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# Impulsive Periodic Boundary Value Problem

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**Abstract**

In the paper we consider the impulsive periodic boundary value problem with a general linear left hand side. The results are based on the topological degree theorems for the corresponding operator equation $(I - F)u = 0$ on a certain set $\Omega$ that is established using properties of strict lower and upper functions of the boundary value problem.

**Key words:** Boundary value problem, topological degree, upper and lower functions, impulsive problem, periodic solution, differential equation.

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## 1 Introduction

In this paper we will study the boundary value problem

$$x'' + a(t)x' + b(t)x = f(t, x, x') \tag{1.1}$$

$$x(t_1+) = J(x(t_1)), \quad x'(t_1+) = M(x'(t_1-)), \tag{1.2}$$

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi). \tag{1.3}$$

We suppose, that $a, b$ are Lebesgue integrable functions on $[0, 2\pi]$ and $f$ fulfils the Carathéodory conditions on $[0, 2\pi] \times \mathrm{R}^2$. Furthermore, we suppose that $t_1 \in (0, 2\pi)$ and

$$\begin{gathered} J, M \text{ are continuous mappings } R \to R \text{ and,} \\ J \text{ is increasing on } R \text{ and } M \text{ is nondecreasing on } R. \end{gathered} \tag{1.4}$$

Our main assertions (Theorem 3.2 and Theorem 3.6) are based on the properties of the Leray–Schauder topological degree. We search an operator problem $u = Fu$ which corresponds to (1.1)–(1.3) and such that operator $I - F$ has nonzero topological degree on a certain set $\Omega$. For establishing $\Omega$ the existence of strict lower and upper functions of the problem is assumed.

We consider two cases of ordering of strict lower and upper functions $\sigma_1$ and $\sigma_2$:

i) The functions are well ordered i.e. $\sigma_1(t) < \sigma_2(t)$ for all $t \in [0, 2\pi]$. In this case, we get the existence of a solution $u$ which lies between the strict lower and upper functions i.e. $\sigma_1(t) < u(t) < \sigma_2(t)$ on $[0, 2\pi]$ (Corollary 3.3).

ii) The functions are in the opposite order i.e. $\sigma_2(t) < \sigma_1(t)$ for all $t \in [0, 2\pi]$. In this case, we get the existence of a solution $u$, at least one point of which lies between the strict functions i.e. $\sigma_2(t_u) < u(t_u) < \sigma_1(t_u)$ for some $t_u \in [0, 2\pi]$ (Corollary 3.7).

This work generalizes the results published in [1],[2] where the equation $x'' = f(t, x, x')$, which is a special case of the equation (1.1), has been studied.

## 1.1 Definitions

$L[0, 2\pi]$ is the Banach space of the Lebesgue integrable functions on $[0, 2\pi]$ with the norm $\|x\|_1 = \int_0^{2\pi} |x(t)|dt$.

$L_\infty[0, 2\pi]$ denotes the Banach space of essentially bounded functions on $[0, 2\pi]$ with the norm $\|x\|_\infty = \operatorname{ess\,sup}\{|x(t)|; t \in [0, 2\pi]\}$.

$C[0, 2\pi]$ and $C^1[0, 2\pi]$ are the spaces of functions continuous on $[0, 2\pi]$ and of functions with continuous first derivatives on $[0, 2\pi]$, respectively.

Similarly, $AC[0, 2\pi]$ and $AC^1[0, 2\pi]$ denote spaces of functions absolutely continuous on $[0, 2\pi]$ and of functions with absolutely continuous first derivatives on $[0, 2\pi]$, respectively.

Let $t_1 \in (0, 2\pi)$. Then $\widetilde{C}^1[0, 2\pi]$ means the set of functions

$$u(t) = \begin{cases} u_1(t) \text{ for } 0 \le t \le t_1 \\ u_2(t) \text{ for } t_1 < t \le 2\pi \end{cases},$$

where $u_1 \in C^1[0, t_1]$ and $u_2 \in C^1[t_1, 2\pi]$. $\widetilde{AC}^1[0, 2\pi]$ specifies the set of functions $u \in \widetilde{C}^1[0, 2\pi]$ with absolutely continuous first derivatives on $(0, t_1)$ and on $(t_1, 2\pi)$. For $u \in \widetilde{C}^1[0, 2\pi]$ we establish

$$u'(0) = \lim_{\tau \to 0+} u'(\tau), \quad u'(2\pi) = \lim_{\tau \to 2\pi-} u'(\tau),$$

$$u'(t_1) = \lim_{\tau \to t_1-} u'(\tau),$$

$$\|u\|_{\widetilde{C}^1} = \|u\|_\infty + \|u'\|_\infty.$$

Moreover, for $u \in \widetilde{C}^1[0, 2\pi]$ and $t \in (0, 2\pi)$ we will use notations

$$\Delta u(t) = u(t+) - u(t), \quad \Delta u'(t) = u'(t+) - u'(t). \tag{1.5}$$

$\widetilde{C}^1[0, 2\pi]$ with the norm $\| \cdot \|_{\widetilde{C}^1}$ is the Banach space.

**Definition 1.1** By a solution of the impulsive problem (1.1)–(1.3) we call $u \in \widetilde{AC}^1[0, 2\pi]$ which fulfils the equation (1.1) for a.e. $t \in [0, 2\pi]$ and satisfies conditions (1.2) and (1.3).

By a solution of the problem (1.1), (1.3) (without impulses) we call $u \in AC^1[0, 2\pi]$ which fulfils the equation (1.1) for a.e. $t \in [0, 2\pi]$ and satisfies conditions (1.3).

