Hans-Görg Roos; Martin Stynes

Necessary conditions for uniform convergence of finite difference schemes for convection-diffusion problems with exponential and parabolic layers

Applications of Mathematics, Vol. 41 (1996), No. 4, 269–280

Persistent URL: http://dml.cz/dmlcz/134326

NECESSARY CONDITIONS FOR UNIFORM CONVERGENCE OF FINITE DIFFERENCE SCHEMES FOR CONVECTION-DIFFUSION PROBLEMS WITH EXPONENTIAL AND PARABOLIC LAYERS

HANS-GÖRG ROOS, Dresden, MARTIN STYNES, Cork

(Received April 14, 1995)

Summary. Singularly perturbed problems of convection-diffusion type cannot be solved numerically in a completely satisfactory manner by standard numerical methods. This indicates the need for robust or ε-uniform methods. In this paper we derive new conditions for such schemes with special emphasize to parabolic layers.

Keywords: numerical analysis, convection-diffusion problems, boundary layers, uniform convergence

AMS classification: 65 N

1. INTRODUCTION

It is well-known that classical numerical methods on uniform grids yield satisfactory numerical solutions for singularly perturbed problems only if one uses an unacceptably large number of grid points. As a result, various "upwind" techniques have been proposed for the solution of convection-diffusion problems. Careful examination of numerical results from upwind schemes shows however that, for fixed (small) values of the perturbation parameter, the maximum pointwise error usually increases as the mesh is refined—because of the presence of layers—until the mesh diameter is comparable in size to the perturbation parameter. This behaviour is clearly unsatisfactory. It prompts a natural question: is it possible to construct numerical methods that are *robust*, i.e., that behave uniformly well for *all* values of the perturbation parameter and mesh diameter?

In this paper we discuss new conditions that robust schemes must satisfy.

2. Exponential layers

We first consider the convection-diffusion problem

$$(1)\qquad \begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu = f \quad &\text{on } \Omega := (0,1)\times(0,1) \\ u = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

where $0 < \varepsilon \leqslant 1$, $b = (b_1, b_2) > (0,0)$ on $\bar{\Omega}$ and $c \geqslant 0$ on $\bar{\Omega}$. We assume that the data of the problem are smooth.

Here ε is a perturbation parameter. Problem (1) is characterized by the existence of *exponential boundary layers* at the sides $x = 1$ and $y = 1$ of $\bar{\Omega}$. These layers cause serious instabilities in standard difference schemes.

Consider solving (1) on a square mesh with mesh width $h = 1/N$ (N a positive integer) and assume that b_1 and b_2 are positive constants with $c = 0$. Suppose that we have a nine-point difference scheme for (1) and that at each mesh point (x_i, y_j) the scheme can be written as

$$(2)\qquad \sum_{\nu,\mu=-1}^{1} a_{\nu\mu} u_{i+\nu,j+\mu} = h\tilde{f}_{ij},$$

where each $a_{\nu\mu}$ and $\tilde{f}_{ij}$ depend only on the ratio h/ε (all schemes of which we are aware have this form).

We say that such a scheme is *uniformly convergent* (with respect to ε) of order α if, for some positive constants C and α that are independent of ε and of the mesh, its solution $\{u_{ij}\}$ satisfies

$$(3)\qquad |u_{ij} - u(x_i, y_j)| \leqslant Ch^{\alpha}$$

for all i and j. (This definition is, of course, with respect to the discrete maximum norm, but other choices are possible; see [13].)

Necessary conditions for convergence, uniformly with respect to the perturbation parameter, of finite difference schemes for convection-diffusion problems are known in several cases [8], [4], [13]. Such conditions are useful both for proving that particular schemes cannot be uniformly convergent and for aiding the construction of new schemes that may be uniformly convergent.

