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NONLINEAR BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS

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The main aim of this paper is to show how the Poincaré method of small parameter (originally used by Poincaré for seeking periodic solutions of a system of ordinary differential equations [1]) may be generalized for solving such boundary-value problems that are near to a boundary-value problem (not necessarily linear) whose solution is known. We restrict ourselves to two-point boundary-value problems and, in these, to two extreme possibilities of a noncritical case or a totally critical one. The restrictions are made for the sake of simplicity of notation. Two special cases, viz. the boundary-value problem for a system of ordinary differential equations and the seeking for periodic solutions of weakly nonlinear wave equation are explained in more detail.

1 An outline of the general theory

In the following t denotes time, $x = (x_1, x_2, \ldots, x_n)$ denotes the vector of space variables (naturally, in problems with ordinary differential equations $n = 0$), u denotes the dependent variable (it may be also a vector) and ε denotes a scalar small parameter which attains values from the interval $\mathfrak{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$. Further, let $P(u)$, $R(u, \varepsilon)$ be differential operators which transform functions $u(x, t)$ from a Banach space (briefly B-space) $\mathfrak{U}$ into another B-space $\tilde{\mathfrak{U}}$ for each $\varepsilon \in \mathfrak{E}$. We suppose that the operator P is of the first order with respect to t and R is not of higher order with respect to x than P is. Let us denote $\mathfrak{U}_t$ the B-space (with an appropriate norm) which is formed by functions $u(x, t)$, t any number from the interval $\mathfrak{T} = \langle t_1, t_2 \rangle$. Let $B(p, q)$, $B_1(p, q, \varepsilon)$ be operators which transform functions p and q from $\mathfrak{U}_t$ into a B-space $\tilde{\mathfrak{U}}_t$ for each $\varepsilon \in \mathfrak{E}$.

Let us investigate the problem

$$P(u) = \varepsilon R(u, \varepsilon) \tag{1.1}$$

or in a more explicit form

$$\begin{gathered} P(u(\varepsilon))(x, t) = \varepsilon R(u(\varepsilon), \varepsilon)(x, t)), \\ B(u(x, t_1), u(x, t_2)) + \varepsilon B_1(u(x, t_1), u(x, t_2), \varepsilon) = 0. \end{gathered} \tag{1.2}$$

The conditions (1.2) will be called essential boundary conditions. Usually, we do not seek a solution of (1.1), (1.2) for all x but only for x from some open region $\mathfrak{X}$, whose boundary will be denoted by $\mathfrak{F}$. (The closure of $\mathfrak{X}$ is denoted by $\overline{\mathfrak{X}}$.) As a rule, then there are given some conditions which the solution of (1.1), (1.2) has to fulfil for $x \in \mathfrak{F}$. These conditions will be called unessential boundary conditions. For the sake of simplicity, we shall take a special case of unessential boundary conditions which appear very frequently, viz.

$$u(x, t)\big|_{x \in \mathfrak{F}} = 0 \tag{1.3}$$

or briefly $u(\mathfrak{F}, t) = 0$. (For ordinary differential equations the condition (1.3) has no meaning, of course.) The boundary-value problem given by equation (1.1), (1.2) and (1.3) will be denoted $(\mathscr{B})$. We shall denote $(\mathscr{M})$ the mixed problem associated to $(\mathscr{B})$ and given by equations (1.1), (1.3) and

$$u(x, t_1) = c(x) \tag{1.4}$$

where $c(x) \in \mathfrak{U}_t$.

As it is usual in the theory of small parameter we consider besides the problems $(\mathscr{B})$ and $(\mathscr{M})$ also the limit problems $(\mathscr{B}_0)$ and $(\mathscr{M}_0)$ respectively, given by equations

$$P(u_0) = 0\,, \tag{1.5}$$

$$B(u_0(x, t_1), u_0(x, t_2)) = 0\,, \tag{1.6}$$

$$u_0(\mathfrak{F}, t) = 0 \tag{1.7}$$

and (1.5), (1.7) and

$$u_0(x, t_1) = c(x)\,, \tag{1.8}$$

respectively.

Obviously, a necessary condition that the problem $(\mathscr{B})$ have a solution is that the problem $(\mathscr{B}_0)$ have a solution. Of course, we suppose that this takes place but we have to distinguish three cases:

i) noncritical case when the problem $(\mathscr{B}_0)$ has a unique solution;

ii) totally critical case when every solution of $(\mathscr{M}_0)$ for arbitrary $c(x) \in \mathfrak{U}_t$ is a solution of $(\mathscr{B}_0)$;

iii) critical case when the initial values $c(x)$ are determined by conditions (1.6) to some extent but not fully.

In this general outline we shall consider only the first two extreme cases.

It is relatively easy to solve the noncritical case and it may be shown under very weak assumptions that the problem $(\mathscr{B})$ has also a solution for sufficiently small ε. In fact, let $U(c)(\varepsilon)(x, t)$ be the solution of $(\mathscr{M})$ and hence $U(c)(0)(x, t) \equiv U_0(c)(x, t)$ is the solution of $(\mathscr{M}_0)$. By choosing P, R and c smooth enough, we get the operator U as smooth as necessary. Putting $U(c)(\varepsilon)(x, t)$ into (1.2) we obtain the necessary and sufficient condition which $c(x)$ must fulfil that $U(c)$ be a solution of $(\mathscr{B})$, i.e.

$$\begin{aligned} \mathfrak{B}(c)(\varepsilon)(x) \equiv B(U(c)(\varepsilon)(x, t_1), U(c)(\varepsilon)(x, t_2)) + \\ + \varepsilon B_1(U(c)(\varepsilon)(x, t_1), U(c)(\varepsilon)(x, t_2), \varepsilon) = 0\,. \end{aligned} \tag{1.9}$$

Letting here $\varepsilon \to 0$, the equation (1.9) yields the necessary condition for $c(0)(x) = c_0(x)$:

$$\mathfrak{B}_0(c_0)(x) \equiv B(U_0(c_0)(x, t_1), U_0(c_0)(x, t_2)) = 0\,. \tag{1.10}$$

But this is the necessary and sufficient condition that the problem $(\mathscr{B}_0)$ have a solution and by our above assumption it follows that the equation (1.10) has a unique solution $c_0 = c_0^*$. Now, by well known general theorems of functional analysis (see e.g. [2]) we have the result:

If the operator $\mathfrak{B}(c)$ has the Fréchet differential in the neighbourhood of c_0^* and there exists the inverse operator to the Fréchet derivative $\mathfrak{B}'_c$ at the point c_0^*, then there exists ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ there exists the unique solution $U(c^*(\varepsilon))(\varepsilon)(x, t) \in \mathfrak{U}$ of the problem $(\mathscr{B})$ such that $c^*(0) = c_0^*$.

