S. G. Mihlin

Variational methods of solving linear and nonlinear boundary value problems

In: (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Prague in September 1962. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1963. pp. 77--92.

Persistent URL: http://dml.cz/dmlcz/702175

VARIATIONAL METHODS OF SOLVING LINEAR AND NONLINEAR BOUNDARY VALUE PROBLEMS

S. G. MIHLIN, Leningrad

1 Introduction

Some results concerning variational methods and their applications to linear boundary value problems are given in the monograph [1] which appeared in 1957. In the present report, the results of the author and his pupils L. N. Hagen-Thorn, G. N. Yaskova, I. V. Gel'man, A. Langenbach, S. N. Rosé, M. N. Yakovlev, are described; some results obtained by L. M. Kačanov and V. M. Mitkevič are also considered. These investigations deal with the problem of stability of Ritz procedure and with the application of the variational methods to nonlinear problems.

In this Introduction we give some earlier results which will be used later.

1.1 *The generalized solution and the variational problem.* Let us consider the equation

$$Au = f \tag{1}$$

for unknown u, where A is a linear operator in a given Hilbert space H, and f and u are elements of the same space. We suppose that A is positive definite. This implies that the domain $D(A)$ of the operator A is dense in the space H and that there exists a constant $\gamma > 0$ with

$$(Au, u) \geqq \gamma^2 \|u\|^2 , \quad u \in D(A) ; \tag{2}$$

if H is a real Hilbert space then it is necessary to suppose in addition that A is symmetric.

If A is positive definite, then the equation (2) has a generalized solution which can be constructed in the following way. We introduce (see [2]) a new Hilbert space H_A which is the closure of the set $D(A)$ in the metric generated by the scalar product

$$[u, v] = [u, v]_A = (Au, v) . \tag{3}$$

We term the space H_A an energetic space; the norm in H_A will be denoted by $|u|$ or $|u|_A$. It is known that H_A can be imbedded in H and that the inequality

$$\|u\| \leqq \frac{1}{\gamma} |u| \tag{4}$$

holds.

The functional (u, f) is bounded in H_A. Using the known theorem of F. Riesz we conclude that there exists a unique element $u_0 \in H_A$ such that $(u, f) = [u, u_0]$ for

$u \in H_A$. This element u_0 is the generalized solution of equation (1); more definitely, u_0 satisfies the equation $\tilde{A}u = f$, where $\tilde{A}$ is the so called K. Friedrich's self-adjoint extension of A; the operator $\tilde{A}$ is also M. G. Kreĭn's "hard" extension of A (see [3]).

Let $\{w_k\}$ be a complete orthonormal set in H_A. Then

$$u_0 = \sum_k (f, w_k)\, w_k \,. \tag{5}$$

The generalized solution minimizes the functional

$$F(u) = [u, u] - 2\mathrm{Re}\,(u, f)\,, \tag{6}$$

the domain of this functional being the whole space H_A.

1.2 *Ritz procedure.* Let H be separable; then H_A is also separable. We may construct an approximation of the generalized solution u_0 using the well known Ritz procedure.

Let the sequence v_n, $n = 1, 2, \ldots$ satisfy the following conditions: a) $v_n \in H_A$, $n = 1, 2, \ldots$; b) the elements $v_1, v_2, \ldots, v_n$ are linearly independent for every n; c) the sequence $\{v_n\}$ is complete in H_A. The elements $v_1, v_2, \ldots$ are called coordinate elements, the sequence $\{v_n\}$ is called a coordinate system.

We choose a positive integer n and look for an approximation to u_0 in the form

$$u_n = \sum_{k=1}^{n} a_k^{(n)} v_k \,; \tag{7}$$

the coefficients $a_k^{(n)}$ are to be determined from the condition that the value of $F(u_n)$ be minimal. This condition leads to the following linear algebraic system (Ritz system)

$$\sum_{k=1}^{n} [v_k, v_j]\, a_k^{(n)} = (f, v_j)\,; \quad j = 1, 2, \ldots, n\,, \tag{8}$$

which has a unique solution. Substituting it into formula (7) we obtain the so-called Ritz approximate solution of the equation (1).

The following relations hold:

$$|u_n - u_0| \downarrow 0\,, \tag{9}$$

$$|u_n| \leqq |u_0|\,. \tag{10}$$

The matrix of Ritz system will be called in the following *Ritz matrix*. It coincides with Gram's matrix of the elements $v_1, v_2, \ldots, v_n$ in H_A. The quantities $a_k^{(n)}$ satisfying the system (8) will be called *Ritz coefficients*.

2 Stability of Ritz Procedure

2.1 *General remarks.* If the coordinate system satisfies conditions a) to c) of the Introduction, then Ritz approximate solution u_n converges to the generalized solution u_0 of the equation (1) (see formula (9)). This is true if the approximate solution itself is calculated exactly, without any error. But actually, this is usually not so: Ritz coefficients are derived from the system (8), which is constructed and solved approximately, with some errors. If the order of Ritz system is small, in other words, if we construct a rough approximation, then these errors are not essential and every coordinate system satisfying the conditions a) to c) mentioned above is practically suitable. But if we wish to construct a more exact approximation we are obliged (this is obvious from formula (9)) to use Ritz systems of higher orders, and the errors commited in calculating the matrix and the right-side members of the system (8) may become significant. Thus there arises the problem of stability of Ritz procedure as to small errors of the type mentioned. The main results concerning this problem are given below.