To achieve uniform convergence, the coefficients of the scheme must satisfy the following four conditions [8]:

$$\sum_{\nu,\mu=-1}^{1} a_{\nu\mu} = 0, \tag{4}$$

$$e^{-b_1 h/\varepsilon} \sum_{\mu=-1}^{1} a_{-1,\mu} + \sum_{\mu=-1}^{1} a_{0,\mu} + e^{b_1 h/\varepsilon} \sum_{\mu=-1}^{1} a_{1,\mu} = 0, \tag{5}$$

$$e^{-b_2 h/\varepsilon} \sum_{\nu=-1}^{1} a_{\nu,-1} + \sum_{\nu=-1}^{1} a_{\nu,0} + e^{b_2 h/\varepsilon} \sum_{\nu=-1}^{1} a_{\nu,1} = 0, \tag{6}$$

$$\sum_{\nu,\mu=-1}^{1} a_{\nu\mu} e^{(\nu b_1 + \mu b_2)h/\varepsilon} = 0. \tag{7}$$

In fact, [8] assumes (4), but it can be deduced from the calculations in [8] that yield (5)–(7).

If one weakens the hypothesis of uniform convergence in the discrete maximum norm to uniform convergence of order greater than $1/2$ in the discrete L_2 norm, then [13] the scheme must still satisfy conditions (5) and (6).

Remark 1. Certain authors have derived alternative conditions on the coefficients $a_{\nu\mu}$ by "optimizing" the truncation error of the scheme at (x_i, y_j) according to various criteria. This approach is unfortunately flawed, since the true order of the truncation error may often be found only by considering the truncation error at some point that is not itself a node.

Example 1. As an application of (4)–(7), consider the five-point scheme that upwinds in each coordinate direction by an arbitrary amount. Assume for simplicity that $c \equiv 0$. After multiplication by h, we can write the stencil of the scheme as

$$-\frac{\varepsilon}{h}\begin{bmatrix} \cdot & 1 & \cdot \\ 1 & -4 & 1 \\ \cdot & 1 & \cdot \end{bmatrix} + \frac{b_1}{2}\begin{bmatrix} \cdot & \cdot & \cdot \\ -1-p & 2p & 1-p \\ \cdot & \cdot & \cdot \end{bmatrix} + \frac{b_2}{2}\begin{bmatrix} \cdot & 1-q & \cdot \\ \cdot & 2q & \cdot \\ \cdot & -1-q & \cdot \end{bmatrix},$$

where p and q are upwinding parameters.

The conditions (4)–(7) are satisfied if

$$p = \coth\left(\frac{b_1 h}{2\varepsilon}\right) - \frac{2\varepsilon}{b_1 h} \quad \text{and} \quad q = \coth\left(\frac{b_2 h}{2\varepsilon}\right) - \frac{2\varepsilon}{b_2 h}.$$

The scheme that uses these values of p and q is the famous Il'in-Allen-Southwell scheme. Emel'janov [3] proved its uniform convergence, provided that the data of (1) satisfies certain compatibility conditions.

If we assume that the scheme (2) has higher-order convergence away from layers, then we can derive two further necessary conditions.

We say that u is *smooth* on an open subset S of Ω if its first-order derivatives are bounded independently of ε on S.

Theorem 1. *Assume that* (4) *holds. Assume also that on open subsets S of Ω where u is smooth, the scheme is superlinearly uniformly convergent; that is,*

$$|u_{ij} - u(x_i, y_j)| \leqslant Ch^{\alpha}, \quad \text{where } \alpha > 1, \tag{8}$$

for some positive constants C and α that are independent of ε and of the mesh. Then

$$\sum_{\nu=-1}^{1} a_{\nu,1} - \sum_{\nu=-1}^{1} a_{\nu,-1} = b_2 \tag{9}$$

and

$$\sum_{\mu=-1}^{1} a_{1,\mu} - \sum_{\mu=-1}^{1} a_{-1,\mu} = b_1. \tag{10}$$

Proof. Fix the ratio $h/\varepsilon = C_1$, say. Choose a positive integer i.

We shall examine the behaviour of the scheme near the point $(1/2, 0)$. Assume that $N = 1/h$ is even. For h sufficiently small, we have the outer asymptotic expansion

$$u(\pm h + 1/2, jh) = (u_0 + \varepsilon u_1)(\pm h + 1/2, h) + O(\varepsilon^2) \tag{11}$$

for $j = 0, 1, 2$, where u_0 is the solution of the reduced problem

$$b \cdot \nabla u_0 = f, \quad \text{on } \Omega,$$
$$u_0(x, y) = 0, \quad \text{when } xy = 0,$$

and u_1 is defined in the usual way [2].