The totally critical case is more difficult and "in general" no solution of the problem $(\mathscr{B})$ exists.

Let us find fundamental conditions that such a solution exist. Let us suppose that the solutions of the problems $(\mathscr{M})$ and $(\mathscr{B})$ may be written in the form

$$u(\varepsilon)(x, t) = u_0(x, t) + \varepsilon\, u_1(\varepsilon)(x, t)$$

and that the operators $P(u)$ and $B(p, q)$ have Fréchet differentials with respect to u and p, q, respectively. Then by definition of the F-differential

$$\begin{gathered} P(u_0 + \varepsilon u_1) = P(u_0) + \varepsilon\, P'_u(u_0)\, u_1 + o_1(u_0, \varepsilon u_1)\,, \\ B(u_0(x, t_1) + \varepsilon\, u_1(x, t_1), u_0(x, t_2) + \varepsilon\, u_1(x, t_2)) = B(u_0(x, t_1), u_0(x, t_2)) + \\ + \varepsilon[B'_p(u_0(x, t_1), u_0(x, t_2))\, u_1(x, t_1) + B'_q(u_0(x, t_1), u_0(x, t_2))\, u_1(x, t_2)] + \\ + o_2(u_0(x, t_1), u_0(x, t_2), \varepsilon\, u_1(x, t_1), \varepsilon\, u_1(x, t_2))\,, \end{gathered} \tag{1.11}$$

where

$$\begin{gathered} \lim_{\|\alpha\| \to 0} \frac{\|o_1(u_0, \alpha)\|}{\|\alpha\|} = 0\,, \\ \lim_{\|\beta\| + \|\gamma\| \to 0} \frac{\|o_2(u_0(x, t_1), u_0(x, t_2), \beta, \gamma)\|}{\|\beta\| + \|\gamma\|} = 0\,. \end{gathered} \tag{1.12}$$

Substituting (1.11) into (1.1) and (1.2) and equating terms not containing ε and the remaining ones respectively we find that the problem $(\mathscr{M})$ is equivalent to the problem $(\mathscr{M}_0)$ together with the problem $(\mathscr{M}_1)$ which is given by

$$P'_u(u_0)\, u_1 + \frac{1}{\varepsilon}\, o_1(u_0, \varepsilon u_1) = R(u_0 + \varepsilon u_1, \varepsilon)\,, \tag{1.13}$$

$$u_1(x, t_1) = 0\,, \tag{1.14}$$

$$u_1(\mathfrak{F}, t) = 0 \tag{1.15}$$

and the problem $(\mathscr{B})$ is equivalent to the problem $(\mathscr{B}_0)$ together with the problem $(\mathscr{B}_1)$

which is given by (1.13), (1.15) and

$$(1.16)\quad B'_p(u_0(x, t_1), u_0(x, t_2))\, u_1(x, t_1) + B'_q(u_0(x, t_1), u_0(x, t_2))\, u_1(x, t_2) + \\ + \frac{1}{\varepsilon}\, o_2(u_0(x, t_1), u_0(x, t_2), \varepsilon\, u_1(x, t_1), \varepsilon\, u_1(x, t_2)) + \\ + B_1(u_0(x, t_1) + \varepsilon\, u_1(x, t_1), u_0(x, t_2) + \varepsilon\, u_1(x, t_2), \varepsilon) = 0$$

where

$$\frac{1}{\varepsilon}\, o_1(u_0, \varepsilon u_1),\ \frac{1}{\varepsilon}\, o_2(u_0(x, t_1), u_0(x, t_2), \varepsilon\, u_1(x, t_1), \varepsilon\, u_2(x, t_2))$$

may be continued for $\varepsilon \to 0$ as continuous operators in ε according to (1.12). Let us suppose that for $\varepsilon \in \mathfrak{E}$ there exists the solution of the problem $(\mathscr{M})$ on the interval $\langle t_1, t_2 \rangle$ which may be written in the form

$$(1.17)\qquad u(\varepsilon)(x, t) = U(c)(\varepsilon)(x, t) = U_0(c)(x, t) + \varepsilon U_1(c)(\varepsilon)(x, t)\,,$$

where $U_0(c)$ is evidently the solution of the problem $(\mathscr{M}_0)$ (and by our assumption of the problem $(\mathscr{B}_0)$, too) while $U_1(c)(\varepsilon)$ is the solution of $(\mathscr{M}_1)$. Substituting $U_0(c)$ instead of u_0 into (1.13) and (1.16), letting $\varepsilon \to 0$ and denoting $v = u_1(0)$, $V(c) = U_1(c)(0)$, we get the so-called variational mixed problem $(\mathscr{M}_v)$

$$(1.18)\qquad P'_u(U_0(c))\, v = R(U_0(c), 0)\,,$$

$$(1.19)\qquad v(x, 0) = 0\,,$$

$$(1.20)\qquad v(\mathfrak{F}, t) = 0$$

(whose solution is $V(c)(x, t)$) and the so-called variational boundary-value problem $(\mathscr{B}_v)$ given by (1.18), (1.20) and

(1.21)

$$B'_p(U_0(c)(x, t_1), U_0(c)(x, t_2))\, v(x, t_1) + B'_q(U_0(c)(x, t_1), U_0(c)(x, t_2))\, v(x, t_2) + \\ + B_1(U_0(c)(x, t_1), U_0(c)(x, t_2), 0) = 0\,.$$

The necessary and sufficient condition that $(\mathscr{B}_1)$ and hence $(\mathscr{B})$, too, have a solution evidently reads

$$(1.22)\qquad \mathfrak{C}(c)(\varepsilon)(x) \equiv B'_p(U_0(c)(x, t_1), U_0(c)(x, t_2))\, U_1(c)(x, t_1) + \\ + B'_q(U_0(c)(x, t_1), U_0(c)(x, t_2))\, U_1(c)(x, t_2) + \\ + \frac{1}{\varepsilon}\, o_2(U_0(c)(x, t_1), U_0(c)(x, t_2), \varepsilon\, U_1(c)(x, t_1), \varepsilon\, U_1(c)(x, t_2)) + \\ + B_1(U_0(c)(x, t_1) + \varepsilon\, U_1(c)(x, t_1), U_0(c)(x, t_2) + \varepsilon\, U_1(c)(x, t_2), \varepsilon) = 0\,.$$