2.2 *Strongly minimal systems of elements.* Let H be any Hilbert space. An infinite countable set $\{v_n\}$ of its elements is called a strongly minimal system in this space (see [4]) if the least eigenvalue of the matrix

$$R_n = ((v_j, v_k)_H)_{j,k=1}^{j,k=n} \tag{11}$$

is bounded from below by a positive number independent of n. A strongly minimal system is also simply minimal in the same space.

Theorem 1 (see [5] and [6]). *Let A and B be two self-adjoint positive definite operators, and let the space H_A be imbedded in H_B. If $v_k \in H_A$, $k = 1, 2, \ldots$, and if the system $\{v_k\}$ is strongly minimal in H_B, then it is also strongly minimal in H_A.*

Corollaries. 1) If the conditions of Theorem 1 are fulfilled then every system which is orthonormal in H_B is strongly minimal in H_A. 2) If the operator A is positive definite and a system is strongly minimal in H, then this system is strongly minimal in H_A. In particular, every system orthonormal in H is strongly minimal in H_A.

2.3 *On the stability of Ritz procedure.* Suppose that every energetical product $[v_j, v_k]$ in Ritz system (8) is calculated with some small error $g_{jk} = \bar{g}_{kj}$, and the right-side members (f, v_j) of the same system are calculated with a small error d_j. Instead of (8) we obtain the following system

$$\sum_{k=1}^{n} \{[v_k, v_j]_A + g_{kj}\}\, a_k^{(n)'} = (f, v_j) + d_j\,, \quad j = 1, 2, \ldots, n\,. \tag{12}$$

We denote by G the matrix of elements g_{kj}. This matrix generates an operator in the n-dimensional unitary space; the norm of this operator will be denoted by $\|G\|$. We denote by d, $a^{(n)}$, $a^{(n)'}$ the column-vectors with components

$$\begin{gathered} d_1,\ d_2,\ \ldots,\ d_n\,; \\ a_1^{(n)},\ a_2^{(n)},\ \ldots,\ a_n^{(n)}\,; \\ a_1^{(n)'},\ a_2^{(n)'},\ \ldots,\ a_n^{(n)'}\,, \end{gathered}$$

respectively.

The solution of Ritz system is called stable (relatively to small variations of the matrix and the column of the right-side members of this system), if there holds an inequality

$$\|a^{(n)\prime} - a^{(n)}\| \leqq p\|G\| + q\|d\|, \tag{13}$$

where the numbers p and q do not depend on n.

Theorem 2. *If the coordinate system is strongly minimal in H_A, then the solution of Ritz system is stable.*

Let us explain the significance of Theorem 2. Let m be the positive lower bound of eigenvalues of the matrices R_n; the coordinate system being strongly minimal, m does not depend on n. Suppose that $\|G\| < m$; then

$$\|a^{(n)\prime} - a^{(n)}\| \leqq \frac{m^{-3/2}\|G\| \, |u_0|_A + m^{-1}\|d\|}{1 - m^{-1}\|G\|}. \tag{14}$$

The results of sections 2.1–2.3 are given in [5] and [6].

Theorem 2 is extended in [7] to the case of Bubnov-Galerkin procedure (see [1]).

2.4 *The stability of Ritz approximate solution* (see [6]).

Theorem 3. *If the coordinate system is strongly minimal in H_A, then Ritz approximate solution is stable relatively to small variations of the matrix and the column of the right-side members of Ritz system.*

Let us explain the assertion of Theorem 3. Denote

$$u_n' = \sum_{k=1}^{n} a_k^{(n)\prime} v_k,$$

where the coefficients $a_k^{(n)\prime}$ satisfy the system (12). If the coordinate system is strongly minimal in H_A, then there exist two constants p_1 and q_1 independent of n and satisfying the inequality

$$|u_0 - u_n| \leqq p_1\|G\| + q_1\|d\|.$$

2.5 *On a certain numerical experiment.* Using Ritz procedure, V. M. Mitkevič (see [8]) has solved approximately the problem of bending of a ring-sectorial plate which has the outer arc edge clamped, while the rest part of the boundary is free. The author used two coordinate systems. The first system consists of the functions $w_{ij} = \varrho^{i+1}\Theta^{2j}$, where $\varrho = (r - b)/a$, r and Θ being the polar coordinates; the symbols a and b denote the outer and the inner radii of the plate. The second coordinate system is obtained from the first by orthogonalisation.

As we can demonstrate, the first coordinate system is not strongly minimal in the corresponding energetical space, while the second system is strongly minimal in this space — it follows immediately from Corrolary 2) in section 2.2.

The computations were performed on the electronic computer "Strela" of the Moscow Computing Centre of the Academy of Sciences of USSR; this computer calculates to 9 decimals. There were introduced some special commands in the computing program so as to introduce zeros in some of the last decimal places; this permits to introduce some small errors in Ritz matrix.

12 coordinate functions were used in the both calculations.