Suppose that f is constant, so that $\lim_{h\to 0} \tilde{f}_{ij} = f$. Then from (2),

$$
\begin{aligned}
f &= \lim_{h\to 0} h^{-1} \sum_{\nu,\mu=-1}^{1} a_{\nu\mu} u_{\nu+N/2,1+\mu} \\
&= \lim_{h\to 0} h^{-1} \sum_{\nu,\mu=-1}^{1} a_{\nu\mu} u(\nu h + 1/2, (1+\mu)h), \quad \text{by (8)}, \\
&= \lim_{h\to 0} h^{-1} \sum_{\nu,\mu=-1}^{1} a_{\nu\mu} (u_0 + \varepsilon u_1)(\nu h + 1/2, (1+\mu)h), \quad \text{by (11)}, \\
&= \lim_{h\to 0} h^{-1} \Bigg\{ \big(u_0 + (h/C_1)u_1\big)(1/2, h) \sum_{\nu,\mu=-1}^{1} a_{\nu\mu} \\
&\qquad + h\frac{\partial u_0}{\partial x}(1/2,h)\Big[\sum_{\mu=-1}^{1} a_{1,\mu} - \sum_{\mu=-1}^{1} a_{-1,\mu}\Big] \\
&\qquad + h\frac{\partial u_0}{\partial y}(1/2,h)\Big[\sum_{\nu=-1}^{1} a_{\nu,1} - \sum_{\nu=-1}^{1} a_{\nu,-1}\Big]\Bigg\}, \qquad (12)
\end{aligned}
$$

by a Taylor expansion. Now $\sum_{\nu,\mu=-1}^{1} a_{\nu\mu} = 0$ from (4). Also,

$$
\begin{aligned}
\lim_{h\to 0} \frac{\partial u_0}{\partial x}(1/2,h) &= \frac{\partial u_0}{\partial x}(1/2,0) = 0, \\
\lim_{h\to 0} \frac{\partial u_0}{\partial y}(1/2,h) &= \frac{\partial u_0}{\partial y}(1/2,0) = f/b_2,
\end{aligned}
$$

by the definition of u_0. Thus (12) simplifies to

$$
f = \frac{f}{b_2}\Big[\sum_{\nu=-1}^{1} a_{\nu,1} - \sum_{\nu=-1}^{1} a_{\nu,-1}\Big].
$$

The result (9) follows. A similar analysis at $(0, 1/2)$ yields (10). □

We say that (2) is a *five-point scheme* if $a_{\nu\mu} = 0$ when the product $\nu\mu$ is non-zero.

Corollary 1. *For* (1), *no uniformly convergent five-point difference scheme exists that is superlinearly convergent on open subsets of* Ω *where* u *is smooth.*

P r o o f. Conditions (5) and (10) force the choice

$$
p = \coth\left(\frac{b_1 h}{2\varepsilon}\right) - \frac{2\varepsilon}{b_1 h}
$$

in Example 1. An analogous result holds for q. That is, our hypotheses hold only for the Il'in-Allen-Southwell scheme. Now the matrix of this scheme is of positive type, and it is known (see [6], or adapt the proof of [14]) that, in general, positive-type schemes yield at best first-order convergence when applied to first-order hyperbolic problems. This result can easily be extended to (1) when ε is small. But this means that we have contradicted the hypotheses of Corollary 1. □

Corollary 2. *For the* 1*-dimensional analogue of* (1), *with constant* b *and* f, *and* $c \equiv 0$, *the Il'in-Allen-Southwell scheme is the only uniformly convergent three-point scheme that can be superlinearly convergent on subintervals where* u *is smooth.*

Proof. We can easily imitate our earlier work to derive 1-dimensional analogues of (4), (5) and (10) for three-point schemes. These conditions are satisfied if and only if the scheme is Il'in-Allen-Southwell. □

Remark 2. Theorem 1 also applies to schemes that are defined on piecewise uniform meshes like those of Shishkin [11], [12].