**Definition 1.2** A function $\sigma_1 \in \widetilde{AC}^1[0, 2\pi]$ is a lower function of (1.1)–(1.3) if

$$\sigma_1'' + a(t)\sigma_1' + b(t)\sigma_1 \geq f(t, \sigma_1, \sigma_1') \ for \ a.e. \ t \in [0, 2\pi], \tag{1.6}$$

$$\sigma_1(t_1+) = J(\sigma_1(t_1)), \ \sigma_1'(t_1+) \geq M(\sigma_1'(t_1)), \tag{1.7}$$

$$\sigma_1(0) = \sigma_1(2\pi), \ \sigma_1'(0) \geq \sigma_1'(2\pi). \tag{1.8}$$

**Definition 1.3** A function $\sigma_2 \in \widetilde{AC}^1[0, 2\pi]$ is an upper function of (1.1)–(1.3) if

$$\sigma_2'' + a(t)\sigma_2' + b(t)\sigma_2 \leq f(t, \sigma_2, \sigma_2') \ for \ a.e. \ t \in [0, 2\pi], \tag{1.9}$$

$$\sigma_2(t_1+) = J(\sigma_2(t_1)), \ \sigma_2'(t_1+) \leq M(\sigma_2'(t_1)), \tag{1.10}$$

$$\sigma_2(0) = \sigma_2(2\pi), \ \sigma_2'(0) \leq \sigma_2'(2\pi). \tag{1.11}$$

**Definition 1.4** A lower function $\sigma_1$ of (1.1)–(1.3) is a strict lower function of (1.1)–(1.3) if it is not a solution of (1.1)–(1.3) and there exists $\varepsilon > 0$ such that

$$\sigma_1'' + a(t)y + b(t)x \geq f(t, x, y) \text{ for a.e. } t \in [0, 2\pi] \tag{1.12}$$

and each

$$x \in [\sigma_1(t), \sigma_1(t) + \varepsilon], \ y \in [\sigma_1'(t) - \varepsilon, \sigma_1'(t) + \varepsilon].$$

Similarly, an upper function $\sigma_2$ of (1.1)–(1.3) is a strict upper function of (1.1)–(1.3) if it is not a solution of (1.1)–(1.3) and there exists $\varepsilon > 0$ such that

$$\sigma_2'' + a(t)y + b(t)x \leq f(t, x, y) \text{ for a.e. } t \in [0, 2\pi] \tag{1.13}$$

and each

$$x \in [\sigma_2(t) - \varepsilon, \sigma_2(t)], \ y \in [\sigma_2'(t) - \varepsilon, \sigma_2'(t) + \varepsilon].$$

# 2 Auxiliary problem

In this chapter we will study the auxiliary Dirichlet problem and present assertions which consist of the relation of the strict lower and upper functions to a solution of the auxiliary problem. The assertions will be used in next chapters.

Consider the boundary value problem

$$x'' + a(t)x' + b(t)x = h(t), \tag{2.1}$$

$$x(0) = x(2\pi) = c, \tag{2.2}$$

where $h \in L[0, 2\pi]$ and $c \in \mathrm{R}$ and the corresponding homogeneous problem

$$x'' + a(t)x' + b(t)x = 0, \tag{2.3}$$

$$x(0) = x(2\pi) = 0. \tag{2.4}$$

We study two cases of the problem:

i) The problem (2.3), (2.4) has only the trivial solution. In this case there is the Green function of (2.3), (2.4) and we can prove that there exists an operator $F$ corresponding to (2.1), (2.2) such that every solution $u$ of $x = Fx$ fulfils

$$u(t_1+) = u(t_1) + d, \;\; u'(t_1+) = u'(t_1) + e, \quad d, e \in \mathrm{R}. \tag{2.5}$$

ii) The problem (2.3), (2.4) has the nontrivial solution. In this case we transform the problem to an equivalent form to be able to use the way in i).

**Lemma 2.1** *Let the homogeneous boundary value problem (2.3), (2.4) has only the trivial solution. Then there exists a unique solution* $u \in \widetilde{AC}^1[0, 2\pi]$ *of the impulsive problem (2.1), (2.2), (2.5).*

The solution can be written in the form

$$u = c + \widetilde{g}(t, t_1)d + g(t, t_1)e + \int_0^{2\pi} g(t, s)[h(s) - cb(s)]ds, \tag{2.6}$$

where $g(t, s)$ is the Green function of (2.3),(2.4) and $\widetilde{g}(t, s)$ is a function which fulfills (2.3) for a.e. $t \in [0, s) \cup (s, 2\pi]$ and each fixed $s \in [0, 2\pi]$ and satisfies conditions (2.4) and

$$\widetilde{g}(s+, s) = \widetilde{g}(s, s) + 1, \qquad \left.\frac{\partial \widetilde{g}(t, s)}{\partial t}\right|_{t \to s+} = \left.\frac{\partial \widetilde{g}(t, s)}{\partial t}\right|_{t \to s-} \tag{2.7}$$

for each $s \in (0, 2\pi)$. At first, we need to prove that such function $\widetilde{g}(t, s)$ exists.

**Lemma 2.2** *Let the homogeneous boundary value problem (2.3), (2.4) has only the trivial solution. Then for each fixed* $s \in [0, 2\pi]$ *there exists a function* $\widetilde{g}$ *which fulfills (2.3) for a.e* $t \in [0, 2\pi]$ *and satisfies (2.4), (2.7).*

**Proof** Consider fixed $s \in [0, 2\pi]$ and a problem (2.4),

$$x''(t) + a(t)x'(t) + b(t)x(t) = \widetilde{h}_s(t), \tag{2.8}$$

where

$$\widetilde{h}_s(t) = h_s^*(t) + \frac{1}{2\pi}a(t) - \Big(1 - \frac{1}{2\pi}t\Big)b(t), \qquad h_s^*(t) = \begin{cases} b(t) & \text{for } t \le s \\ 0 & \text{for } s < t \end{cases}.$$