V. M. Mitkevič gives in [8] the value of the ratio of errors of certain unknown quantities to the error of the elements of Ritz matrix. When the first coordinate system was used, this ratio was of the order $10^4 - 10^5$ for Ritz coefficients, of the order $50 - 100$ for the normal displacements and of the order $500 - 1000$ for the bending moments. When the second system was used the errors of the mentioned quantities were of the same order as the errors of the elements of Ritz matrix. The ratio mentioned above varies between the limits 0·10 and 5·31, with the exception of one of Ritz coefficients, for which this ratio was 39.7.

2.6 *On the condition number of Ritz matrices.* It was supposed in the preceding sections that the solution of Ritz system is calculated exactly, so that errors arise only in the process of setting up this system. However, round-off errors are inevitable in the numerical solution of an algebraic system. If the coordinate system is only strongly minimal in H_A, then the condition number of Ritz matrix may increase indefinitely together with its order and the solution of Ritz system (8) may be unstable relatively to round-off errors.

The condition number of Ritz matrix is equal to the ratio of the maximal and minimal eigenvalue of this matrix; the mentioned number is evidently bounded if the eigenvalues of Ritz matrix are bounded from above and below by positive numbers independent of the order of the matrix.

The last condition is fulfilled if the coordinate system is chosen as described below. Let B be a positive definite operator and let the spaces H_A and H_B consist of the same elements, i.e. $D(\sqrt{A}) = D(\sqrt{B})$. If any set orthonormal and complete in H_B is chosen as a coordinate system, then the eigenvalues of Ritz matrices are bounded as described above.

2.7 *On the convergence of the discrepancy to zero* (see [9]). Let A be a positive definite operator, and assume that equation (1) is solved approximately by means of Ritz procedure. Let u_n be the corresponding approximate solution. In general the discrepancy $Au_n - f$ does not converge to zero as $n \to \infty$; indeed if $Au_n - f \to 0$ $(n \to \infty)$ for every coordinate system, then the operator A is bounded.

But if the coordinate system is specially chosen, then the discrepancy $Au_n - f$ may converge to zero even in the case of an unbounded operator A. One method of such a choice is given by the following theorem.

Theorem 4. *Let A and B be self-adjoint positive definite operators with a common domain, and let B have a pure point spectrum, i.e., let the set of the eigenelements of B be complete in the given Hilbert space. If the set of eigenelements of B is chosen as a coordinate system, then $Au_n - f \to 0$ $(n \to \infty)$.*

2.8 *On a rational choice of the coordinate system* (see [10]). Positive definite and self-adjoint operators A and B are called semisimilar if $D(\sqrt{A}) = D(\sqrt{B})$. Let A and B be semisimilar. If any set orthonormal and complete in H_B is chosen as a coordinate system for the equation (1), then Ritz approximation is stable and the condition number of Ritz matrix is bounded independently of its order.

Positive definite self-adjoint operators with a common domain are called similar. Let the operators A and B be similar, and let B have a pure point spectrum. If the set of the eigenelements (which we suppose to be normalized in H_B) of B is chosen as a coordinate system for equation (1), then Ritz approximate solution is stable, the condition number of Ritz matrix is bounded independently of its order and $Au_n - f \to 0$ $(n \to \infty)$.

Example 1. Consider the differential equation of the second order

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)\,u = f(x) \tag{15}$$

with the boundary conditions

$$u(0) = u(1) = 0\,. \tag{16}$$

We suppose for simplicity that the functions $p(x)$, $p'(x)$, $q(x)$, $f(x)$ are continuous on the segment $\langle 0, 1\rangle$, $p(x) \geqq \tilde{p} = \text{const} > 0$, $q(x) \geqq 0$. The operator B, where $Bu = -u''(x)$, $u(0) = u(1) = 0$ is similar to the operator given by formulas (15) and (16). The eigenfunctions of B which are normalized in H_B, are given by the following formula:

$$v_n(x) = \frac{\sqrt{2}}{n\pi}\sin n\pi x\,; \quad n = 1, 2, \ldots$$

If these functions are chosen as the coordinate ones for the problem (15)–(16), then Ritz approximate solution is stable, the condition number of Ritz matrix is bounded and

$$\left\|\frac{\mathrm{d}^2 u_n}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 u_0}{\mathrm{d}x^2}\right\|_{L_2} \to 0 \quad (n \to \infty)\,,$$

where u_0 is the generalized solution of the problem (15)–(16).

Example 2. Now let us consider equation (15) under the boundary conditions

$$u'(0) - a\,u(0) = 0\,, \quad u'(1) + b\,u(1) = 0\,, \quad a > 0\,, \quad b > 0\,. \tag{17}$$

The operator $-\,d^2/dx^2$ under conditions (17) is similar to the operator (15)–(17). But it is difficult to construct the eigenfunctions of the similar operator because this construction is connected with the problem of solving a certain transcendental equation. However, it is easily to indicate a semisimilar operator and a system of functions which is complete and orthonormal in the corresponding energetic space. For example, such a semisimilar operator is given by the formulas

$$Bu = -\frac{d^2u}{dx^2}\,, \quad u'(0) - u(0) = 0\,, \quad u'(1) = 0\,;$$

the corresponding energetical product and energetical norm are

$$[u, v]_B = u(0)\,\overline{v(0)} + \int_0^1 u'(x)\,\overline{v'(x)}\,dx\,, \tag{18}$$

$$\|u\|_B^2 = |u(0)|^2 + \int_0^1 |u'(x)|^2\,dx\,.$$

The set of functions

$$1,\ x,\ \frac{\sqrt{2}}{\pi}\sin \pi x,\ \frac{\sqrt{2}}{2\pi}\sin 2\pi x,\ \ldots,\ \frac{\sqrt{2}}{n\pi}\sin n\pi x,\ \ldots$$

is complete and orthonormal in the metric defined by (18).

