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ON PROPERTIES OF SPECTRAL APPROXIMATIONS

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In this paper, we want to discuss connections between some conditions used in the theory of spectral approximation. For the sake of simplicity we shall restrict ourselves to the following framework: X is a complex Banach space with norm $\|\cdot\|$; X_n, $n \in \mathbb{N}$, is a sequence of finite dimensional subspaces of X; $\Pi_n: X \to X$ are linear projectors with range X_n which converge strongly to the identity; $A: X \to X$ is a linear bounded operator; the linear operators $B_n: X \to X$, uniformely bounded, with range in X_n, are supposed to approximate A; $A_n: X_n \to X_n$ is then defined as the restriction of B_n to X_n (or, given the A_n's, one can, for example, define $B_n = A_n\Pi_n$); B_n will be called the "Galerkin approximation of A" if $B_n = \Pi_n A$. Remark that B_n is compact and has the same eigenvalues and eigensubspaces as A_n (with the exception of o).

We shall use the following notations. If Y and Z are closed subspaces of X, then, for $x \in X$, $\delta(x,Y) = \inf_{y\in Y}\|x-y\|$, $\delta(Y,Z) = \sup_{y\in Y,\|y\|=1} \delta(y,Z)$, $\hat{\delta}(Y,Z) = \max(\delta(Y,Z),\delta(Z,Y))$. For a linear operator C defined on X or X_n, with range in X, we set $\|C\|_n = \sup_{x\in X_n,\|x\|=1} \|Cx\|$.

Let us introduce some properties of approximations of A by A_n or B_n: U) $\lim_{n\to\infty}\|A-B_n\| = o$; A1) $\lim_{n\to\infty} B_n = A$ strongly; A2) $\{B_n x \mid \|x\| \le 1,\ n \in \mathbb{N}\}$ is relatively compact; Z) $\lim_{n\to\infty}\|A-A_n\|_n = o$; R) $\lim_{n\to\infty} \sup_{x\in X_n,\|x\|=1} \delta(Ax,X_n) = o$; V1) $x_n \in X_n$, $\lim_{n\to\infty} x_n = x$ $\Rightarrow \lim_{n\to\infty} A_n x_n = Ax$; V2) for any bounded sequence $x_n \in X_n$, $\{(A-A_n)x_n\}$ is relatively compact ; G) for any $\lambda \in \rho(A)$, for any subsequence $\{x_\alpha\}$ of any bounded sequence $x_n \in X_n$ such that $(A_\alpha-\lambda)x_\alpha$ converges, there exists a converging subsequence $\{x_\beta\}$ of $\{x_\alpha\}$ such that $A(\lim_\beta x_\beta) = \lim_\beta A_\beta x_\beta$.

A2 means that $\{B_n\}$ is collectively compact in the sense of Anselone [1]; Z and R has been studied by the authors in [2]; R means that X_n is "almost" an invariant subspace of A; V1 and V2 imply that A_n is a compact approximation in the sense of Vainikko [8]; G is used, in a more general context, by Grigorieff and others in particular in [4],[5]. Since B_n is compact, note that U or {A1,A2} implies that A is compact.

In the following $\sigma(A)$, $\rho(A)$, $\sigma(A_n)$, $\rho(A_n)$, $\sigma(B_n)$, $\rho(B_n)$ will denote the spectrum and the resolvant sets of A, A_n and B_n. $R_z(A) = (A-z)^{-1}: X \to X$ and $R_z(A_n) = (A_n-z)^{-1}: X_n \to X_n$ are the resolvent operators of A and A_n defined respectively for $z \in \rho(A)$ and $z \in \rho(A_n)$.

Let $\Gamma \subset \rho(A)$ be a Jordan curve; we set $P = -(2\pi i)^{-1} \int_\Gamma R_z(A)dz$ and, if $\Gamma\ \rho(A_n)$, $P_n = -(2\pi i)^{-1} \int_\Gamma R_z(A_n)dz: X_n \to X_n$. P and P_n are the spectral projectors and $E = P(X)$, $E_n = P_n(X_n)$ are the invariant subspaces of A and A_n relative to Γ.

Consider now some spectral properties: S1) for any $z \in \rho(A), \exists N_z \in \mathbb{N}$ and M_z such that $\| R_z(A_n)\|_n \leq M_z$, $n > N_z$; S2) $\forall x \in E$, $\underline{\lim}_{n\to\infty} \delta(x,E_n) = 0$; S3) $\lim_{n\to\infty} \delta(E_n,E) = 0$; S4) if E is finite dimensional, then $\lim_{n\to\infty} \hat{\delta}(E_n,E) = 0$. If X is a Hilbert space and if A and A_n are selfadjoint, for an interval $I \subset \mathbb{R}$, define E_I as the invariant subspace of A relative to I and $E_{In} \subset X_n$ as the invariant subspace of A_n relative to I; we then introduce the property SH): for all intervals I and J, the closure of I being a subset of the interior of J, one has $\lim_{n\to\infty} \delta(E_{In},E_J) = 0$.