Letting here $\varepsilon \to 0$ we get the necessary condition for c_0 that the problem $(\mathscr{B})$ have a solution

$$(1.23)\qquad \mathfrak{C}_0(c_0)(x) \equiv B'_p(U_0(c_0)(x, t_1), U_0(c_0)(x, t_2))\, V(c_0)(x, t_1) + + B'_q(U_0(c_0)(x, t_1), U_0(c_0)(x, t_2))\, V(c_0)(x, t_2) + + B_1(U_0(c_0)(x, t_1), U_0(c_0)(x, t_2), 0) = 0 .$$

It is readily seen that this is also a necessary and sufficient condition that the variational boundary-value problem $(\mathscr{B}_v)$ have a solution. The equation (1.23) may be considered the fundamental condition that a solution of $(\mathscr{B})$ exist. By the same general theorem as above we get the assertion:

If

i) *the equation* (1.23) *has a solution* $c_0 = c_0^* \in \mathfrak{U}_t$;

ii) *the operator* $\mathfrak{C}(c)$ *is* F-*differentiable in the neighbourhood of* c_0^*;

iii) *the* F-*derivative* $\mathfrak{C}'_{0,c_0}(c_0)$ *at the point* $c_0 = c_0^*$ *has an inverse operator,*

then the problem $(\mathscr{B})$ *has a solution* $U(c^*(\varepsilon))(\varepsilon)(x, t)$ *such that* $c^*(0) = c_0^*$.

Being led by the interpretation of the Poincaré method due to Malkin who was the first in the special case of periodic boundary conditions (see [3], [4]) to make use of the boundary-value problem adjoint to the variational b.-v. problem, let us suppose that there is given a b.-v. problem $(\mathscr{B}_a)$, adjoint to the b.-v. problem $(\mathscr{B}_v)$, given by

$$(1.24)\qquad Q(U_0(c))\, w = 0 ,$$

$$(1.25)\qquad C_1(U_0(c)(x, t_1), U_0(c)(x, t_2))\, w(x, t_1) + + C_2(U_0(c)(x, t_1), U_0(c)(x, t_2))\, w(x, t_2) = 0 ,$$

$$(1.26)\qquad w(\mathfrak{F}, t) = 0 ,$$

where $w(x, t) \in \mathfrak{W}$, $w(x, \tau) \in \mathfrak{W}_t$, τ any fixed point from $\langle t_1, t_2 \rangle$, $\mathfrak{W}$ and $\mathfrak{W}_t$ being B-spaces, Q and C_1, C_2 are operators transforming $\mathfrak{W}$ and $\mathfrak{W}_t$, respectively, into B-spaces $\tilde{\mathfrak{W}}$ and $\tilde{\mathfrak{W}}_t$, respectively (for any $c \in \mathfrak{U}_t$). $(\mathscr{B}_a)$ is understood to be adjoint to $(\mathscr{B}_v)$ in the following sense: Let the scalar products $w \,.\, \tilde{u}$, $\tilde{w} \,.\, u$, ... and similar expressions, where $w \in \mathfrak{W}$, $\tilde{w} \in \tilde{\mathfrak{W}}$, $u \in \mathfrak{U}$, $\tilde{u} \in \tilde{\mathfrak{U}}$ and so on, be defined and let the following Green's formula hold:

$$(1.27)\qquad \iint_{\mathfrak{X} \times \mathfrak{T}} [w P'_u v - (Q w) \,.\, v]\, \mathrm{d}X\, \mathrm{d}t = = \int_{\mathfrak{X}} \{[\tilde{C}_1\, w(x, t_1) + \tilde{C}_2\, w(x, t_2)]\, [B'_p\, v(x, t_1) + B'_q\, v(x, t_2)] + + [C_1\, w(x, t_1) + C_2\, w(x, t_2)]\, [\tilde{B}'_p\, v(x, t_1) + \tilde{B}'_q\, v(x, t_2)]\}\, \mathrm{d}X ,$$

where $\mathrm{d}X = \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n$ and $\tilde{B}'_p$, $\tilde{B}'_q$ and $\tilde{C}_1$, $\tilde{C}_2$ respectively, define so-called complementary boundary conditions and complementary adjoint boundary conditions, respectively. (Without explaining details we assume that complementary boundary conditions, adjoint boundary conditions and complementary adjoint boundary conditions have similar properties as these conditions have in the case

$n = 0$. See [5].) Finally, let us suppose that the following assertion holds: The problem $(\mathscr{B}_v)$ has a solution if and only if

$$(1.28) \qquad \iint_{\mathfrak{X}\times\mathfrak{T}} w^* R(U_0(c), 0)\, \mathrm{d}X\, \mathrm{d}t =$$
$$= - \int_{\mathfrak{X}} [\tilde{C}_1\, w^*(x, t_1) + \tilde{C}_2\, w^*(x, t_2)]\, B_1(U_0(c)(x, t_1), U_0(c)(x, t_2), 0)\, \mathrm{d}X\,,$$

where $w^*(x, t)$ is any solution of $(\mathscr{B}_a)$. (The necessity of this condition follows immediately from (1.27), but its sufficiency is proved in some special cases only.)

Hence, if the problem $(\mathscr{B}_a)$ is defined and has the mentioned properties, the two conditions (1.23) and (1.28) must be equivalent. Unfortunately, we are not able to prove it in the generality in which we sketched this outline. Here, there are two main reasons why we cannot do so: (i) it is not known how to determine explicitely the adjoint b.-v. problem. (Even the manner in which in functional analysis an adjoint differential system (1.24) is defined in the dual space does not coincide with the manner in which we define it in special cases in classical analysis.) (ii) We do not know a more explicit form of the operators $V(w, R, c)$ and $\tilde{V}(w, R, B)$ which define solutions $V(R, c)(x, t)$ and $\tilde{V}(R, B)(x, t)$ of the mixed variational problem and of the variational b.-v. problem, respectively.

Let us remind the reader that two most familiar cases in which we know how to answer both these questions are two-point b.-v. problems for ordinary differential equations (see e.g. [5]) and, partially, for linear hyperbolic partial differential operators of the second order (here, $\tilde{V}(w, R, B)$ is not known for general boundary conditions) (see, e.g. [6]).