3. Parabolic layers

For a problem with a *parabolic boundary layer*, however, it is an open problem whether or not a uniformly convergent method on an equidistant mesh exists. (Here we define uniform convergence analogously to Section 2.) We note that, for a parabolic initial-boundary value problem of reaction-diffusion type in one space dimension, whose solution has a parabolic layer, Shiskin [10] proved the following remarkable result: *If a six-point scheme uses only standard functions on an equidistant mesh and satisfies the discrete maximum principle, then it cannot be uniformly convergent.* The difficulty is caused by the parabolic layer in the solution, not by the parabolic nature of the differential equation. Therefore one would expect a similar negative result to hold also for any elliptic convection-diffusion problem whose solution has a parabolic layer.

We shall consider the model problem

$$
\begin{aligned}
Lu := \varepsilon\Delta u - u_y = f, \quad & \text{on } \Omega = (0,1)\times(0,1), \\
\text{where } u = 0 \quad \text{for } x = 0, \quad & u = g_1 \quad \text{for } x = 1, \\
u = 0 \quad \text{for } y = 0, \quad & u = g_2 \quad \text{for } y = 1,
\end{aligned}
\tag{13}
$$

and we shall specify g_1 and g_2 later. Once again, $0 < \varepsilon \leqslant 1$ and we assume that f is smooth.

We first study the asymptotic behaviour of u as $\varepsilon \to 0$. Let u_0 be the solution of the *reduced problem*

$$-(u_0)_y = f(x,y) \quad \text{on } \Omega, \quad u_0\big|_{y=0} = 0.$$

Clearly $u_0(x,y) = -\int_0^y f(x,t)\,\mathrm{d}t$, so u_0 is smooth. This property is worth noting because it is not shared by the solution of the reduced problem associated with (1), which will generally be smooth only under additional assumptions on the data in a neighbourhood of the corner (0,0).

In general u has a boundary layer along the side $x = 0$ of $\bar{\Omega}$. Introducing $\xi := x/\sqrt{\varepsilon}$, we require the layer correction $v(\xi, y)$ to solve

$$\begin{aligned} &v_{\xi\xi} - v_y = 0 \quad \text{on } \Omega, \\ &\text{with } v(\xi,0) = 0, \\ &\qquad v(0,y) = -u_0(0,y) = h(y), \text{ where } h(y) := \int_0^y f(0,t)\,\mathrm{d}t. \end{aligned} \tag{14}$$

The equations (14) form an initial-boundary value problem of parabolic type, so at $x = 0$ we say that we have a *parabolic layer*. Now the solution $v(\xi, y)$ of (14) decreases exponentially to zero as $\xi \to \infty$. The smoothness of v in the neighbourhood of the corner $(0,0)$ depends on the behaviour of $h^{(l)}(0)$, for $l = 0, 1, \ldots$.

We now assume in addition that

$$f(0,y) = 2y. \tag{15}$$

Then $h(y) = y^2$, and we have $h(0) = h'(0) = 0$. Hence v_{yy} is bounded at (0,0), but higher-order derivatives with respect to y have a singularity at the origin.

We define

$$g_1 := (u_0 + v)\big|_{x=1} \quad \text{and} \quad g_2 := (u_0 + v)\big|_{y=1}. \tag{16}$$

We make these choices to exclude the development of layers at $x = 1$ (where, in general, one finds a second parabolic layer) and $y = 1$ (here, in general, an ordinary exponential layer occurs). Consequently they also exclude an overlap of parabolic and exponential layers at $(0,1)$ and $(1,1)$, which would complicate the asymptotic structure (see [7] and [9]). We obtain

Lemma 1. *There exists a constant C independent of ε such that*

$$|u(x,y) - (u_0(x,y) + v(x,y))| \leqslant C\varepsilon \quad \textit{for all } (x,y) \in \bar{\Omega}. \tag{17}$$

Proof. For the problem (13) with the choices (15) and (16), the zero-order compatibility conditions are fulfilled, so (13) has a classical solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ (see [5]). Setting $u_{as} := u_0 + v$, the asymptotic approximation u_{as} satisfies the given boundary conditions. The smoothness of u, u_0 and v allows us to compute

$$L(u - u_{as}) = -\varepsilon \Delta u_0 - \varepsilon v_{yy} \quad \text{on } \Omega,$$

and we note that Δu_0 and v_{yy} are bounded independently of ε. The comparison principle for second-order elliptic operators then immediately yields (17). □