S1, which is a property of stability, implies the upper semi-continuity of the spectrum and garantees the meaningfullness of the approximated spectrum $\sigma(A_n)$. S2 has little importance for application; however S3 garantees the meaningfullness of all the elements of the approximate invariant subspace E_n. If Γ contains only an eigenvalue $\lambda \in \sigma(A)$ of algebraic finite multiplicity, S1 and S4 imply that λ is stable in the sense of Kato ([6],p.437). For the selfadjoint case, SH is a refinement of S3.

Proposition 1: a) U $\Rightarrow$ {A1, A2, Z, R, V1, V2, G, S1, S2, S3, S4}; b) {A1, A2} $\Rightarrow$ {R, V1, V2, G, S1, S2, S4}; {A1, A2} $\not\Rightarrow$ S3; if A and B_n are selfadjoint {A1, A2} $\Rightarrow$ U; c) Z $\Rightarrow$ {R, V1, V2, G, S1, S2, S3, S4}; for the selfadjoint case, Z $\Leftrightarrow$ SH $\Leftrightarrow$ {V1, V2}; d) if A_n is the Galerkin approximation of A, R $\Leftrightarrow$ Z $\Leftrightarrow$ V2; e) {V1, V2} $\Rightarrow$ {G, S1, S2, S4}, V2 $\Rightarrow$ R; {V1, V2} $\not\Rightarrow$ S3; f) G $\Leftrightarrow$ {V1, S1}; G $\not\Rightarrow$ S2; G $\not\Rightarrow$ R, G $\not\Rightarrow$ S3; G $\not\Rightarrow$ S4.

Most statements of Proposition 1 can be obtained directly or with little work from known results in the litterature; for b), see Anselone [1]; for c), d), see Descloux, Nassif, Rappaz [2],[3]; for e), see Vainikko [8]; for f), see, for example, Grigorieff[4], Jeggle [5]. However let us verify in e) that V2 $\Rightarrow$ R: suppose R false; $\exists \varepsilon > 0$, the sequence $x_n \in X_n$, $n \in \mathbb{N}$, $\| x_n\| = 1$ and a subsequence $\{x_\alpha\}$ of $\{x_n\}$ such that $\delta(Ax_\alpha,X_\alpha) \geq \varepsilon$; V2 implies the existence of $y \in X$ and of a subsequence $\{x_\beta\}$ of $\{x_\alpha\}$ such that $\lim_\beta (A-A_\beta)x_\beta = y$; setting $Z_\beta = A_\beta x_\beta + \Pi_\beta y \in X_\beta$, one has $\lim_{\beta\to\infty}(Ax_\beta - Z_\beta) = 0$, which is a contradiction. We verify in c) that {V1, V2} $\Rightarrow$ Z in the selfadjoint case: suppose Z false; there exist $\varepsilon > 0$, the sequence $x_n \in X_n$, $n \in \mathbb{N}$, $\| x_n\| = 1$ and a subsequence $\{x_\alpha\}$ of $\{x_n\}$ such that $\|(A-A_\alpha)x_\alpha\| \geq \varepsilon$; V2 implies the existence of $y \in X$ and of a subsequence $\{x_\beta\}$ of $\{x_\alpha\}$ such that $\lim_\beta (A-A_\beta)x_\beta = y$; denoting by (,) the scalar product in X, one has by V1: $\varepsilon^2 \leq \|y\|^2 = \lim_\beta((A-A_\beta)x_\beta,\Pi_\beta y) = \lim_\beta(x_\beta,(A-A_\beta)\Pi_\beta y) = 0$;

contradiction. Note that the last property we have verified is in fact a particular case of the following result: let X^*, X_n^*, A^*, A_n^*, Π_b^* be the adjoint spaces of X, X_n and the adjoint operators of A, A_n, Π_n; X_n^* is identified as a subspace of X^* by the map $\varphi_n \in X_n^* \to \varphi \in X^*$ with $\varphi(x) = \varphi_n(\Pi_n x)$ $\forall x \in X$; then the three properties V2, Π_n^* converges strongly to the idendity in X^*, for all converging sequences $x_n \in X_n^*$ one has $\lim\limits_{n\to\infty} A_n^* x_n^* = A^*(\lim\limits_{n\to\infty} x_n^*)$, imply Z.