Finally, let us note that a class of autonomous (i.e. such that $R(u, \varepsilon)$ does not depend explicitely on t) b.-v. problems requires a special treatment. Clearly, $u(x, t)$ being a solution of an autonomous problem (1.1), (1.3), $u(x, t + h)$, where h is any sufficiently small real number, is also a solution of the same problem. Now if the b.-v. problem $(\mathscr{B})$ has the property that $u^*(x, t)$ being a solution of $(\mathscr{B})$, $u^*(x, t + h)$ is also a solution of $(\mathscr{B})$, we say that $(\mathscr{B})$ has the property $(\mathscr{P})$. (Evidently, every b.-v. problem with periodic boundary conditions has the property $(\mathscr{P})$.) Thus, we may choose in problems with the property $(\mathscr{P})$ the initial time in such a way that $u^*(x, t)$ fulfils an additional condition. On the other hand, by generalization of well known results of nonlinear mechanics we may expect that an autonomous b.-v. problem with the property $(\mathscr{P})$ has a solution for $\varepsilon \in (0, \varepsilon_1)$, $\varepsilon_1 > 0$ only if we seek it in a time-interval whose length is a function of ε. Hence, we substitute the condition (1.2) by

$$(1.2') \qquad B(u(x, t_1), u(x, t_2 + \tau(\varepsilon))) + \varepsilon_1\, B_1(u(x, t_1), u(x, t_2 + \tau(\varepsilon)), \varepsilon) = 0\,,$$

where $\tau(\varepsilon)$ is continuous and $\tau(0) = 0$.

2 Boundary-value problems for a system of ordinary differential equations

In this section, let us consider the following b.-v. problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} - f(t, u) = \varepsilon\, g(t, u, \varepsilon)\,, \tag{2.1}$$

$$b(u(t_1), u(t_2)) + \varepsilon\, b_1(u(t_1), u(t_2), \varepsilon) = 0\,, \tag{2.2}$$

where u, f, g, b and b_1 are n-vectors. This problem was studied in [5]. As the B-space $\mathfrak{U}$ we take here the space C^1 with the norm $\|u\| = \max_{1 \leq i \leq n} \max_{t \in \mathfrak{T}} (|u_i(t)|, |\mathrm{d}u_i(t)/\mathrm{d}t|)$. (In the sequel we shall number equations by the same numbers as the corresponding equations in section 1 and thus, the numbers of the equations which have no sense in this case will be omitted.)

The associated initial problem $(\mathscr{M})$ (which is here identical with the mixed problem in the general case) is given by (2.1) and

$$u(t_1) = c\,, \tag{2.4}$$

where c is an n-dimensional constant vector. The limit boundary-value problem $(\mathscr{B}_0)$ and the limit initial problem $(\mathscr{M}_0)$ are given by

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} - f(t, u_0) = 0\,, \tag{2.5}$$

$$b(u_0(t_1), u_0(t_2)) = 0 \tag{2.6}$$

and (2.5) with

$$u_0(t_1) = c\,, \tag{2.8}$$

respectively.

The solution of the initial problem $(\mathscr{M})$ has the form

$$u(\varepsilon)(t) = U(c)(\varepsilon)(t) = \mu(t, \gamma(t, c, \varepsilon))\,, \tag{*}$$

where $\mu(t, c)$ is the solution of $(\mathscr{M}_0)$ and $\gamma(t, c, \varepsilon)$ is the solution of the integral equation

$$\gamma(t, c, \varepsilon) = c + \varepsilon \int_{t_1}^{t} \left[\frac{\mathrm{D}\mu}{\mathrm{D}c}(\vartheta, \gamma(\vartheta, c, \varepsilon))\right]^{-1} g(\vartheta, \mu(\vartheta, \gamma(\vartheta, c, \varepsilon)), \varepsilon)\, \mathrm{d}\vartheta\,,$$

where $\mathrm{D}\mu/\mathrm{D}c$ denotes the functional matrix (Jacobi's matrix) of partial derivatives of the components of μ with respect to the components of c. (Formula (*) is obtained by the variation-of-constants method.) We verify easily that $U(c)(\varepsilon)(t) \in C^{1,0,1}$ if $f(t, u) \in C^{0,1}$ and $g(t, u, \varepsilon) \in C^{0,1,0}$. Since the existence of F-differential of $U(c)(\varepsilon)$ with respect to c is here equivalent to the existence of the first partial derivatives of u we see easily that under our assumptions $U(c)(\varepsilon)$ has the F-differential of the first order.

Putting (*) into (2.2) we get

(2.9) $\mathfrak{B}(c)(\varepsilon) \equiv b(\mu(t_1, c), \mu(t_2, \gamma(t_2, c, \varepsilon))) + \varepsilon\, b_1(\mu(t_1, c), \mu(t_2, \gamma(t_2, c, \varepsilon)), \varepsilon) = 0$

and letting $\varepsilon \to 0$

$$(2.10) \qquad \mathfrak{B}_0(c_0) \equiv b(u_0(t_1, c_0), u_0(t_2, c_0)) \equiv b(\mu(t_1, c_0), \mu(t_2, c_0)) = 0\,.$$

In the noncritical case (2.10) has a unique solution $c_0 = c_0^*$. If the vectors $b(p, q)$ and $b_1(p, q, \varepsilon)$ have the first partial derivatives with respect to p and q, then $\mathfrak{B}(c)$ has the F-differential with respect to c. If the jacobian of (2.10) is nonvanishing at the point $c_0 = c_0^*$, then there exists the inverse operator to the F-derivative of $\mathfrak{B}(c)$ at the point $c = c_0^*$, whence the existence of a solution $U(c^*(\varepsilon))(\varepsilon)(t)$ of the b.-v. problem ($\mathscr{B}$) follows under the mentioned conditions for sufficiently small ε. (In [5] the existence of such a solution is proved under less restrictive assumptions by making use of succesive approximations instead of the general implicit function theorem.)