Example 3. We proceed to the case of m variables, and consider the Dirichlet problem

$$-\sum_{j,k=1}^{m} \frac{\partial}{\partial x_j}\left(A_{jk}\frac{\partial u}{\partial x_k}\right) + Cu = f(x)\,; \quad u|_S = 0\,, \tag{19}$$

where the differential operator is elliptic nondegenerate; S denotes the boundary of a finite region D. We suppose for simplicity that the function A_{jk} and C are sufficiently smooth and that $C \geqq 0$.

Let us suppose that the formula

$$x' = x'(x) \tag{20}$$

realises a one-to-one mapping of the region D onto a certain region D', and that the eigenfunction of the Dirichlet problem for Laplace's equation and the region D' are known. We also suppose that the function (20) is sufficiently smooth and that the Jacobian $J = D(x')/D(x)$ is bounded from below by a positive constant. We suppose finally that the boundary S' of the region D' consists of a finite number of sufficiently smooth surfaces.

The transformation (20) changes the problem (19) in the following one:

$$(21) \qquad -\sum_{j,k=1}^{m} \frac{\partial}{\partial x'_j}\left(A'_{jk}\frac{\partial u}{\partial x'_k}\right) + CJu = Jf\,; \quad u|_S = 0\,.$$

The operator of the problem (21) is semisimilar to the operator

$$(22) \qquad -\Delta u = 0\,, \quad u|_{S'} = 0\,;$$

if the set of the eigenfunctions of the operator (22) is chosen as a coordinate system for the problem (21), then Ritz approximate solution is stable and the condition number of Ritz matrix is bounded.

We may indicate two cases when the operators of the problems (21) and (22) are similar: 1) the boundary S' is sufficiently smooth, 2) $m = 2$ and the boundary S' consists of a finite number of arcs, each of which is sufficiently smooth and at each corner the inner angle is less than π. In these cases we may conclude that

$$\left\|\sum_{j,k=1}^{m} \frac{\partial}{\partial x'_j}\left(A'_{jk}\frac{\partial u_n}{\partial x'_k}\right) + CJu_n - Jf\right\|_{L_2} \to 0 \quad (n \to \infty)\,,$$

where u_n is Ritz approximate solution. It follows from the last relation that

$$\left\|\frac{\partial^2 u_n}{\partial x'_j\,\partial x'_k} - \frac{\partial^2 u_0}{\partial x'_j\,\partial x'_k}\right\|_{L_2} \to 0 \quad (n \to \infty)\,,$$

where u_0 denotes the generalized solution of the problem (21).

3 Variational Methods in Nonlinear Problems

3.1 *The variational problem in Hilbert space* (see [11], [12] and [13]).

Theorem 5. *Let P be a nonlinear operator which acts in a real Hilbert space H and is defined on a dense linear set M. Let the following conditions be fulfilled: 1) $PO = 0$; 2) Gateaux's differential $P'(u)\,h$ exists for every $u, h \in M$ and this differential is linear with respect to h; the element $P'(u)\,h$ is continuous in any two-dimensional plane containing the fixed point u; 3) the operator $P'(u)$ is symmetric and positive for every $u \in M$, i.e., for any u, h_1, h_2, $h \in M$ the relations*

$$(P'(u)\,h_1, h_2) = (h_1, P'(u)\,h_2)\,,$$
$$(P'(u)\,h, h) > 0\,, \quad h \neq 0$$

are true. If the equation

$$(23) \qquad Pu = f\,, \quad f \in H$$

has a solution, then this solution is unique and minimizes the functional

$$(24) \qquad F(u) = \int_0^1 (Ptu, u)\,\mathrm{d}t - (f, u)\,.$$

Conversely, if there exists an element of the set M which minimizes the functional (24), *then this element satisfies the equation* (23).

Theorem 6. *Let the conditions of Theorem 5 hold and let there exist a constant* $\gamma > 0$ *such that*

$$(P'(u)\, h, h) \geqq \gamma^2 \|h\|^2 . \tag{25}$$

Then the functional (24) *is bounded from below and every sequence which minimizes this functional converges in H.*

This limit, the existence (and uniqueness) of which follows from Theorem 6, is called the generalized solution of the problem of minimizing the functional (24).

