachines in steady flow of an incompressible fluid.

The complex fluid velocity due to a lattice of profiles in two-dimensional, irrotational, incompressible flow satisfies the equation [42]

$$u(z) = u^* - \frac{1}{2a\mathrm{i}} \int_\Gamma \frac{u(\xi)\,\mathrm{d}\xi}{\tan\left[\frac{\pi(\xi - z)}{a}\right]} \tag{4.1}$$

in which

$$u^* = \frac{v_1 \mathrm{e}^{\mathrm{i}\omega_1} + v_2 \mathrm{e}^{\mathrm{i}\omega_2}}{2} \tag{4.2}$$

where Γ is the contour of the profile with length L, $z \notin L$. We apply Plemelj formulae [1], [43], [44], when $z \to \xi^* \in \Gamma$:

$$u(\xi^*) = 2u^* - \frac{1}{a\mathrm{i}} \int_\Gamma \frac{u(\xi)\,\mathrm{d}\xi}{\tan\left[\frac{\pi(\xi - \xi^*)}{a}\right]}. \tag{4.3}$$

Furthermore, the expression $\tan\left[\frac{\pi(\xi-\xi^*)}{a}\right]$ in (4.3) can be approximated by the formula [45]

$$1/\tan\left[\frac{\pi(\xi-\xi^*)}{a}\right] = \frac{a}{\pi}\left[\frac{1}{\xi-\xi^*} + \sum_{n=1}^{\infty}\left[\frac{1}{\xi-\xi^*-na} + \frac{1}{\xi-\xi^*+na}\right]\right]. \tag{4.4}$$

On the other hand, the points $\xi^* \pm na$ are outside the contour Γ, and therefore we have

$$\int_\Gamma \frac{\mathrm{d}\xi}{\xi-(\xi^* \pm na)} = 0. \tag{4.5}$$

Hence, from (4.5) we obtain

$$\frac{1}{a\mathrm{i}}\int_\Gamma \frac{\mathrm{d}\xi}{\tan\left[\frac{\pi(\xi-\xi^*)}{a}\right]} = 1. \tag{4.6}$$

By combining (4.1) and (4.6) one has

$$u(\xi^*) = u^* - \frac{1}{2a\mathrm{i}}\int_\Gamma \frac{u(\xi)-u(\xi^*)}{\tan\left[\frac{\pi(\xi-\xi^*)}{a}\right]}\,\mathrm{d}\xi. \tag{4.7}$$

Moreover, the contour Γ has to be a streamline, and therefore the following condition must be satisfied:

$$\frac{u(\xi)}{v(\lambda)} = \frac{\mathrm{d}\lambda}{\mathrm{d}\xi} \tag{4.8}$$

Relation (4.8) can be also written as follows:

$$u(\xi) = \frac{v(\lambda)}{\delta(\lambda)} \tag{4.9}$$

with

$$\delta(\lambda) = \mathrm{d}\xi/\mathrm{d}\lambda \tag{4.10}$$

Hence, from (4.9) one has

$$\xi(\lambda) - \xi(\lambda^*) = \int_{\lambda^*}^{\lambda} \delta(\lambda)\,\mathrm{d}\lambda. \tag{4.11}$$

Moreover, if we write $L\lambda$ instead of λ, then (4.7) reduces to the non-linear singular integral equation

$$A\delta \equiv u^* \delta(\lambda^*) - \frac{L}{2a\mathrm{i}} \int_0^1 \frac{\big[v(\lambda)\delta(\lambda^*) - \delta(\lambda)v(\lambda^*)\big]}{\tan\Big[\frac{\pi L}{a}\big(\xi(\lambda) - \xi(\lambda^*)\big)\Big]}\,\mathrm{d}\lambda = v(\lambda^*) \tag{4.12}$$

in which L is determined by the relation

$$L = \frac{a(v_1 \cos\omega_1 - v_2 \cos\omega_2)}{\int_0^1 v(\lambda)\,\mathrm{d}\lambda} \tag{4.13}$$

and the following conditions have to be satisfied for the function δ:

$$\begin{aligned} \int_0^1 \delta(\lambda)\,\mathrm{d}\lambda &= 0, \\ \big|\delta(\lambda)\big| &= 1, \\ \delta(0) &= \delta(1). \end{aligned} \tag{4.14}$$

The existence of solutions of this non-linear singular integral equation was proved by the theorems of the previous sections.

In order to solve numerically the non-linear singular integral equation (4.12), we shall use the non-linear programming method. Therefore, equation (4.12) is approximated by the formula

$$\delta_n(\lambda) = \sum_{h=-n}^{n} c_h \exp[2\mathrm{i}h\pi\lambda]. \tag{4.15}$$

Hence, we require the values of c_h in order to minimize the function

$$B(P) = \int_0^1 \big|A\delta_n - v(\lambda^*)\big|^2\,\mathrm{d}\lambda + \int_0^1 \big(|\delta_n(\lambda)| - 1\big)^2\,\mathrm{d}\lambda \tag{4.16}$$

with

$$P = P(c_h^{(1)}, c_h^{(2)}), \tag{4.17}$$

$$c_h = c_h^{(1)} + c_h^{(2)}. \tag{4.18}$$

Finally, the initial values of c_h are easily obtained, when an initial profile is given. Otherwise, we have to solve a more general minimization problem by using some general algorithms.

5. Conclusions

An analysis of non-linear singular integral equations has been presented by investigating some basic theorems. These concern the existence of solutions for the general case when the non-linear singular integral equations are defined on Banach spaces.

Some basic results have been obtained when the non-linear singular integral equations of the first and the second kind are defined over finite sets of contours. Moreover, the existence and uniqueness of solutions have been studied when the non-linear singular integral equations of the second kind are defined over a finite number of arbitrarily ordered arcs.

The theory of non-linear singular integral equations presented in this report is an extension of the theory of linear finite-part singular integral equations and linear multidimensional singular integral equations developed by E. G. Ladopoulos [1]–[9].

The non-linear singular integral equations are widely used in many problems of mathematical physics and especially in problems on an applied character. In the present communication a two-dimensional fluid mechanics application has been described, by considering the form of the lattice of profiles of turbomachines in the two-dimensional flow of an incompressible fluid, this problem being reduced to the solution of a non-linear singular integral equation.

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