The model problem (13) has exactly one layer of parabolic type and we can explicitly solve the layer correction problem (14)–(15). Setting

$$\Phi(\xi, y) = 1 - \operatorname{erf}(\xi/2\sqrt{y}) = 1 - \frac{2}{\pi} \int_0^{\xi/2\sqrt{y}} e^{-\alpha^2}\, d\alpha = \operatorname{erfc}(\xi/2\sqrt{y})$$

(Φ is the solution of (14) when $h(y) \equiv 1$), we obtain

$$v(\xi, y) = 2 \int_0^y \int_0^\tau \Phi(\xi, \mu)\, d\mu\, d\tau. \tag{18}$$

This representation follows easily from the following observation: if Φ^* is a solution of (14), then

$$\Phi^{**}(\xi, y) := \int_0^y \Phi^*(\xi, \mu)\, d\mu$$

is also a solution of (14); only the boundary condition at $\xi = 0$ is altered.

Consider a square mesh with mesh width $h = 1/N$. Suppose that at each mesh point (x_i, y_j), we have a nine-point difference scheme for (13) that can be written as

$$\sum_{\nu,\mu=-1}^{1} a_{\nu\mu} u_{i+\nu,j+\mu} = h\tilde{f}_{ij}, \tag{19}$$

where each $a_{\nu\mu}$ and $\tilde{f}_{ij}$ now depend only on the ratio $\varrho := h/\sqrt{\varepsilon}$, which corresponds to the nature of the parabolic layer (cf. (2)).

A scheme for (13) is said to be *uniformly convergent* in the discrete maximum norm if its solution $\{u_{i,j}\}$ satisfies

$$\lim_{h \to 0} |u(x_i, y_j) - u_{i,j}| = 0, \quad \text{for all } i \text{ and } j.$$

Suppose that the scheme (19) is uniformly convergent. Fix ϱ, i and j. Then

$$
\begin{aligned}
0 = \lim_{h\to 0} h\tilde{f}_{i,N-j} &= \lim_{h\to 0} \sum_{\nu,\mu=-1}^{1} a_{\nu\mu} u_{i+\nu,N-j+\mu} \\
&= \lim_{h\to 0} \sum_{\nu,\mu=-1}^{1} a_{\nu\mu} u(x_{i+\nu}, y_{N-j+\mu}),
\end{aligned}
$$

since the scheme is uniformly convergent. Hence, using (17),

$$
\begin{aligned}
0 &= \sum_{\nu,\mu=-1}^{1} a_{\nu\mu}\left(u_0(0,1) + v((i+\nu)\varrho, 1)\right) \\
&= \sum_{\nu,\mu=-1}^{1} a_{\nu\mu}\left(-1 + v((i+\nu)\varrho, 1)\right).
\end{aligned}
$$

Assume that (4) holds, viz., that $\sum a_{\nu\mu} = 0$ (this will be true if the scheme is consistent). Introducing the abbreviation $V_i := v(i\varrho, 1)$, we then obtain

Theorem 2. *If the scheme* (19) *applied to the problem* (13) *with smooth data is uniformly convergent in the discrete maximum norm and* (4) *holds, then for any fixed i we have*

$$
V_{i-1} \sum_{\mu=-1}^{1} a_{-1,\mu} + V_i \sum_{\mu=-1}^{1} a_{0,\mu} + V_{i+1} \sum_{\mu=-1}^{1} a_{1,\mu} = 0. \tag{20}
$$

Remark 2. Note how (20) resembles condition (5) earlier.

In fact, (20) implies the following more specific result.