Theorem 7. *Let the conditions of the Theorem 5 hold and let there exist positive constants β and γ such that*

$$(P'(u)\, h, h) \geqq \beta(P'(0)\, h, h) \geqq \gamma^2 \|h\|^2 \tag{26}$$

for any $u, h \in M$. Then the generalized solution of the problem of minimizing the functional (24) *belongs to the space H_A where $A = P'(0)$.*

3.2 *The variational problem in Sobolev spaces* (see [14]). Let E be a Banach space with a weakly compact sphere. A functional $F(u)$ with domain E is said to be essentially convex if the inequality

$$f(au + bv) < a\, f(u) + b\, f(v)\,, \quad u \neq v$$

is true for any elements $u, v \in E$ and any positive numbers a, b with $a + b = 1$.

Theorem 8. *Let the functional $F(u)$ with domain E be increasing, lower semicontinuous and essentially convex. Then this functional is bounded from below; and there exists a unique element $u_0 \in E$ such that*

$$F(u_0) = \inf_{u \in E} F(u)\,.$$

Every sequence which is a minimizing one for the functional $F(u)$ converges weakly to u_0.

Now suppose that the functional F has a linear Gateaux's differential in every point $u \in E$ and that the domain of grad F is not empty. We also suppose that the remainder $k(u, h)$ of Gateaux's differential satisfies the inequality

$$k(u, h) \geqq \alpha \|h\|^{1+\varepsilon}\,, \quad \alpha, \varepsilon \text{ positive constants}\,.$$

Then every minimizing sequence converges strongly to u_0.

We shall consider the problem of minimizing the functional

$$F(u) = \int_D G(x_1, x_2, \ldots, x_m; \ldots, u^{i,j}_{j_1,\ldots,j_m}, \ldots)\, dx\,, \tag{27}$$

$$i = 1, 2, \ldots, N\,; \quad j = 0, 1, \ldots, l\,; \quad j_1 + \ldots + j_m = j\,,$$

where

$$u^{i,j}_{j_1,\ldots,j_m} = \frac{\partial^j u^i}{\partial x_1^{j_1} \ldots \partial x_m^{j_m}}\,;$$

the minimum is to be found on the set of functions which satisfy the boundary conditions

$$u^{i,j}_{j_1,\ldots,j_m}\big|_S = 0\,; \quad i = 1, 2, \ldots, N,\ j = 0, 1, 2, \ldots, l-1\,. \tag{28}$$

We suppose that the function G is defined for any $x \in D$ and for any values of the variables $u^{i,j}_{j_1,\ldots,j_m}$.

Important cases of this problem are investigated in the classical works of S. N. Bernstein and L. Tonelli and also in some papers by Ch. Morrey and A. G. Sigalov. Here we shall formulate only one theorem proved by I. V. Gel'man [14]. The symbol $\mathring{W}^{(l)}_\alpha(D)$ will denote the set of functions which belong to the Sobolev space $W^{(l)}_\alpha(D)$ and satisfy the boundary conditions (28).

Theorem 9. *Let the function G have the following properties:*

1) *This function and its derivatives of the first and second order with respect to the variables $u^{i,j}_{j_1,\ldots,j_m}$ are continuous in the whole domain of the function G.*

2) *The inequality*

$$K_1 \sum_{i=1}^{N} \Big(\sum_{l_1+\ldots+l_m=l} |u^{i,l}_{l_1,\ldots,l_m}|^2 \Big)^{\alpha/2} \leqq G \leqq \tag{29}$$
$$\leqq K_2 \sum_{i=1}^{N} \Big(\sum_{j=0}^{l} \sum_{j_1+\ldots+j_m=j} |u^{i,j}_{j_1,\ldots,j_m}|^2 \Big)^{\alpha/2}$$

is true in the same domain; the symbols K_1, K_2, α denote constants, $K_1 > 0$, $K_2 > 0$, $\alpha > 1$.

3) *The quadratic form*

$$\sum_{i,k,j,s=1}^{N} \sum_{\substack{j_1+\ldots+j_m=j \\ s_1+\ldots+s_m=s}} \frac{\partial^2 G}{\partial u^{i,j}_{j_1,\ldots,j_m} \partial u^{k,s}_{s_1,\ldots,s_m}} t^{i,j}_{j_1,\ldots,j_m} t^{k,s}_{s_1,\ldots,s_m}$$

is positive definite.

Then there exists a unique element $u_0 \in \mathring{W}^{(l)}_\alpha(D)$ such that

$$F(u_0) = \inf_{u \in \mathring{W}^{(l)}_\alpha(D)} F(u)\,;$$

every sequence which is a minimizing one for the functional F converges weakly to u_0.

The proof follows easily from Theorem 8; the semicontinuity of the functional F follows from a theorem of V. I. Kazimirov [15].

If $\alpha > 2$ in (29) and the inequalities

$$\left|\frac{\partial G}{\partial u^{k,s}_{s_1,\ldots,s_m}}\right| \leqq K_3 \sum_{i=1}^{N} \Big(\sum_{j=0}^{l} \sum_{j_1+\ldots+j_m=j} |u^{i,j}_{j_1,\ldots,j_m}|^2\Big)^{(\alpha-1)/2}, \quad K_3 > 0\,;$$

$$\left|\frac{\partial^2 G}{\partial u^{k,s}_{s_1,\ldots,s_m}\,\partial u^{r,t}_{t_1,\ldots,t_m}}\right| \leqq K_4 \sum_{i=1}^{N} \Big(\sum_{j=0}^{l} \sum_{j_1+\ldots+j_m=j} |u^{i,j}_{j_1,\ldots,j_m}|^2\Big)^{(\alpha-2)/2}, \quad K_4 > 0\,;$$

are true, then every minimizing sequence converges strongly in $W^{(l)}_\alpha(D)$.