Since the corresponding homogeneous problem has only the trivial solution and $\widetilde{h}_s \in L[0, 2\pi]$ then there exists a solution $w_s \in AC^1[0, 2\pi]$

$$w_s(t) = \int_0^{2\pi} g(t, \tau)\widetilde{h}_s(\tau)d\tau$$

satisfying for a.e. $t \in [0, 2\pi]$

$$w_s''(t) + a(t)w_s'(t) + b(t)w_s(t) = h_s^*(t) + \frac{1}{2\pi}a(t) - \Big(1 - \frac{1}{2\pi}t\Big)b(t),$$

$$w_s(0) = 0, \quad w_s(2\pi) = 0.$$

Denote

$$u_s(t) = \begin{cases} w_s(t) - \frac{1}{2\pi}t & \text{for } t \le s \\ w_s(t) + 1 - \frac{1}{2\pi}t & \text{for } t > s \end{cases}.$$

Then $u_s \in AC^1([0, 2\pi] \backslash \{s\})$ and

$$u_s''(t) + a(t)u_s'(t) + b(t)u_s(t) = w_s''(t) + a(t)\left[w_s'(t) - \frac{1}{2\pi}\right]$$

$$+ b(t)\left[w_s(t) - \tfrac{1}{2\pi}t\right] = h_s^*(t) - b(t) = 0$$

for a.e. $t \in (0, s)$ and

$$u_s''(t) + a(t)u_s'(t) + b(t)u_s(t) = w_s''(t) + a(t)\left[w_s'(t) - \frac{1}{2\pi}\right]$$

$$+ b(t)\left[w_s(t) + 1 - \tfrac{1}{2\pi}t\right] = h_s^*(t) = 0$$

for a.e. $t \in (s, 2\pi)$. Moreover

$$\Delta u_s(s) = w_s(s) + 1 - \frac{1}{2\pi}s - \left[w_s(s) - \frac{1}{2\pi}s\right] = 1,$$

$$\Delta u_s'(s) = w_s'(s) - \frac{1}{2\pi} - \left[w_s'(s) - \frac{1}{2\pi}\right] = 0,$$

$$u_s(0) = 0, \quad u_s(2\pi) = 0.$$

Hence we can define $\widetilde{g}(t, s) = u_s(t)$ for each fixed $s \in [0, 2\pi]$. □

**Proof of Lemma 2.1** Now, we prove that $u$ given by (2.6) is a solution of (2.1), (2.2), (2.5). For fixed $t_1 \in [0, 2\pi]$ we denote

$$\phi(t) = g(t, t_1), \quad \widetilde{\phi}(t) = \widetilde{g}(t, t_1),$$

$$u_1(t) = \widetilde{\phi}(t)d + \phi(t)e, \quad u_2(t) = \int_0^{2\pi} g(t, s)[h(s) - cb(s)]\, ds.$$

In view to properties of functions $g, \widetilde{g}$ we have $\phi, \widetilde{\phi} \in \widetilde{AC}^1[0, 2\pi]$ and

$$\Delta\phi(t_1) = 0, \quad \Delta\phi'(t_1) = 1,$$

$$\Delta\widetilde{\phi}(t_1) = 1, \quad \Delta\widetilde{\phi}'(t_1) = 0.$$

Then $u_1 \in \widetilde{AC}^1[0, 2\pi]$ is a solution of (2.3)–(2.5). Moreover $u_2 \in AC^1[0, 2\pi]$ is a solution of (2.4),

$$x'' + a(t)x' + b(t)x = h(t) - cb(t)$$

i.e. $u_2 + c$ is a solution of the problem (2.1), (2.2) without impulses. Thus $u = c + u_1 + u_2 \in \widetilde{AC}^1[0, 2\pi]$ is a solution of the impulsive problem (2.1), (2.2), (2.5). □

**Lemma 2.3** *Let*

$$b(t) \le 0 \text{ for a.e. } t \in [0, 2\pi] \text{ and } \int_0^{2\pi} b(t)\, dt \ne 0. \tag{2.9}$$

*Then the homogeneous boundary value problem (2.3), (2.4) has only the trivial solution.*

**Proof** On the contrary, suppose that there exists a nontrivial solution $u$ of (2.3), (2.4). Since $-u$ is a solution of (2.3), (2.4), as well, without loss of generality we can suppose that there exists a maximum point

$$\max_{t \in J} u(t) = u(t_M) > 0, \quad u'(t_M) = 0, \quad t_M \in (0, 2\pi).$$

Then, with respect to (2.4), there exists $t_0 \in (t_M, 2\pi)$ such that $u(t) > 0$ for all $t \in (t_M, t_0)$ and $u'(t_0) < 0$. On the contrary

$$u'(t_0) = -e^{-A(t_0)} \int_{t_M}^{t_0} e^{A(s)} b(s) u(s)\, ds \ge 0,$$

where $A(t) = \int_{t_M}^{t} a(s)\, ds$, a contradiction. □

**Lemma 2.4** *Let (2.9) be fulfilled, let* $\sigma_2(t)$ *be a strict upper function of the problem (1.1)–(1.3) and let* $\widetilde{f}(t,x,y)$ *satisfy Carathéodory conditions on* $[0,2\pi]\times \mathrm{R}^2$ *and*

$$\widetilde{f}(t,x,y) > f(t,\sigma_2,y) \quad \textit{for a.e. } t\in[0,2\pi],\ x>\sigma_2,\ y\in \mathrm{R}. \tag{2.10}$$

*Then*

$$u(t)\le \sigma_2(t) \tag{2.11}$$

*is valid for* $t\in[0,2\pi]$ *for every solution* $u$ *of (1.2),*

$$x''+a(t)x'+b(t)x=\widetilde{f}(t,x,x'), \tag{2.12}$$

*which fulfils*

$$u(0)=u(2\pi)\le\sigma_2(0). \tag{2.13}$$

**Proof** Denote $v(t)=u(t)-\sigma_2(t)$ for $t\in[0,2\pi]$.

(i) Let there exist $t_0\in(0,t_1)\cup(t_1,2\pi)$ such that $v(t_0)=\max\{v(t): t\in(0,t_1)\cup(t_1,2\pi)\}>0$, $v'(t_0)=0$. Then there exists $\delta>0$ such that $v(t)>0$, $|v'(t)|<\varepsilon$ for all $t\in(t_0,t_0+\delta)$, where $\varepsilon$ is from (1.13) and so for a.e. $t\in(t_0,t_0+\delta)$

$$\begin{aligned} v''(t)&=u''(t)-\sigma_2''(t)=\widetilde{f}(t,u(t),u'(t))-a(t)u'(t)-b(t)u(t)-\sigma_2''(t)\\ &>f(t,\sigma_2(t),u'(t))-a(t)u'(t)-b(t)u(t)-\sigma_2''(t)\ge -b(t)(u(t)-\sigma_2(t))\ge 0.\end{aligned}$$

Hence, $v'(t)>0$ and $v(t)>v(t_0)$ for each $t\in(t_0,t_0+\delta)$, a contradiction.