Corollary 3. *The necessary condition* (20) *for uniform convergence can be satisfied only if*

$$
\sum_{\mu=-1}^{1} a_{-1,\mu} = \sum_{\mu=-1}^{1} a_{0,\mu} = \sum_{\mu=-1}^{1} a_{1,\mu} = 0. \tag{21}
$$

Hence no five-point scheme (as defined in Section 1) exists that is consistent with (13) *and uniformly convergent.*

Proof. The necessary condition (20) is of the form

$$
AV_{i-1} + BV_i + DV_{i+1} = 0, \tag{22}
$$

where A, B and D are constants. The solution of such a difference equation, if $ABD \neq 0$, is a linear combination of exponentials $e^{k_1 i}$ and $e^{k_2 i}$, for certain constants k_1 and k_2. It is again exponential in i if exactly two of A, B and D are nonzero. But for large values of i, well-known properties of the erfc function (see [1], 7.1.23) show that there exists a constant d with

$$V_i \sim e^{-d\,i^2} \quad \text{for large } i.$$

Hence V_i cannot satisfy (22) unless $A = B = D = 0$, i.e., (21) holds.

For a five-point scheme, (21) immediately yields $a_{-1,0} = a_{1,0} = 0$. Thus the scheme reduces to a three-point scheme. But a scheme with fewer than five points cannot be consistent with (13). □

We expect that, likewise, the class (19) of nine-point schemes contains no uniformly convergent scheme, but at present we cannot prove so general a result. Consider, however, a standard discretization of (13) with a fitting factor $\sigma = \varepsilon^{-1/2}\sigma^*(\varrho)$; this can be rewritten in the form

$$\frac{\sigma^*}{\varrho} \sum_{\nu,\mu=-1}^{1} a^*_{\nu\mu} u_{i+\nu,j+\mu} + \sum_{\nu,\mu=-1}^{1} b^*_{\nu\mu} u_{i+\nu,j+\mu} = h\tilde{f}_{ij},$$

where $a^*_{\nu,\mu}$ and $b^*_{\nu,\mu}$ are independent of ϱ (they come from the discretizations of Δu and u_y respectively). Then for uniform convergence we expect some form of exponential fitting, but (21) results in $\sigma^*(\varrho) = C\varrho$. That is, any scheme that satisfies (21) cannot reflect the behaviour of the solution of the differential equation in the parabolic layer.

Returning to schemes of the form (19), let us observe that for problem (13) we can derive an analogue of Theorem 1 on open subsets S of Ω where u is smooth.

Theorem 3. *Assume that* (4) *holds. Assume also that on open subsets S of Ω where u is smooth, the scheme* (19) *is superlinearly uniformly convergent. Then*

(23) $$\sum_{\nu=-1}^{1} a_{\nu,1} - \sum_{\nu=-1}^{1} a_{\nu,-1} = -1$$

and

(24) $$\sum_{\mu=-1}^{1} a_{1,\mu} - \sum_{\mu=-1}^{1} a_{-1,\mu} = 0.$$

Proof. The argument is similar to the proof of Theorem 1. Instead of the asymptotic expansion (11), we use

$$u = u_0 + O(\varepsilon)$$

in S, where u_0 is now of course the reduced solution of (13).

Fixing ϱ, we obtain for constant f, analogously to the proof of Theorem 1, the condition (23).

By choosing $f(x, y) = x$ and considering $u(1/2 \pm h, 1/2 \pm h)$, we obtain the second necessary condition (24). □

Similar ideas can also be used to analyse schemes other than (19), as we now show.

Let us for a moment suppose that we use different mesh sizes h_x and h_y in the x- and y-directions and consider the fitted scheme

$$\begin{aligned} \kappa(\varrho)(u_{i-1,j} - 2u_{ij} + u_{i+1,j}) + \frac{\varepsilon}{h_y^2}(u_{i,j-1} - 2u_{ij} + u_{i,j+1}) \\ - \frac{1}{h_y}(u_{i,j} - u_{i,j-1}) = f(x_i, y_j) \end{aligned} \tag{25}$$

with $\varrho = h_x/\sqrt{\varepsilon}$. This upwind scheme is fitted in the $x-$direction in an attempt to achieve uniform convergence; in particular, the choice

$$\kappa(\varrho) = \frac{1}{\varrho^2}$$

yields standard upwinding.

Now we assume that (25) is uniformly convergent, and follow the line of argument of Theorem 2 to arrive at a contradiction.