I. V. Gel'man (see [16]) has proved a similar theorem on the existence of the solution of the variational problem (27)–(28) in some Orlicz spaces; the asymptotic behavior of function G may also be different from that specified above.

3.3 *Ritz procedure* (see [17]). One usually construcs the minimizing sequence by means of Ritz procedure. Let us look for the minimum of a functional F with a linear domain. We choose a coordinate system $\{v_n\}$ which satisfies the usual conditions:

a) all the elements v_k belong to the domain of F;

b) the elements $v_1, v_2, \ldots, v_n$ are linearly independent for all n;

c) the coordinate system is complete in a certain metric space containing the domain of F.

The set $D(F)$ is linear and therefore

$$u_n = \sum_{k=1}^{n} a_k v_k \in D(F)$$

for any values of the constants $a_1, a_2, \ldots, a_n$; consequently $F(u_n)$ is defined. The expression $F(u_n)$ is a function of a finite number of variables $a_1, a_2, \ldots, a_n$. Let us find the values of this variables for which the function $F(u_n)$ attains its minimum; for this purpose it will suffice to solve the system of equations

$$\frac{\partial F(u_n)}{\partial a_j} = 0\,; \quad j = 1, 2, \ldots, n \tag{30}$$

and to verify that the so obtained values of a_k actually minimize $F(u_n)$. If we then substitute these values of a_k in the expression for u_n we obtain the so called Ritz approximate solution of the given variational problem.

Theorem 10. *Let the functional F be increasing, lower semicontinuous in a certain metric space and continuously differentiable on every finite-dimensional linear subdomain of its domain. Then:* 1) *F is bounded from below*; 2) *Ritz approximate solution can be constructed for every n*; 3) *the sequence of Ritz approximate solutions is a minimizing one for the given functional F.*

3.4 *The functionals of the theory of plasticity and their generalization* (see [11], [12] and [13]). Some problems of the theory of plasticity may be reduced to the problem of minimizing a functional of the following kind

$$F(u) = \int_D \left(\sum_{j=1}^{p} \int_0^{\tau_j^2(u)} \varrho_j(\xi)\, d\xi \right) dx - \int_D fu\, dx\,, \quad f \in L_2(D)\,. \tag{31}$$

Here ϱ_j are non-negative functions of ξ determined on the interval $0 \leqq \xi < \infty$, D is a finite region in the coordinate space, $\tau_j^2(u)$ are non-negative quadratic forms of the function u and its derivatives which orders do not surpass a certain number. We suppose that the inequalities

$$\varrho_j(\xi) \geqq \varrho_0 = \text{const} > 0\,,$$

$$\int_D \tau_j^2(u)\, dx \geqq \gamma^2 \int_D u^2\, dx\,; \quad \gamma = \text{const} > 0$$

are true at least for one value i of the index j; the last inequality must be fulfilled if the function u satisfies the boundary conditions of the given problem.

For example, in the case of the elastic-plastic torsion problem we have

$$F(u) = \iint_D \left(\frac{1}{2} \int_0^{T_2} \bar{g}(\xi)\, d\xi - \omega u \right) dx\, dy\,,$$

where $T^2 = u_x^2 + u_y^2$, D is the region of the cross-section of the given rod, $\omega = \text{const}$; the function u must vanish at the boundary of the cross section D which is supposed to be simply connected. Finally, the function $\bar{g}(\xi)$ satisfies the inequalities $\bar{g}'(\xi) > 0$, $\bar{g}(\xi) > G^{-1}$, where G is the shear-modulus of the material in the elastic state.

The functional (31) satisfies the conditions of theorems 5–7. Therefore the functional (31) attains its minimum and Ritz approximate solutions converge to the minimizing element.

3.5 *Procedure of L. M. Kačanov for solving non-linear Ritz systems.* L. M. Kačanov [18] has developed his procedure in relation to a certain problem of the theory of plasticity; S. N. Rosé [19] has examined this procedure theoretically and extended it to functionals (31).

Let us solve the problem of minimizing (31) by Ritz procedure. We choose any coordinate system satisfying the conditions of section 3.2, put

$$u_n = \sum_{k=1}^{n} a_k v_k \tag{32}$$

and form the equations

$$\frac{\partial F(u_n)}{\partial a_j} = 0\,, \quad j = 1, 2, \ldots, n\,. \tag{33}$$