In the totally critical case

$$\mathfrak{B}_0(c_0) \equiv b(\mu(t_1, c_0), \mu(t_2, c_0)) \equiv 0\,.$$

Under the above assumption, the solution (*) may be written in the form

$$(2.17) \qquad U(c)(\varepsilon)(t) \equiv \mu(t, c) + \\ + \varepsilon \int_0^1 \frac{\mathrm{D}\mu}{\mathrm{D}c}\left(t, c + \alpha\varepsilon \int_{t_1}^t \left[\frac{\mathrm{D}\mu}{\mathrm{D}c}\right]^{-1} . \, g \, \mathrm{d}\vartheta\right) \mathrm{d}\alpha \,. \int_{t_1}^t \left[\frac{\mathrm{D}\mu}{\mathrm{D}c}\right]^{-1} g \, \mathrm{d}\vartheta\,.$$

Denoting

$$P(u) = \frac{\mathrm{d}u}{\mathrm{d}t} - f(t, u)\,,$$

the F-differential of P exists at every point $u \in \mathfrak{U}$ under the above assumptions and

$$\mathrm{D}[P(u); v] \equiv \frac{\mathrm{d}v}{\mathrm{d}t} - \frac{\mathrm{D}f}{\mathrm{D}u}(t, u)\, v\,.$$

Similarly,

$$\mathrm{D}[b(u(t_1), u(t_2)); v] \equiv \frac{\mathrm{D}b}{\mathrm{D}p}(u(t_1), u(t_2))\, v(t_1) + \frac{\mathrm{D}b}{\mathrm{D}q}(u(t_1), u(t_2))\, v(t_2)\,.$$

Thus, we find easily the variational b.-v. problem

$$(2.18) \qquad \frac{\mathrm{d}v}{\mathrm{d}t} - \frac{\mathrm{D}f}{\mathrm{D}u}(t, \mu(t, c))\, v = g(t, \mu(t, c), 0)\,,$$

$$(2.19) \qquad \frac{\mathrm{D}b}{\mathrm{D}p}(\mu(t_1, c), \mu(t_2, c))\, v(t_1) + \frac{\mathrm{D}b}{\mathrm{D}q}(\mu(t_1, c), \mu(t_2, c))\, v(t_2) + \\ + b_1(\mu(t_1, c), \mu(t_2, c), 0) = 0$$

and the variational initial problem $(\mathscr{M}_v)$, given by (2.18) and

$$v(t_1, c) = 0\,. \tag{2.21}$$

Putting the solution

$$V(c)(t) = \frac{\mathrm{D}\mu}{\mathrm{D}c}(t, c) \int_{t_1}^{t} \left[\frac{\mathrm{D}\mu}{\mathrm{D}c}(\vartheta, c)\right]^{-1} g(\vartheta, \mu(\vartheta, c), 0)\, \mathrm{d}\vartheta$$

of the problem $(\mathscr{M}_v)$ into (2.19), we get the necessary condition

(2.23)

$$\mathfrak{C}_0(c_0) \equiv \frac{\mathrm{D}b}{\mathrm{D}q}(\mu(t_1, c_0), \mu(t_2, c_0)) \frac{\mathrm{D}\mu}{\mathrm{D}c}(t_2, c_0) \int_{t_1}^{t_2} \left[\frac{\mathrm{D}\mu}{\mathrm{D}c}(\vartheta, c_0)\right]^{-1} g(\vartheta, \mu(\vartheta, c_0), 0)\, \mathrm{d}\vartheta + $$
$$+ b_1(\mu(t_1, c_0), \mu(t_2, c_0), 0) = 0\,.$$

Let us suppose that the equation (2.23) has a unique solution $c_0 = c_0^*$. For the sake of brevity we do not write the expression for the operator $\mathfrak{C}(c)$ corresponding to (1.22). Omitting details let us only note that to be able to prove the F-differentiability of the operator it is necessary to strengthen our assumptions about f and b, viz. $f(t, u) \in C^{0,2}$ and $b(p, q) \in C^{2,2}$. To insure the existence of the inverse operator to $\mathfrak{C}'_c(c)$ at the point $c_0 = c_0^*$ it is sufficient to suppose that the jacobian of (2.23) at the point c_0^* is nonvanishing. If all these conditions are fulfilled we may assert that the b.-v. problem $(\mathscr{B})$ has a solution $U(c^*(\varepsilon))(\varepsilon)(t)$, $c^*(0) = c_0^*$, for sufficiently small ε.

The b.-v. problem $(\mathscr{B}_a)$ adjoint to $(\mathscr{B}_v)$ is well known from classical theory of differential equations (see e.g. [5]), and it reads

$$\frac{\mathrm{d}w'}{\mathrm{d}t} + w' \frac{\mathrm{D}f}{\mathrm{D}u}(t, \mu(t, c)) = 0\,, \tag{2.24}$$

$$w'(t_1)\, C_1(c) + w'(t_2)\, C_2(c) = 0\,, \tag{2.25}$$

where $'$ denotes the transposition of a matrix or a vector and the matrices C_1, C_2 fulfil the conditions

$$\operatorname{rank}(C_1 : C_2) = n\,,$$

$$-\frac{\mathrm{D}b}{\mathrm{D}p}(\mu(t_1, c), \mu(t_2, c))\, C_1(c) + \frac{\mathrm{D}b}{\mathrm{D}q}(\mu(t_1, c), \mu(t_2, c))\, C_2(c) \equiv 0\,.$$

In the theory of b.-v. problems it is shown that if the homogeneous variational b.-v. problem is totally critical (which occurs in our case) then the adjoint b.-v. problem is also totally critical. The multiplication of elements $w \in \mathfrak{W}$ and $v \in \mathfrak{U}$ (where $\mathfrak{W}$ is evidently formed by n-rows vectors whose components are of class C^1 and $\|w'(t)\| = \max_i \max_{t \in \mathfrak{T}} (|w_i(t)|, |\mathrm{d}w_i(t)/\mathrm{d}t|)$) being defined as the usual scalar multiplication

$$w'(t)\, v(t) = \sum_{i=1}^{n} w_i(t)\, v_i(t)$$

and $W(t)$ being the fundamental $n \times n$ matrix of solutions of $(\mathscr{B}_a)$ the classical necessary and sufficient condition for the existence of a solution of $(\mathscr{B}_v)$ reads

$$(2.28) \qquad \int_{t_1}^{t_2} W(t, c)\, g(t, \mu(t, c), 0)\, \mathrm{d}t = -\left[W(t_1, c)\, \tilde{C}_1 + W(t_2, c)\, \tilde{C}_2\right] . \\ . \, b_1(\mu(t_1, c), \mu(t_2, c), 0)\,,$$

where $\tilde{C}_1, \tilde{C}_2$ are $n \times n$ matrices defining so-called complementary boundary conditions to adjoint boundary conditions (2.25). It may be shown (but we cannot perform it here) that the conditions (2.28) and (2.23) are really equivalent (see [5], Remark 3.4).