Thus, assume as in (15) that $f(0, y) = 2y$; fix ϱ, i and j (replacing j by $N - j$ as in the earlier proof); take $h = h_x = h_y$ and let $h \to 0$ in (25). As before, the uniform convergence allows us first to replace each $u_{i+\nu,N-j+\mu}$ by $u(x_{i+\nu}, y_{N-j+\mu})$ and second, using (17), to bring $u_0 + v$ into the game. Recall that the solution u_0 of the reduced problem satisfies $-(u_0)_y = f$ and that the discretization of the first-order derivative reflects this on the discrete level as $h_y \to 0$. We thereby obtain

$$\kappa(\varrho) \sum_{\nu=-1}^{1} v((i+\nu)\varrho, 1) - \frac{\partial v(i\varrho, y)}{\partial y}\Big|_{y=1} = 0. \tag{26}$$

The condition (26) leads us to the following formula for the fitting factor:

$$\kappa(\varrho) = \frac{\partial v(i\varrho, y)}{\partial y}\Big|_{y=1} / \{V_{i-1} - 2V_i + V_{i+1}\}. \tag{27}$$

The behaviour of the erfc function (see the derivation of Corollary 3 and note that v_y behaves asymptotically like v) implies that, at least when i is large, the right-hand side here depends on i, while the left-hand side is independent of i. This contradiction proves

Theorem 4. *For problem* (13), *there exist no uniformly convergent fitted scheme of the form* (25).

References

[1] *Abramowitz, M., Stegun, I.A.*: Handbook of mathematical functions. National Bureau of Standards, 1964.

[2] *Eckhaus, W.*: Matched asymptotic expansions and singular perturbations. North-Holland, Amsterdam, 1973.

[3] *Emel'janov, K.V.*: A difference scheme for a three-dimensional elliptic equation with a small parameter multiplying the highest derivative. Boundary value problems for equations of mathematical physics, USSR Academy of Sciences, Ural Scientific Centre. 1973, pp. 30–42. (In Russian.)

[4] *Guo, W.*: Uniformly convergent finite element methods for singularly perturbed parabolic problems. Ph.D. Dissertation, National University of Ireland, 1993.

[5] *Han, H., Kellogg, R.B.*: Differentiability properties of solutions of the equation $-\varepsilon^2\Delta u + ru = f(x,y)$ in a square. SIAM J. Math. Anal., *21* (1990), 394–408.

[6] *Lax, P.D.*: On the stability of difference approximations to solutions of hyperbolic equations with variable coefficients. Comm. Pure Appl. Math. *14* (1961), 497–520.

[7] *Lelikova, E.F.*: On the asymptotic solution of an elliptic equation of the second order with a small parameter effecting the highest derivative. Differential Equations *12* (1976), 1852–1865. (In Russian.)

[8] *Roos, H.-G.*: Necessary convergence conditions for upwind schemes in the two-dimensional case. Int. J. Numer. Meth. Eng. *21* (1985), 1459–1469.

[9] *Shih, S.D., Kellogg, R.B.*: Asymptotic analysis of a singular perturbation problem. SIAM J. Math. Anal., *18* (1987), 1467–1511.

[10] *Shishkin, G.I.*: Approximation of the solutions of singularly perturbed boundary-value problems with a parabolic boundary layer. U.S.S.R. Comput. Maths. Math. Physics *29* (1989), 1–10.

[11] *Shishkin, G.I.*: Grid approximation of singularly perturbed boundary value problems with convective terms. Sov. J. Numer. Anal. Math. Modelling *5* (1990), 173–187.

[12] *Shishkin, G.I.*: Methods of constructing grid approximations for singularly perturbed boundary value problems. Sov. J. Numer. Anal. Math. Modelling *7* (1992), 537–562.

[13] *Stynes, M., Tobiska, L.*: Necessary L_2-uniform conditions for difference schemes for two-dimensional convection-diffusion problems. Computers Math. Applic. *29* (1995), 45–53.

[14] *Yserentant, H.*: Die maximale Konsistenzordnung von Differenzapproximationen nichtnegativer Art. Numer. Math. *42* (1983), 119–123.

Authors' addresses: *Hans-Görg Roos*, Institut f. Numer. Mathematik, TU Dresden, D-01062 Dresden, Germany, e-mail: `roos@math.tu-dresden.de`; *Martin Stynes*, Mathematics Department, University College Cork, Ireland, e-mail: `stynes@ucc.ie`.