The system (33) will be solved as follows. We substitute the functions $\varrho_j(\xi)$ in the integral (31) by some constants $\varrho_j^{(0)}$ such that $\varrho_j^{(0)} > 0$; let $F_1(u)$ be the so obtained quadratic functional:

$$F_1(u) = \int_D \sum_{j=1}^{p} \varrho_j^{(0)} \tau(u) \, dx - (f, u) .$$

Now put

$$u_{n1} = \sum_{k=1}^{n} a_{k1} v_k$$

and determine the coefficients a_{k1} so that the value of $F_1(u_{n1})$ be minimal; this leads to a certain linear algebraic system. The coefficients a_{k1} being determined, we substitute $\varrho_j(\xi)$ by $\varrho_j(\tau_j^2(u_{n1}))$ in (31) and obtain a new quadratic functional

$$F_2(u) = \int_D \sum_{j=1}^{p} \varrho_j(\tau_j^2(u_{n1})) \, \tau_j^2(u) \, dx - (f, u) .$$

Put

$$u_{n2} = \sum_{k=1}^{n} a_{k2} v_k$$

and determine the coefficients a_{k2} so that the value of $F_2(u_{n2})$ be minimal; this also leads to a linear algebraic system. This process must be repeated infinitely. The set of solutions

$$(a_{1s}, a_{2s}, \ldots, a_{ns}) , \quad s = 1, 2, \ldots$$

is compact and every limit point of this set is a solution of the system (33).

3.6 *Reduction to Cauchy problem* (see [20]). Here we shall develop another procedure for solving non-linear Ritz systems; this procedure is based on a reduction of the mentioned system to a certain system of ordinary differential equations. The idea of this reduction is given for example in D. F. Davidenko's paper [21].

Consider the system of equations

(34) $$f_i(a_1, a_2, \ldots, a_n) = 0 , \quad i = 1, 2, \ldots, n .$$

Choose some functions $F_i(a_1, a_2, \ldots, a_n, t)$ such that

$$F_i(a_1, a_2, \ldots, a_n, 1) = f_i(a_1, a_2, \ldots, a_n)$$

and that the expression $F_i(a_1, a_2, \ldots, a_n, 0)$ is sufficiently simple; let

(35) $$F_i(a_1, a_2, \ldots, a_n, 0) = a_i .$$

The equations

(36) $$F_i(a_1, a_2, \ldots, a_n, t) = 0 , \quad i = 1, 2, \ldots, n$$

determine $a_1, a_2, \ldots, a_n$ as functions of t; for our purposes it suffices to know the values of these functions at $t = 1$. On differentiating the equations (36), we obtain a system of ordinary differential equations of the first order

$$\sum_{k=1}^{N} \frac{\partial F_i}{\partial a_k} \frac{\partial a_k}{\partial t} + \frac{\partial F_i}{\partial t} = 0\,, \quad i = 1, 2, \ldots, n\,; \tag{37}$$

the relations (35) give the initial values of the unknown functions a_i:

$$a_i\big|_{t=0} = 0\,. \tag{38}$$

Let the Cauchy problem (37)–(38) have a solution on the segment $0 \leqq t \leqq 1$. We shall construct this solution by some approximate method; then on putting $t = 1$, we shall obtain the solution of (34).

Let the system (34) be a Ritz system. Then the Cauchy problem mentioned above has a solution on the segment $0 \leqq t \leqq 1$, if certain conditions, described below, are fulfilled.

Consider a functional $F(u)$ with a linear domain which is dense in some real Hilbert space H. Let $F(u)$ be the potential of a certain operator $P(u)$, i.e., let the relation

$$\frac{\mathrm{d}}{\mathrm{d}s} F(u + sh)\big|_{s=0} = (P(u), h)$$

hold for any elements $u, h \in D(F)$.

Suppose that the Gateaux's differential $P'(u)\, h$ of the operator $P(u)$ is symmetric and uniformly positive definite, i.e., there exists a positive number γ independent of both u and h which satisfies the inequality

$$(P'(u)\, h, h) \geqq \gamma^2 \|h\|^2\,.$$

Set up the problem of minimizing F and solve it by means of Ritz procedure. We obtain the system of equations

$$(P(u_n), v_j) = 0\,, \quad j = 1, 2, \ldots, n\,, \tag{39}$$

where

$$u_n = \sum_{k=1}^{n} a_k v_k$$

and v_k are the coordinate elements. Note that we shall obtain the same system (39) if we apply the Bubnov-Galerkin procedure (see [1]) to the equation $P(u) = 0$, which is the Euler-Lagrange equation for the functional F.

In order to reduce our problem to a Cauchy problem we choose the system (36) in the form

$$a_j + t[(P(u_n), v_j) - a_j] = 0\,, \quad j = 1, 2, \ldots, n\,.$$

Then we obtain the following Cauchy problem

$$\frac{\mathrm{d}a_j}{\mathrm{d}t} + (F(u_n), v_j) - a_j + \tag{40}$$

$$+ t \sum_{k=1}^{n} \left[(P'(u_n) v_k, v_j) - \delta_{jk} \frac{da_k}{dt} \right] = 0 ,$$

$$a_j|_{t=0} = 0 , \quad j = 1, 2, \ldots, n .$$

The determinant Δ_n of the matrix consisting of the coefficients of the derivatives in (40) is non-zero and the system (40) may be reduced to the form

$$\frac{da_j}{dt} = \frac{\Delta_n^{(j)}}{\Delta_n} = g_j(a_1, a_2, \ldots, a_n, t) , \quad j = 1, 2, \ldots, n .$$

Suppose the following conditions are fulfilled:

1) The functions $(P(u_n), v_j)$ and $(P'(u_n) v_k, v_j)$ of the variables $a_1, a_2, \ldots, a_n$ are continuous for all values of their arguments and may increase at infinity in such a manner that

$$(41) \qquad |(P(u_n), v_j)| \leqq p_m(|a_1|, |a_2|, \ldots, |a_n|) ,$$
$$|(P'(u_n) v_k, v_j)| \leqq p_{m-1}(|a_1|, |a_2|, \ldots, |a_n|) ,$$

where p_m and p_{m-1} are polynomials of degrees m and $m - 1$ respectively, and m is a positive integer.

