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LINEAR DIFFERENTIAL EQUATIONS - GLOBAL THEORY

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I. Introduction

Investigations of linear differential equations started in the last century in works of E.E. Kummer, E. Laguerre, F. Brioschi, G. H. Halphen, P. Stäckel, S. Lie, E.J. Wilczynski and others. As pointed out by G. Birkhoff [1], their results were of local character, and till the middle of this century there were only isolated results of a global nature.

In the last 30 years O. Borůvka [2], [3] deeply developed the theory of global properties of linear differential equations of the second order. In this paper we give some basic facts from a global approach to linear differential equations of the n-th order.

II. Global transformations

Let

$P: \quad y^{(n)}+p_{n-1}(x)y^{(n-1)}+ \dots +p_0(x)y = 0 \quad \text{on } I\subset \mathbb{R},$ and

$Q: \quad z^{(n)}+q_{n-1}(t)z^{(n-1)}+ \dots +q_0(t)z = 0 \quad \text{on } J\subset \mathbb{R}$

be two linear differential equations of the n-th order, $n \geq 2$, with real continuous coefficients. We say that P is globally transformable into Q, if there exist functions f and h, $f: J \to \mathbb{R}$, $f \in C^n(J)$, $f(t) \neq 0$ on J, $h(J) = I$, $h \in C^n(J)$, $dh(t)/dt \neq 0$ on J, such that

(1) $$z(t) = f(t).y(h(t))$$

is a solution of Q whenever y is a solution of P.

For n-tuples $\underline{y}$ and $\underline{z}$ of linearly independent solutions of P and Q, resp., the condition (1) can be replaced by

(2) $$\underline{z}(t) = A.f(t).\underline{y}(h(t))$$

for a suitable real constant regular matrix A. In such a situation we shall briefly write

$$\alpha P = Q$$

to express that P is globally transformable into Q by a transformation α (consisting of A, f, and h in the sense of (2)).

Furthermore, let Q be globally transformable into an equation R by β, i.e. $\beta Q = R$. Then we define $(\beta * \alpha) P = \beta(\alpha(P)) = R$. The relation of "global transformability" is an equivalence relation and it divides the set of all linear differential equations into classes of globally equivalent equations.

All linear differential equations as objects and global transformations as morphisms with the composition "$*$" form a special category, an Ehresmann groupoid (i.e. a category every morphism of which has an inverse).

All transformations that transform an equation, say P, into itself form a group, so-called a stationary group S(P) of P. A typical result obtained by combining methods of the theory of categories and functional equations is the following

Theorem 1. Stationary groups of any two globally equivalent linear differential equations are conjugate. The groups are not trivial if and only if a linear differential equation with periodic coefficients belongs to the same equivalence class.

For further details see [6] , [8] .

III. Zeros of solutions

The following theorem, first introduced in [5], enables us to see into essence of possible distributions of zeros of solutions often without necessity of complicated calculations and analytic constructions.

Theorem 2. Let P be a linear differential equation and $\underline{y}$ be a curve in n-dimensional vector space formed by n linearly independent solutions of P. To each solution of P there exists a hyperplane going through the origin of the space such that zeros of the solution are parameters of intersections of the hyperplane with the curve (including multiplicities).

The theorem remains true if euclidean space is taken, and instead of $\underline{y}$ its central projection onto the unit sphere is considered. Then powerful topological tools can be applied to solve some open problems and to construct linear differential equations with prescribed properties of zeros of their solutions, [5], [7].

IV. Canonical forms

The well-known Laguerre-Forsyth canonical form is not global in the sense that not every linear differential equation even of the third order can be globally transformed into an equation of that form (see [1]). It can be shown that the so-called Halphen form is not global either. Neither of these forms can be made global by restricting ourselves to the class of linear differential equations where some smoothness (or even analyticity) of coefficients is required. Using our general approach we have several ways how to suggest global canonical forms. One of the possibilities is described in [5]:

Consider the central projection $\underline{v}$ on the unit sphere of an n-tuple $\underline{y}$ of linearly independent solutions of an equation P. Then introduce a lenght parametrization into $\underline{v}$ to get a curve $\underline{u}$. Linear differential equations with these $\underline{u}$ as n-tuples of solutions are global canonical equations. In this way we get

Theorem 3. $u'' + u = 0$ on some $I \subset \mathbb{R}$,

$u''' - \frac{a'}{a} u'' + (1 + a^2)\, u' - \frac{a'}{a} u = 0$ on some $I \subset \mathbb{R}$, $a \in C^1(I)$, $a > 0$,

are global canonical forms.

For $n \geqq 4$ and more details see [5].

There is also another way how to construct global canonical forms. The following theorem introduces one of the possible forms obtained by the method, [9].

Theorem 4.

$$u^{(n)} + u^{(n-2)} + s_{n-3}(x)u^{(n-3)} + \dots + s_0(x)u = 0 \qquad \text{on } I \subset \mathbb{R},$$

is a global canonical form.

V. Effective conditions for global equivalence

Combining our approach with some results of G.H. Halphen and with Cartan s method of moving frame we get effective conditions for global equivalence of two given linear differential equations of the n-th order, $n \geqq 3$, with the exception of so-called iterative equations. The problem for the iterative equations can be reduced to linear differential equations of the second order. Conditions for global equivalence of these equations are given in [2].

By effectivity we mean that the conditions are expressible in terms of coefficients of given equations. If the equations are globally equivalent, then the corresponding global transformation can also be constructed effectively from their coefficients.

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