In [5] the critical case (not necessarily totally critical) and the case of an autonomous b.-v. problem with the property $(\mathscr{P})$ (see sec. 1 above) are investigated. Besides, theorems for special cases of quasilinear problems and problems with periodic boundary conditions are stated.

The study of perturbed b.-v. problems for ordinary differential equations cannot be taken for accomplished in several directions. In the first place in some critical cases not even the system (2.23) determines a unique solution $c_0 = c_0^*$ though a unique solution of the problem $(\mathscr{B})$ exists. For periodic boundary conditions such a situation was investigated in two different cases by I. G. Malkin [4] and W. S. Loud [7]. In the second place the problem may have a k-parametric ($1 \leqq k \leqq n$) family of solutions. By a slight modification of the method used above we can get an existence theorem, too. From the point of view of mechanics it is interesting to put this situation into relation with the existence of first integrals of a certain type. This was performed for periodic boundary conditions by D. C. Lewis jun. [8]. In the third place we can expect that in some cases the solution of $(\mathscr{B})$ may be expressed approximately as an polynomial in powers of $\varepsilon^{p/q}$, p and q positive integers, $q \neq 1$. Such a case was studied in [4] again for periodic boundary conditions (systems near to Lyapunov's systems). Finally, it may happen that a solution of $(\mathscr{B}_0)$ can let arise several solutions of $(\mathscr{B})$ (branching points) (see [9]). All these possibilities ought to be investigated in problems with general two-point boundary conditions. Another task consists in transferring as many results as possible from two-point b.-v. problems to general b.-v. problems.

3 Periodic solutions of a weakly nonlinear wave equation

In this section we give some of our results which have not yet been published. Let us investigate the existence of periodic solutions of a weakly nonlinear string of the length π and clamped at both ends. This problem is described by the equations

$$(3.1) \qquad u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon)\,,$$

$$(3.2) \qquad u(x, T) - u(x, 0) = 0\,, \quad u_t(x, T) - u_t(x, 0) = 0\,,$$

$$(3.3) \qquad u(0, t) = u(\pi, t) = 0\,,$$

where the function f is T-periodic in t. (To write (3.1) and (3.2) in agreement with sec.1, we ought to substitute them by the hyperbolic system

$$(3.1') \qquad \frac{\partial u}{\partial t} = u_1 , \quad \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = \varepsilon f(t, x, u, u_1, u_2, \varepsilon) , \quad \frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial x} = 0 ,$$

where obviously

$$u_2 = \frac{\partial u}{\partial x}$$

and by boundary conditions

$$(3.2') \qquad u(x, T) - u(x, 0) = 0 , \quad u_1(x, T) - u_1(x, 0) = 0 .$$

Obviously, to have then the possibility to put (3.3) into accordance with (1.3) we have to generalize (1.3) in

$$(1.3') \qquad G(u(\mathfrak{F}, t)) = 0 ,$$

where G is an operator; then we can leave (3.3) without change. In the sequel we shall retain the above form of writing.)

In this case, let us denote $\mathfrak{T} = \langle 0, T \rangle$, $\overline{\mathfrak{X}} = \langle 0, \pi \rangle$. (Thus, the boundary $\mathfrak{F}$ of $\overline{\mathfrak{X}}$ consists of the points 0 and π.) First, the function $f(t, x, u, v, w, \varepsilon)$ is defined for $t \in \mathfrak{T}$, $x \in \overline{\mathfrak{X}}$, $u, v, w \in \mathfrak{R}$, $\varepsilon \in \mathfrak{E}$, where $\mathfrak{R} = (-\infty, \infty)$ and $\mathfrak{E} = \langle 0, \varepsilon_0 \rangle$. Let us suppose that the function f fulfils on the planes $x = 0$ and $x = \pi$ such conditions that after being continued in x on the whole interval $\mathfrak{R}$ by the formulas

$$f(t, x, u, v, w, \varepsilon) = - f(t, -x, -u, -v, w, \varepsilon) = f(t, x + 2\pi, u, v, w, \varepsilon)$$

it remains continuous with its partial derivatives of the second order with respect to x, u, v and w in the whole space $\mathfrak{M} = \mathfrak{T} \times \mathfrak{R}^4 \times \mathfrak{E}$.

The mixed problem $(\mathscr{M})$ associated with $(\mathscr{B})$ is given by equations (3.1), (3.3) an d

$$(3.4) \qquad u(x, 0) = \varphi(x) , \quad u_t(x, 0) = \psi(x) .$$

First, the functions φ and ψ are again given on the interval $\overline{\mathfrak{X}}$ only. Let us suppose $\varphi(x) \in C^2$, $\psi(x) \in C^1$, $\varphi(0) = \varphi(\pi) = \varphi''(0) = \varphi''(\pi) = \psi(0) = \psi(\pi) = 0$. Then continuing φ and ψ nto $\mathfrak{R}$ as odd and 2π-periodic functions, $\varphi \in C^2$ and $\psi \in C^1$ on the whole $\mathfrak{R}$.

Under these conditions it may be easily verified that a solution $u^*(x, t)$ of $(\mathscr{M})$ is odd and 2π-periodic in the variable x. We seek a classical solution of $(\mathscr{B})$ and $(\mathscr{M})$, i.e. a solution o class C^2 in x and t. Therefore, we choose as the B-space $\mathfrak{U}$ the space of elements $u(x, t)$ which are of class C^2 and with norm

$$\|u\| = \max (|u|, |u_t|, |u_x|, |u_{tt}|, |u_{tx}|, |u_{xx}|) ,$$

where the maximum is sought over the set $\mathfrak{R} \times \mathfrak{T}$. We find easily that under the above assumptions there exists a unique solution $u^*(\varepsilon)(x, t)$ of the problem $(\mathscr{M})$ on the interval $\mathfrak{T}$ for sufficiently small ε. Before going over to the problem $(\mathscr{B})$ let us notify that the equations (3.1), (3.2) and (3.4) are equivalent to the integro-differential equation

$$(**)\qquad u(x, t) = s(x + t) - s(-x + t) + \\ + \tfrac{1}{2}\varepsilon \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, z, u(z, \vartheta), u_t(z, \vartheta), u_x(z, \vartheta), \varepsilon)\, dz\, d\vartheta$$

where

$$s(\xi) = \frac{1}{2}\left[\varphi(\xi) + \int^{\xi} \psi(z)\, dz\right]$$

and the functions φ, ψ and f are subjected to the conditions mentioned above. Hence, the function s is 2π-periodic, of class C^2 and is uniquely determined up to an additive constant. On the other hand the functions φ and ψ are determined uniquely by s as

$$\varphi(\xi) = s(\xi) - s(-\xi)\,, \quad \psi(\xi) = s'(\xi) - s'(-\xi)\,.$$

From the equation (**) it follows readily that the solution of $(\mathscr{M})$ may be written as

$$u(\varepsilon)(x, t) = U(s)(\varepsilon)(x, t)\,,$$

where $U(s)$ has the F-differential of the first order.