2) The inequality

$$(42) \qquad (P'(u_n) h, h) \geqq N(\sum_{k=1}^{n} a_k^2)^{(m-1)/2} \|h\|^2 , \quad N = \text{const} > 0$$

holds.

Then the Cauchy problem (40) has a solution in the segment $0 \leqq t \leqq 1$.

One can show (see [22]) that the inequalities (41) and (42) are fulfilled with $m = 1$ in the case of the functional (31), if the functions $\varrho_j(\xi)$ satisfy some additional conditions.

REFERENCES

[1] S. G. Mihlin: Variational methods in mathematical physics. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow 1957 (Russian); German translation: Variationsmethoden der mathematischen Physik, Berlin 1962.

[2] K. Friedrichs: Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differenzialoperatoren. Math. Ann. 109, *4—5* (1934), 465—487.

[3] M. G. Kreĭn: The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. Mat. Sb. 20 (62), *3* (1947), 431—490 (Russian), 491—495 (English summary).

[4] A. T. Taldykin: Systems of elements of a Hilbert space and series formed from them. Mat. Sb. N. S. 29 (71), *1* (1951), 79—120 (Russian).

[5] S. G. Mihlin: On the stability of Ritz method. Dokl. Akad. Nauk SSSR 135, *1* (1960), 16—19 (Russian).

[6] S. H. Mihlin: On certain conditions of the stability of Ritz method. Vestnik Leningrad. Univ. 13, *3* (1961), 40—50 (Russian), 51 (English summary).

[7] G. N. YASKOVA and M. N. YAKOVLEV: Certain conditions for stability of the Petrov-Galerkin method. Trudy Mat. Inst. Steklov. *66* (1962), 182—189 (Russian).
[8] V. M. MITKEVIČ: Applications of Ritz method to the problem of bending of a console sectorial plate. Sbornik Trudov Laborat. Gidravlič. mašin Akad. Nauk USSR *9* (1961), (Russian).
[9] S. G. MIHLIN: On Ritz method. Dokl. Akad. Nauk SSSR (N.S.) 106, *3* (1956), 391—394 (Russian).
[10] S. G. MIHLIN: On a rational choice of the coordinate functions in Ritz method. Ž. Vyčisl. Mat. Fiz. *2*, *3* (1962), 437—444 (Russian).
[11] A. LANGENBACH: On the application of a variational principle to some non-linear differential equations. Dokl. Akad. Nauk SSSR 121, *2* (1958), 214—217 (Russian).
[12] A. LANGENBACH: Variationsmethoden in der nichtlinearen Elastizität- und Plastizitätstheorie. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe. IX, 1959/1960.
[13] A. LANGENBACH: On some nonlinear operators of the theory of elasticity in Hilbert space. Vestnik Leningrad. univ. *1* (1961), 38—50 (Russian), 50 (English summary).
[14] I. V. GEĽMAN: The minimum problem for a non-linear functional. Leningrad. Gos. Ped. Inst. Uč. Zap. *166* (1958), 255—263 (Russian).
[15] V. I. KAZIMIROV: On the semi-continuity of integrals of the calculus of variations. Uspehi Mat. Nauk (N.S.) 11 (69), *3* (1956), 125—129 (Russian).
[16] I. V. GEĽMAN: Imbedding theorems of certain linear normed spaces. Izv. Vysš. Uč. Zaved. Matematika *4* (17), (1960), 55—66 (Russian).
[17] S. G. MIHLIN: On Ritz method in non-linear problems. Dokl. Akad. Nauk SSSR 142, *4* (1962), 792—793 (Russian).
[18] L. M. KAČANOV: On the variational methods of solving problems in plasticity theory. Prikl. Mat. Meh. 23, *3* (1959), 616—617 (Russian).
[19] S. N. ROSÉ: On the convergence of L. M. Kačanov's method. Vestnik Leningrad. Univ. 19 *4* (1961), 170—174 (Russian), 174 (English summary).
[20] L. N. HAGEN-THORN and S. G. MIHLIN: On the solvability of non-linear Ritz systems. Dokl. Akad. Nauk SSSR 138, *2* (1961), 256—260 (Russian).
[21] D. F. DAVIDENKO: On a new method of numerical solution of systems of non-linear equations. Dokl. Akad. Nauk SSSR (N.S.) 88, *4* (1953), 601—602 (Russian).
[22] L. N. HAGEN-THORN: On the solvability of Ritz systems for the functionals of plasticity theory. Trudy Mat. Inst. Steklov. *66* (1962), 190—195 (Russian).