(ii) Now, we suppose that $v(t_1)>v(t)$ for all $t\in(0,t_1)$ and $v(t_1)>0$. Then $u'(t_1)-\sigma_2'(t_1)=v'(t_1)\ge 0$ and $u(t_1)>\sigma_2(t_1)$. From the properties of $J$ and $M$ we get

$$v(t_1+)=J(u(t_1))-J(\sigma_2(t_1))>0,\quad v'(t_1+)=M(u'(t_1))-M(\sigma_2'(t_1))\ge 0.$$

Let $v'(t_1+)>0$. Then in view to (2.13) there is a maximum point $t_0\in(t_1,2\pi)$ and $v(t_0)>0$ which contradicts to (i). Then $v'(t_1+)=0$ and there exists $\beta\in(t_1,2\pi)$ such that $v'(\beta)<0$, $v(t)>0$, $|v'(t)|<\varepsilon$ for all $t\in(t_1,\beta)$, where $\varepsilon$ is from (1.13) and then

$$v''(t)=u''(t)-\sigma_2''(t)=\widetilde{f}(t,u(t),u'(t))-a(t)u'(t)-b(t)u(t)-\sigma_2''(t)\ge 0,$$

for a.e. $t\in(t_1,\beta)$ and hence $v'(\beta)\ge 0$, a contradiction.

(iii) Suppose that $v(t_1+)>v(t)$ for all $t\in(t_1,2\pi]$ and $v(t_1+)>0$. Then $u'(t_1+)-\sigma_2'(t_1+)=v'(t_1+)\le 0$ and $u(t_1+)>\sigma_2(t_1+)$. If $v'(t_1+)=0$ then we get a contradiction as in (i). Hence $v'(t_1+)<0$. From the properties of functions $J,M$ we get

$$v(t_1)=u(t_1)-\sigma_2(t_1)>0,\quad v'(t_1)=u'(t_1)-\sigma_2'(t_1)<0.$$

In view to (2.13) there exists a maximum point $t_0\in(0,t_1)$ such that $v(t_0)>0$, a contradiction with (i). □

**Lemma 2.5** *Let (2.9) be fulfilled and $\sigma_2$ be a strict upper function of the problem (1.1)–(1.3). Then*

$$u(t) < \sigma_2(t) \text{ on } [0, 2\pi] \tag{2.14}$$

*is valid for every solution $u$ of (1.1)–(1.3) which satisfies (2.11).*

**Proof** Denote $v(t) = u(t) - \sigma_2(t)$ for $t \in [0, 2\pi]$.

(i) Let there exist $t_0 \in (0, t_1)$, such that $v(t_0) = \max\{v(t) : t \in (0, t_1)\} = 0$. Then $v'(t_0) = 0$ and there exist $\alpha, \beta$ such that $0 \le \alpha < t_0 < \beta \le t_1$ and $-\varepsilon < v(t) \le 0$, $|v'(t)| < \varepsilon$ for each $t \in (\alpha, \beta)$ . From the property of the strict upper function, we get for a.e. $t \in (\alpha, \beta)$

$$v''(t) = u''(t) - \sigma_2''(t) = f(t, u(t), u'(t)) - a(t)u'(t) - b(t)u(t) - \sigma_2''(t) \ge 0$$

and so $v'(t) \ge 0$, $v(t) \ge 0$ for $t \in (t_0, \beta)$ and $v'(t) \le 0$, $v(t) \ge 0$ for $t \in (\alpha, t_0)$. With respect to (2.11) it is possible only if $v(t) = v'(t) = 0$ for $t \in (\alpha, \beta)$ where $\alpha = 0, \beta = t_1$. From (1.3), (1.11) we get $v(2\pi) = v(0) = 0$, $v'(2\pi) \le v'(0) = 0$ i.e. $v(2\pi) = v'(2\pi) = 0$ and hence, we obtain $v(t) = v'(t) = 0$ for $t \in (t_1, 2\pi]$, as well. Then $u(t) = \sigma_2(t)$ for $t \in [0, 2\pi]$, which contradicts to the definition of the strict upper function. In the case $t_0 \in (t_1, 2\pi)$, we can use the same arguments to get a contradiction.

(ii) Let $v(t) < v(t_1) = 0$ for $t \in [0, t_1)$. Then $v'(t_1) \ge 0$ and

$$v(t_1+) = J(u(t_1)) - J(\sigma(t_1)) = 0, \quad v'(t_1+) = M(u'(t_1)) - M(\sigma'(t_1)) \ge 0.$$

If $v'(t_1+) > 0$ then there exists $\gamma_1 \in (t_1, 2\pi)$ such that $v(\gamma_1) > 0$, a contradiction. Thus $v'(t_1+) = v(t_1+) = 0$ and so $v'(t) = v(t) = 0$ for $t \in (t_1, 2\pi]$. Using boundary value conditions we get $v(t) = 0$ for $t \in [0, t_1)$, as well. Then $u(t) = \sigma_2(t)$ for $t \in [0, 2\pi]$, a contradiction.