We also prove the negative statements of Proposition 1 by examples. Let $X = \ell^2$ with scalar product (,) and canonical basis $e_1, e_2, ..$; note $Y_n = \text{span}(e_1, e_2, .., e_n)$; Π_n will be the orthogonal projector on Y_n. We show that {A1, A2} $\not\Rightarrow$ S3 (and consequently {V1, V2} $\not\Rightarrow$ S3, G $\not\Rightarrow$ S3); set $X_n = Y_n$; the operators $Ax = (x, e_1)e_1$ and $B_n x = (x, e_1 + e_n)e_1$ verify {A1, A2}; but $e_1 - e_n$ is an eigenvector of $A_n \equiv B_n$ (restricted to X_n) for the eigenvalue o. The following example will show that even in the Galerkin selfadjoint case, G $\not\Rightarrow$ R and G $\not\Rightarrow$ S4; set $X_n = Y_{2n}$, $Ax = \sum\limits_{n=1}^{\infty} (x, e_{2n})e_{2n+1} + (x, e_{2n+1})e_{2n}$, $A_n = \Pi_n A$ (restricted to X_n); clearly property R is not verified; furthermore $\sigma(A) = \{-1, o, 1\}$ where o is an eigenvalue of multiplicity 1 of A, $\sigma(A_n) = \sigma(A_n)$ $(n \geq 2)$ where o is an eigenvalue of multiplicity 2 of A_n so that S4 is not verified; since A_n is selfadjoint $\| R_z(A_n) \| = 1/(\text{distance } (z, \sigma(A_n))$, S1 is verified; since A_n is a Galerkin approximation, V1 is satisfied and by proposition 1f, one has also G. (An example of a differential operator illustrating the same situation is contained in Rappaz [7] p. 71).

Remarks: Condition Z appears as a generalization of U, whereas {V1, V2} is generalization of {A1, A2}. G is essentially equivalent to the stability conditions S1. For practical applications, {A1, A2} has been used in connection with integral operators (see Anselone [1]), {V1, V2} and G have been used in connection with finite difference methods for compact operators (see Vainikko [9], Grigorieff [4]; condition Z has been verified in connection with Galerkin finite element methods for non compact operators of plasma physics (see Descloux, Nassif, Rappaz [2]).

Proposition 1 does not exhaust the list of relations between the different properties we have introduced. We mention another one.

Proposition 2: Let X be a Hilbert space, Π_n be the orthogonal projector from X onto X_n, A be compact. A_n is given and we set $B_n = A_n \Pi_n$; then Z $\Rightarrow$ U.

Proof: From the realation $A - B_n = (A - A_n)\Pi_n + A(I - \Pi_n)$, one has $\|A - B_n\| \leq \| A - A_n \|_n + \|A(I - \Pi_n)\|$;

by Z, $\lim_{n\to\infty} \|A - A_n\|_n = o$, since A and consequently its adjoint A^* are compact, since $\lim_{n\to\infty} \Pi_n = I$ strongly, one has $\lim_{n\to\infty} \|A(I - \Pi_n)\| = \lim_{n\to\infty} \|(I - \Pi_n)A^*\| = o$.

Finally, we show for the typical situation of integral operators with continuous kernel that the properties {A1, A2} can be "transformed" in uniform convergence. To be specific, let $K: [o,1] \times [o,1] \to \mathbb{C}$ be a continuous kernel, X be either $C^o[o,1]$ or $L^2(o,1)$, $A: X \to X$ be the integral operator defined by $(Ax)(t) = \int_o^1 K(t,\tau)x(\tau)d\tau$. Let for $n \in \mathbb{N}$, $h = 1/n$, $t_i = ih$; for $X = C^o[o,1]$, we approximate A by the trapezoïdal rule and define $B_n: X \to X$ by $(B_nx)(t) = \sum_{j=1}^{n} \frac{h}{2}\{K(t,t_{j-1})x(t_{j-1}) + K(t,t_j)x(t_j)\}$. A and B_n then satisfy properties $\{A_1,A_2\}$ (see Anselone [1]).

Proposition 3: For the above situation, there exists the operator $C_n: X \to X$, where $X = L^2(o,1)$ such that $\sigma(C_n) = \sigma(B_n)$ and $\lim_{n\to\infty} \|A - C_n\| = o$.

Proof: By proposition 2, it suffices to construct a subspace $X_n \subset L^2(o,1)$ and an operator $A_n: X_n \to X_n$ such that $\sigma(A_n) \cup \{o\} = \sigma(B_n)$ and $\lim_{n\to\infty} \|A - A_n\|_n = o$. Choose X_n as the set of continuous piecewise linear function relative to the mesh $\{t_i\}$; for $x \in X_n$, A_nx is then defined as the interpolant of B_nx in X_n; using the uniform continuity of K, one obtains easily that $\lim_{n\to\infty} \|A - A_n\|_n = o$. (For more details see Descloux, Nassif, Rappaz [3]).

Remark: Proposition 3 is still valid when B_n is obtained by other classical integration formulae, for example Newton cotes or Gauss-Legendre; one has only to define convenient subspaces X_n.

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