The limit b.-v. problem $(\mathscr{B}_0)$ reads

$$(3.5)\qquad u_{0tt} - u_{0xx} = 0\,,$$

$$(3.6)\qquad u_0(x, T) - u_0(x, 0) = 0\,, \quad u_{0t}(x, T) - u_{0t}(x, 0) = 0\,,$$

$$(3.7)\qquad u_0(0, t) = u_0(\pi, t) = 0\,,$$

whereas the limit mixed problem $(\mathscr{M}_0)$ is given by (3.5), (3.7) and

$$(3.8)\qquad u_0(x, 0) = \varphi(x)\,, \quad u_{0t}(x, 0) = \psi(x)$$

or, which is the same

$$(3.8')\qquad u_0(x, 0) = s(x) - s(-x)\,, \quad u_{0t}(x, 0) = s'(x) - s'(-x)\,.$$

In the first place let us suppose that $T/2\pi$ is an irrational number. Then we find easily that the equations

$$s(x + T) - s(-x + T) - s(x) + s(-x) = 0\,,$$
$$s'(x + T) - s'(-x + T) - s'(x) + s'(-x) = 0$$

have only solutions $s(x) =$ const. and hence $(\mathscr{B}_0)$ has a unique solution, viz. $u_0^*(x, t) \equiv \equiv 0$. Thus, we have to deal with a noncritical case. Nevertheless a T-periodic solution of $(\mathscr{B})$ does not exist in general, since it may be shown that an operator inverse to the operator $\mathfrak{B}(s)$, defined by the just quoted equations, does not always exist.

In the second place, let $T = 2\pi N$, N a positive integer. (We omit the case $T/2\pi = N/M$, N and M positive integers, $M \neq 1$, which is critical but not totally.) This case is totally critical, since, as we know, $s(x)$ is 2π-periodic and thus every solution of $(\mathscr{M}_0)$ is a solution of $(\mathscr{B}_0)$. Putting the solution $U(s)(\varepsilon)(x, t)$ of the problem $(\mathscr{M})$ into (3.2) we find easily that these two conditions are equivalent to an only equation

$$(3.22) \qquad \mathfrak{C}(s)(\varepsilon)(x) \equiv \int_0^{2\pi N} f(\vartheta, x - \vartheta, U(s)(\varepsilon)(x - \vartheta, \vartheta), \frac{\partial}{\partial t} U(s)(\varepsilon)(x - \vartheta, \vartheta), \frac{\partial}{\partial x} U(s)(\varepsilon)(x - \vartheta, \vartheta), \varepsilon)\, d\vartheta = 0 .$$

Letting $\varepsilon \to 0$ we get the necessary condition for s_0, viz.

$$(3.23) \qquad \mathfrak{C}_0(s_0)(x) \equiv \int_0^{2\pi N} f(\vartheta, x - \vartheta, s_0(x) - s_0(-x + 2\vartheta), s_0'(x) - s_0'(-x + 2\vartheta), s_0'(x) + s_0'(-x + 2\vartheta), 0)\, d\vartheta = 0 .$$

Under the above assumptions the operator $\mathfrak{C}(s)$ is F-differentiable. Let us suppose that (3.23) has a 2π-periodic solution $s_0 = s_0^*(x) \in C^2$ and let $\mathfrak{C}'_{0s}(s)$ have an inverse operator at the point $s_0 = s_0^*$. Then, it may be proved that the problem $(\mathscr{B})$ has for sufficiently small ε a unique solution or, that (3.1), (3.3) has a unique $2\pi N$-periodic solution $U(s^*(\varepsilon))(\varepsilon)(x, t) \in \mathfrak{U}$, where $s^*(0) = s_0^*$.

The variational problem $(\mathscr{B}_v)$ reads

$$(3.18) \qquad v_{tt} - v_{xx} = f(t, x, U_0(s_0)(x, t), \frac{\partial}{\partial t} U_0(s_0)(x, t), \frac{\partial}{\partial x} U_0(s_0)(x, t), 0) = F(t, x),$$

$$(3.19) \qquad v(x, T) - v(x, 0) = 0, \quad v_t(x, T) - v_t(x, 0) = 0,$$

$$(3.20) \qquad v(0, t) = v(\pi, t) = 0,$$

where $U_0(s_0)(x, t) = s_0(x + t) - s_0(-x + t)$. Defining the b.-v. problem $(\mathscr{B}_a)$ adjoint to $(\mathscr{B}_v)$ as

$$(3.24) \qquad w_{tt} - w_{xx} = 0,$$

$$(3.25) \qquad w(x, T) - w(x, 0) = 0, \quad w_t(x, T) - w_t(x, 0) = 0,$$

$$(3.26) \qquad w(0, t) = w(\pi, t) = 0,$$

it may be shown (for an arbitrary function $F(t, x)$ of class $C^{0,1}$ and T-periodic in t) that the following theorem holds: The problem $(\mathscr{B}_v)$ has a solution only if the function $F(t, x)$ is orthogonal on $\mathfrak{X} \times \mathfrak{T}$ to every solution $w^*(x, t)$ of the problem $(\mathscr{B}_a)$, i.e. only if

$$(3.28^*) \qquad \int_0^{\pi} \int_0^{T} w^*(z, \vartheta) F(\vartheta, z)\, dz\, d\vartheta = 0 .$$

The case $T = 2\pi/\alpha$, α being irrational number, shows that this condition (opposite to a similar theorem for b.-v. problems for ordinary differential equations) is not sufficient. In fact, the condition (3.28*) is fulfilled trivially in this case although the solution of the problem $(\mathscr{B}_v)$ need not exist. On the contrary, if $T = 2\pi N$, N a natural number, $w^*(x, t) = s(x + t) - s(-x + t)$ where s is an arbitrary 2π-periodic function of the class C^2 and it may be verified that the condition (3.28*) is equivalent to

$$\int_0^{2\pi N} F(\vartheta, x - \vartheta)\, d\vartheta = 0 .$$

Hence substituting for $F(t, x)$ the expression from (3.18) into the last equation we get the necessary condition for s_0 which is precisely the condition (3.23).