(iii) Now, let $v(t) < v(t_1+) = 0$ for $t \in (t_1, 2\pi]$. Then $v'(t_1+) \le 0$. Suppose $v'(t_1+) = 0$. Then there is $\beta \in (t_1, 2\pi]$ such that $0 > v(t) > -\varepsilon$ and $|v'(t)| < \varepsilon$ for $t \in (t_1, \beta)$ where $\varepsilon > 0$ is the constant from Definition 1.4. Thus, we get $v'(t) = 0$ for all $t \in (t_1, \beta)$ with $\beta = 2\pi$ and the same result we get on $[0, t_1)$, a contradiction. Then $v'(t_1+) < 0$ and from the properties of functions $J$ and $M$ we obtain $v(t_1) = 0$, $v'(t_1) < 0$. Hence there exists $\gamma_2 \in (0, t_1)$ such that $v(\gamma_2) > 0$, a contradiction.

(iv) Let $v(0) = v(2\pi) = 0$. From (1.2), (1.11) we get $v'(0) = v'(2\pi) = 0$. We get a contradiction as in (i). □

**Lemma 2.6** *Let (2.9) be fulfilled, let $\sigma_1(t)$ be a strict lower function of the problem (1.1)–(1.3) and let $\widetilde{f}(t, x, y)$ satisfy Carathéodory conditions on $[0, 2\pi] \times \mathrm{R}^2$ and*

$$\widetilde{f}(t, x, y) < f(t, \sigma_1, y) \text{ for a.e. } t \in [0, 2\pi],\ x < \sigma_1,\ y \in \mathrm{R}. \tag{2.15}$$

*Then*

$$u(t) \ge \sigma_1(t) \tag{2.16}$$

*is valid for* $t \in [0, 2\pi]$ *for every solution* $u$ *of (2.12), (1.2) which fulfils*

$$u(0) = u(2\pi) \geq \sigma_1(0). \tag{2.17}$$

**Proof** We can use similar arguments as in the proof of Lemma 2.4. □

**Lemma 2.7** *Let (2.9) be fulfilled, let* $\sigma_1(t)$ *be a strict lower function of the problem (1.1)–(1.3). Then*

$$u(t) > \sigma_1(t) \text{ on } [0, 2\pi] \tag{2.18}$$

*is valid for every solution* $u$ *of (1.1)–(1.3) which satisfies (2.16).*

**Proof** We can use similar arguments as in the proof of Lemma 2.5. □

We can rewrite the periodic conditions (1.3) to the equivalent form of Dirichlet type boundary conditions

$$x(0) = x(0) + x'(0) - x'(2\pi),\ x(2\pi) = x(0) + x'(0) - x'(2\pi). \tag{2.19}$$

In view to Lemma 2.1 and (2.19), we can rewrite problem (1.1)–(1.3) to the operator form

$$\begin{aligned} (Fx)(t) &= x(0) + x'(0) - x'(2\pi) \\ &+ \int_0^{2\pi} g(t,s)\big[f(s,x(s),x'(s)) - \big(x(0)+x'(0)-x'(2\pi)\big)b(s)\big]\,ds \\ &+ \widetilde{g}(t,t_1)[J(x(t_1)) - x(t_1)] + g(t,t_1)[M(x'(t_1)) - x'(t_1)], \quad t \in [0,2\pi]. \end{aligned} \tag{2.20}$$

The operator $F : \widetilde{C}^1[0, 2\pi] \to \widetilde{C}^1[0, 2\pi]$ is completely continuous (see [2], Lemma 3.1) and every fixed point $u \in \widetilde{C}^1[0, 2\pi]$ of F is a solution of (1.1)–(1.3).

Now, assume that problem (2.3), (2.4) has a nontrivial solution. Then we choose an arbitrary $\mu \in (-\infty, 0)$ and instead of (1.1) we will use the equation

$$x'' + a(t)x' + \mu x = f_\mu(t, x, x'), \tag{2.21}$$

where

$$f_\mu(t, x, x') = f(t, x, x') + (\mu - b(t))x. \tag{2.22}$$

Then in view to Lemma 2.3 the corresponding homogeneous problem

$$x'' + a(t)x' + \mu x = 0, \tag{2.23}$$

(2.4) has only the trivial solution and hence we can rewrite problem (2.21), (1.2), (1.3) to the operator form

$$\begin{aligned} (F_\mu x)(t) &= x(0) + x'(0) - x'(2\pi) \\ &+ \int_0^{2\pi} g_\mu(t,s)\big[f_\mu(s,x(s),x'(s)) - \big(x(0)+x'(0)-x'(2\pi)\big)\mu\big]\,ds \\ &+ \widetilde{g}_\mu(t,t_1)[J(x(t_1)) - x(t_1)] + g_\mu(t,t_1)[M(x'(t_1)) - x'(t_1)], \quad t \in [0,2\pi], \end{aligned} \tag{2.24}$$

where $g_\mu$ is the Green function of (2.23), (2.2) and $\widetilde{g}_\mu$ is a function which fulfils (2.23) for a.e. $t \in [0,s) \cup (s,2\pi]$ and each fixed $s \in [0,2\pi]$ satisfies (2.4) and

$$\widetilde{g}_\mu(s+,s) = \widetilde{g}_\mu(s,s) + 1, \quad \frac{\partial \widetilde{g}_\mu(t,s)}{\partial t}\bigg|_{t=s+} = \frac{\partial \widetilde{g}_\mu(t,s)}{\partial t}\bigg|_{t=s-}. \tag{2.25}$$

The operator $F_\mu : \widetilde{C}^1[0,2\pi] \to \widetilde{C}^1[0,2\pi]$ is completely continuous and every its fixed point $u \in \widetilde{C}^1[0,2\pi]$ is a solution of (1.1)–(1.3).

## 3 Strict lower and upper functions and topological degree

**Lemma 3.1** *Suppose that $r_0 \in (0,\infty)$, $p \in L[0,2\pi]$, $q \in L_\infty[0,2\pi]$ , $p,q$ are positive a.e. on $[0,2\pi]$. Then there exists $r^* \in (0,\infty)$ such that for each $x \in \widetilde{AC}^1[0,2\pi]$ fulfilling (1.2), (1.3),*

$$\|x\|_\infty < r_0 \tag{3.1}$$

*and*

$$|x'' + a(t)x' + b(t)x(t)| \le (1+|x'|)(p(t) + q(t)|x'|) \tag{3.2}$$

*for a.e. $t \in [0,2\pi]$, the estimate*

$$\|x'\|_\infty < r^* \tag{3.3}$$

*is valid.*

**Proof** Let (3.1), (3.2) be valid. In view to the mean value theorem there exist $\tau_1 \in [0,t_1), \tau_2 \in (t_1,2\pi]$ such that

$$|x'(\tau_1)| \le \frac{2r_0}{t_1}, \quad |x'(\tau_2)| \le \frac{2r_0}{2\pi - t_1}.$$

Denote

$$\begin{aligned} A(t) &= \exp[\textstyle\int_0^t a(\tau)\,d\tau\,], \\ y(t) &= A(t)x'(t), \quad \widetilde{A} = \|A\|_\infty, \\ N &= \|p\|_1 + 2\|q\|_\infty r_0 + \|b\|_1 r_0, \\ \widetilde{r} &> \max\Big\{A(\tau_1)\tfrac{2r_0}{t_1}, A(\tau_2)\tfrac{2r_0}{2\pi-t_1}\Big\}. \end{aligned} \tag{3.4}$$

(i) At first, suppose $x'(t_{sup}) = \sup\{x'(t) : t \in [0,2\pi]\} > 0$.

Assume $0 \le t_{sup} \le t_1$. Then there are $\alpha, \beta$ such that $0 \le \alpha < \beta \le t_1$ and $t_{sup} \in [\alpha,\beta]$ and such that $x'(t) > 0$ for each $t \in [\alpha,\beta]$. From (3.2) we get for a.e. $t \in [\alpha,\beta]$

$$|x''(t) + a(t)x'(t)| \le (1 + x'(t))[p(t) + q(t)x'(t)] + |b(t)|r_0,$$

$$|[A(t)x'(t)]'| \leq A(t)(1 + x'(t))(p(t) + q(t)x'(t)) + A(t)|b(t)|r_0,$$

$$\left|\frac{y'(t)}{\widetilde{A} + y(t)}\right| \leq \frac{|[A(t)x'(t)]'|}{A(t)(1 + x'(t))} \leq p(t) + q(t)x'(t) + |b(t)|r_0,$$

$$(3.5) \qquad -p(t) - q(t)x'(t) - |b(t)|r_0 \leq \frac{y'(t)}{\widetilde{A} + y(t)} \leq p(t) + q(t)x'(t) + |b(t)|r_0.$$

Let $\tau_1 \leq t_{sup}$. Then we can choose $\tau_1$ such that $x'(\tau_1) \geq 0$ and $x'(t) > 0$ on $(\tau_1, t_{sup})$. Then by integrating of the right hand inequality of (3.5) on $(\tau_1, t_{sup})$, we get

$$\ln\left(\frac{\widetilde{A} + y(t_{sup})}{\widetilde{A} + y(\tau_1)}\right) \leq \|p\|_1 + 2\|q\|_\infty r_0 + \|b\|_1 r_0 = N,$$

$$(3.6) \qquad x'(t_{sup}) \leq \frac{1}{A(t_{sup})}[(\widetilde{A} + \widetilde{r})e^N - \widetilde{A}].$$

Let $\tau_1 \geq t_{sup}$. Then we can choose $\tau_1$ such that $x'(\tau_1) \geq 0$ and $x'(t) > 0$ on $(t_{sup}, \tau_1)$. Then we get (3.6) by integrating of the left hand inequality of (3.5). Similarly we get (3.6) with $\tau_2$ instead of $\tau_1$ for $t_1 < t_{sup} \leq 2\pi$ .

Assume $x'(t_1+) > x'(t)$ for each $t \in (t_1, 2\pi]$. Then there exists $\beta \in (t_1, 2\pi)$ such that $x'(t) > 0$ on $(t_1, \beta)$. Thus (3.5) is valid for each $t \in (t_1, \beta)$. We can choose $\tau_2$ such that $x'(\tau_2) \geq 0$ and $x'(t) > 0$ on $(t_1, \tau_2)$. By integrating of the left hand inequality of (3.5) on $(t_1, \tau_2)$ we get

$$x'(t_1+) \leq \frac{1}{A(t_1)}[(\widetilde{A} + \widetilde{r})e^N - \widetilde{A}].$$

(ii) Now, suppose $x'(t_{inf}) = \inf\{x'(t) : t \in [0, 2\pi]\} < 0$.

Assume $0 \leq t_{inf} \leq t_1$. Then there are $\alpha, \beta$ such that $0 \leq \alpha < \beta \leq t_1$ and $t_{inf} \in [\alpha, \beta]$ and such that $x'(t) < 0$ for each $t \in [\alpha, \beta]$. From (3.2) we get for a.e. $t \in [\alpha, \beta]$

$$|x''(t) + a(t)x'(t)| \leq (1 - x'(t))[p(t) - q(t)x'(t)] + |b(t)|r_0,$$

$$|[A(t)x'(t)]'| \leq A(t)(1 - x'(t))(p(t) - q(t)x'(t)) + A(t)|b(t)|r_0,$$

$$\left|\frac{y'(t)}{\widetilde{A} - y(t)}\right| \leq \frac{|[A(t)x'(t)]'|}{A(t)(1 - x'(t))} \leq p(t) - q(t)x'(t) + |b(t)|r_0,$$

$$(3.7) \qquad -p(t) + q(t)x'(t) - |b(t)|r_0 \leq \frac{y'(t)}{\widetilde{A} - y(t)} \leq p(t) - q(t)x'(t) + |b(t)|r_0.$$