By means of the described method it was proved that the problem $(\mathscr{B})$ has a solution when

$$f(t, x, u, u_t, u_x, \varepsilon) = \alpha u - \beta u^3 + (\gamma_1 \cos t + \gamma_2 \sin t) \sin x ,$$

where α, β, γ_i are constants and $\alpha\beta > 0$.

Now, let us say a few words about the b.-v. problem $(\mathscr{B})$ given by (3.1), (3.2), (3.3) if it is autonomous, i.e. if the function f does not depend on t explicitly. Clearly, this problem has the property $(\mathscr{P})$. Since u^*_{0x} (or u^*_{xt}) of every nontrivial solution $u^*_0(x, t)$ of the problem $(\mathscr{B}_0)$ assumes in the point (0, 0) values in closed intervals having 0 as an interior point we may require (for fixing the initial time) that the sought solution $u(x, t)$ of $(\mathscr{B})$ fulfils the condition $u_x(0, 0) = 0$ or $u_{xt}(0, 0) = 0$ or which is the same that $s'(0) = 0$ or $s''(0) = 0$. (We shall prefer the latter possibility.) On the other hand, seeking a solution of $(\mathscr{B})$ near to a $2\pi N$-periodic solution of $(\mathscr{B}_0)$, let us write its period in the form $T = 2\pi N + \varepsilon\, \omega(\varepsilon)$. Then we find (instead of (3.23)) the following necessary condition for s_0 and $\omega_0 = \omega(0)$:

$$(3.30)\quad \omega_0\, s_0''(x) + \int_0^{2\pi N} f(x - \vartheta, s_0(x) - s_0(-x + 2\vartheta), s_0'(x) - s_0'(-x + 2\vartheta), s_0'(x) + s_0'(-x + 2\vartheta), 0)\, d\vartheta = 0$$

to which we have to add, say,

$$(3.31)\quad s_0''(0) = 0 .$$

Supposing that the system (3.30), (3.31) has a solution $\omega_0 = \omega_0^*$, $s_0 = s_0^*(x)$, s_0^* being 2π-periodic, we may prove under suitable conditions concerning the properties of the operator $\mathfrak{D}(s, \omega)$ which corresponds to the operator $\mathfrak{C}(s)$ in the nonautonomous case, an existence theorem for the problem $(\mathscr{B})$. We succeeded to prove that the problem $(\mathscr{B})$ with

$$f = u + \varepsilon(1 - \alpha u^2)\, u_t$$

has a two-parametric family of solutions $u^*(x, t + h; \omega)$, where ω lies in a certain open interval. On the other hand, with help of (3.30) it may be shown that the problem

($\mathscr{B}$) with

$$u_{tt} - u_{xx} = \varepsilon(1 - u^2)\, u_t \tag{3.32}$$

or

$$u_{tt} - (1 + \varepsilon u_x^2)\, u_{xx} = 0 \tag{3.33}$$

cannot have any classical periodic solutions.

For the equation (3.32), J. Kurzweil has proved a stronger result, viz. that it cannot have any generalized (only continuous) periodic solution. The equation (3.33) (which was investigated by R. Grammel [10] with only apparently different nonessential boundary conditions) is not of the type studied above but it may be shown that the corresponding necessary condition for s_0 and ω_0 is the same as if we would write formally $f = -u_x^2 u_{xx}$ in (3.1). (The existence of periodic solutions in t for the equation $u_{tt} - u_{xx} = \alpha u - \beta u^3$ for an infinite string was investigated by G. Petiau in [11].)

The existence of periodic solutions of a weakly nonlinear wave equation was studied by several Russian mathematicians ([12]–[19]). The papers [12]–[17] are not available in Czechoslovakia. From [18] we know that they deal mainly with the equation

$$u_{tt} - u_{xx} = h(t, x) + \varepsilon f(u)$$

where $f(u)$ is a polynomial in u and that they consider a special critical case (but not the totally critical one). In [18] Karp considers the equation

$$u_{tt} - u_{xx} = h(t, x) + \varepsilon f(t, x, u, u_t)\,. \tag{3.34}$$

He makes use of the method of wave regions, i.e. he transforms the equation (3.34) together with the conditions of periodicity into a integro-differential equation with a kernel which is constant on a finite number of regions (and just these regions are called wave regions). In [19] a procedure is given for calculating periodic solution of the equation

$$\frac{\partial}{\partial x}\left(p\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\,\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(p\,\frac{\partial u}{\partial z}\right) - qu - \varrho\,\frac{\partial^2 u}{\partial t^2} =$$
$$= \varepsilon f(x, y, z, u, \varepsilon)\,,$$

where

$$p(x, y, z)\,, \quad q(x, y, z)\,, \quad \varrho(x, y, z)\,, \quad f(x, y, z, u, \varepsilon)$$

are analytic functions of mentioned variables; the proof is omitted.

American mathematicians B. A. Fleischman and R. A. Ficken investigate in several papers from which the most important is [20] the equation

$$u_{tt} - u_{xx} + 2\kappa u_t + \alpha u = f(t, x) - \varepsilon u^3\,, \quad \kappa > 0\,, \quad \varepsilon > 0\,.$$

They make use of the fixed point theorem. A more general equation

$$u_{tt} - u_{xx} + g(t, x, u, u_t) = h(t, x, u)$$

(where the function g ensures the dissipativity of the system) was treated by G. Prodi in [21] with help of Fourier series and the fixed point theorem. (Prodi introduces a new kind of generalized solution of the quoted equation.) In [22] F. M. Stewart claims to have proved the existence of 2π-periodic solutions of the equation

$$u_{tt} - u_{xx} = \varepsilon[u^3 + f(x) \sin t]$$

by means of the Fourier method but till now the proof has not been published. In [23] L. Cesari proves the existence of solutions periodic in y of the equation

$$u_{xy} = f(x, y, u, u_x, u_y)$$

in a sufficiently narrow stripe along the x-axis.

The reader may find further bibliography on some related problems in [24], [25] and [11].

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