Let $\tau_1 \le t_{inf}$. Then we can choose $\tau_1$ such that $x'(\tau_1) \le 0$ and $x'(t) < 0$ on $(\tau_1, t_{inf})$. By integrating of the right hand inequality of (3.7) on $(\tau_1, t_{inf})$, we get

$$-\ln\left(\frac{\widetilde{A} - y(\tau_1)}{\widetilde{A} - y(t_{inf})}\right) \le \|p\|_1 + 2\|q\|_\infty r_0 + \|b\|_1 r_0 = N,$$

$$x'(t_{inf}) \ge -\frac{1}{A(t_{inf})}[(\widetilde{A} + \widetilde{r})e^N - \widetilde{A}]. \tag{3.8}$$

Let $\tau_1 \ge t_{inf}$.Then we can choose $\tau_1$ such that $x'(\tau_1) \le 0$ and $x'(t) < 0$ on $(t_{inf}, \tau_1)$. By integrating of the left hand inequality of (3.7) on $(t_{inf}, \tau_1)$ we get (3.8), as well. Similarly we get (3.8) with $\tau_2$ instead of $\tau_1$ for $t_1 < t_{inf} \le 2\pi$.

Assume $x'(t_1+) < x'(t)$ for each $t \in (t_1, 2\pi]$. Then there exists $\beta \in (t_1, 2\pi)$ such that $x'(t) < 0$ on $(t_1, \beta)$. Thus (3.7) is valid for each $t \in (t_1, \beta)$. We can choose $\tau_2$ such that $x'(\tau_2) \ge 0$ and $x'(t) > 0$ on $(t_1, \tau_2)$. By integrating of the left right inequality of (3.7) on $(t_1, \tau_2)$ we get

$$x'(t_1+) \ge -\frac{1}{A(t_1)}[(\widetilde{A} + \widetilde{r})e^N - \widetilde{A}].$$

Hence for

$$r^* > \frac{1}{\min\{A(t) : t \in [0, 2\pi]\}}[(\widetilde{A} + \widetilde{r})e^N - \widetilde{A}]$$

the inequality (3.3) is valid. □

**Theorem 3.2** *Let* $\sigma_1, \sigma_2 \in \widetilde{AC}^1[0, 2\pi]$ *be strict lower and upper functions of (1.1)–(1.3) such that*

$$\sigma_1(t) < \sigma_2(t) \ \ for\ t \in [0, 2\pi] \tag{3.9}$$

*and let there exist functions* $p, q \in L[0, 2\pi]$ *positive a.e. on* $[0, 2\pi]$ *such that*

$$|f(t, x, y)| \le (1 + |y|)(p(t) + q(t)|y|) \tag{3.10}$$

*for a.e.* $t \in [0, 2\pi]$ *and all* $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times \mathrm{R}$. *Then*

$$d[I - F_\mu, \Omega] = 1$$

*for any* $\mu \in (-\infty, 0)$ *and* $F_\mu$ *defined by (2.24), where*

$$\Omega = \{x \in \widetilde{C}^1[0, 2\pi] : \sigma_1(t) < x(t) < \sigma_2(t) \ on\ [0, 2\pi], \tag{3.11}$$

$$\sigma_1(t_1+) < x(t_1+) < \sigma_2(t_1+),\ \|x'\|_\infty < C\},$$

$$C > \left[1 + \left(\|\sigma_1\|_{\widetilde{C}_1} + \|\sigma_2\|_{\widetilde{C}_1}\right) \max\left\{\frac{1}{t_1}; \frac{1}{2\pi - t_1}\right\}\right] e^{\|p\|_1 + 2\|q\|_\infty r_0 + \|b\|_1 r_0},$$

$$r_0 = \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}.$$

**Proof** Let us choose a constant $C$ satisfying (3.11) and define auxiliary functions

$$\alpha(t,x) = \begin{cases} \sigma_1(t) & \text{for } x < \sigma_1(t) \\ x & \text{for } \sigma_1(t) \le x \le \sigma_2(t) \\ \sigma_2(t) & \text{for } \sigma_2(t) < x \end{cases}, \tag{3.12}$$

$$\beta(y) = \begin{cases} -C & \text{for } \; y < -C \\ y & \text{for } \; -C \le y \le C \\ C & \text{for } C < y \end{cases}, \tag{3.13}$$

$$\overline{f}_\mu(t,x,y) = f_\mu(t,x,\beta(y)) = f(t,x,\beta(y)) + (\mu - b(t))x, \tag{3.14}$$

$$\widetilde{f}_\mu(t,x,y) = \begin{cases} \overline{f}_\mu(t,\sigma_1(t),y) - \frac{\sigma_1(t)-x}{1+\sigma_1(t)-x} & \text{for } x < \sigma_1(t) \\ \overline{f}_\mu(t,x,y) & \text{for } \sigma_1(t) \le x \le \sigma_2(t) \\ \overline{f}_\mu(t,\sigma_2(t),y) + \frac{x-\sigma_2(t)}{1+x-\sigma_2(t)} & \text{for } \sigma_2(t) < x \end{cases}, \tag{3.15}$$

and an operator

$$\begin{aligned}(\widetilde{F}_\mu x)(t) &= \alpha(0, x(0) + x'(0) - x'(2\pi)) + \widetilde{g}(t,t_1)[J(\alpha(t_1, x(t_1))) \\ &\quad - \alpha(t_1, x(t_1))] + g(t,t_1)[M(\beta(x'(t_1-))) - \beta(x'(t_1-))] \\ &\quad + \int_0^{2\pi} g(t,s)[\widetilde{f}_\mu(s,x(s),x'(s)) - \alpha(0,x(0)+x'(0)-x'(2\pi))b(s)]\,ds\end{aligned} \tag{2.24}$$

for $t \in [0, 2\pi]$ and $\mu \in (-\infty, 0)$. We can see that $\widetilde{f}_\mu$ fulfills the Carathéodory conditions on $[0,2\pi] \times \mathrm{R}^2$ and $\alpha : \mathrm{R}^2 \to \mathrm{R}$, $\beta$: $\mathrm{R} \to \mathrm{R}$ are continuous mappings. Therefore $\widetilde{F}_\mu : \widetilde{C}^1[0,2\pi] \to \widetilde{C}^1[0,2\pi]$ is completely continuous. Consider the parameter system of operator equations

$$x - \lambda \widetilde{F}_\mu x = 0, \quad \lambda \in [0,1]. \tag{3.17}$$

With respect to (3.12)–(3.16), we can show that there is $r \in (0, \infty)$ that

$$\|\widetilde{F}_\mu x\|_{\widetilde{C}^1} \le r \text{ for } x \in \widetilde{C}^1[0,2\pi]. \tag{3.18}$$

Hence, there is $\rho > 0$ such that for any $\lambda \in [0,1]$ every solution of (3.17) lies inside the set

$$K(\rho) = \{x \in \widetilde{C}^1[0,2\pi]; \|x\|_{\widetilde{C}^1} < \rho\}.$$

Then $G = I - \lambda \widetilde{F}_\mu$ is a homotopic map on $\overline{K(\rho)} \times [0,1]$ and

$$d[I - \widetilde{F}_\mu, K(\rho)] = d[I \;, K(\rho)] = 1.$$

Therefore there exists a solution $u$ of (3.17) with $\lambda = 1$. In view to (3.16), $u$ is a solution of

$$x'' + a(t)x' + \mu x = \widetilde{f}_\mu(t, x, x'), \tag{3.19}$$

$$x(t_1+) = \widetilde{J}(x(t_1)),\ x'(t_1+) = \widetilde{M}(x'(t_1)), \tag{3.20}$$

$$x(0) = x(2\pi) = \alpha(0, x(0) + x'(0) - x'(2\pi)), \tag{3.21}$$

where

$$\begin{aligned} \widetilde{J}(x) &= x + J(\alpha(t, x)) - \alpha(t, x) \text{ for } x \in \mathrm{R}, \\ \widetilde{M}(y) &= y + M(\beta(y)) - \beta(y) \text{ for } y \in \mathrm{R}. \end{aligned} \tag{3.22}$$

With respect to (3.12) we have

$$\sigma_1(0) \le \alpha(0, u(0) + u'(0) - u'(2\pi)) \le \sigma_2(0),$$

$$\widetilde{f}_\mu(t, x, y) > \overline{f}_\mu(t, \sigma_2, y) \text{ for a.e. } t \in [0, 2\pi], \text{ each } x \in (\sigma_2, \infty), y \in \mathrm{R},$$

$$\widetilde{f}_\mu(t, x, y) < \overline{f}_\mu(t, \sigma_1, y) \text{ for a.e. } t \in [0, 2\pi], \text{ each } x \in (-\infty, \sigma_1), y \in \mathrm{R}.$$

In view to (3.22), $\widetilde{J}$ and $\widetilde{M}$ satisfy conditions (1.4). Let $\varepsilon > 0$ be from Definition 1.4. Since $\|\sigma_1\|_{\widetilde{C}^1} + \|\sigma_2\|_{\widetilde{C}^1} < C$, then there exists $\varepsilon_1 < \varepsilon$ such that $\|\sigma_1\|_\infty + \varepsilon_1 < C$ and $\|\sigma_2\|_\infty + \varepsilon_1 < C$. Then $\overline{f}_\mu(t, x, y) = f_\mu(t, x, y)$ for $(x, y) \in [\sigma_1(t), \sigma_1(t) + \varepsilon_1] \times [\sigma_1'(t) - \varepsilon_1, \sigma_1'(t) + \varepsilon_1]$ i.e. $\sigma_1$ fulfils condition (1.12), (1.7), (1.8) with $\overline{f}_\mu(t, x, y)$ instead of $f(t, x, y)$. Hence $\sigma_1$ is a strict lower function of

$$x'' + a(t)x' + \mu x = \overline{f}_\mu(t, x, x'), \tag{3.23}$$

(1.2), (1.3). Similarly $\overline{f}_\mu(t, x, y) = f_\mu(t, x, y)$ for $(x, y) \in [\sigma_2(t) - \varepsilon_1, \sigma_2(t)] \times [\sigma_2'(t) - \varepsilon_1, \sigma_2'(t) + \varepsilon_1]$ i.e. $\sigma_2$ fulfils conditions (1.13), (1.10), (1.11) with $\overline{f}_\mu(t, x, y)$ instead of $f(t, x, y)$. Hence $\sigma_2$ is a strict upper function of (3.23), (1.2), (1.3). In view to (3.12) and (3.21) we have $\sigma_1(0) \le u(0) = u(2\pi) \le \sigma_2(0)$. Then using lemmas 2.4-2.7 with $\overline{f}_\mu(t, x, y)$ and $\widetilde{f}_\mu(t, x, y)$ instead of $f(t, x, y)$ and $\widetilde{f}(t, x, y)$ we get

$$\sigma_1(t) < u(t) < \sigma_2(t) \text{ on } [0, 2\pi]. \tag{3.24}$$

Furthermore, $\widetilde{f}(t, x, y)$ fulfills (3.10) and thus from Lemma 3.1 we get

$$\|u'\|_\infty < C.$$

Moreover, in view to (3.15) for a.e. $t \in [0, 2\pi]$ we have

$$\widetilde{f}_\mu(t, u(t), u'(t)) = f_\mu(t, u(t), u'(t)).$$

Then we get that $u$ is a solution of the equation (2.21) and satisfies condition (1.2) and $u(0) = u(2\pi)$. Now, we need to prove the second condition in (1.3) i.e. we prove that

$$u'(0) = u'(2\pi).$$

Especially, we prove

$$\sigma_1(0) \le u(0) + u'(0) - u'(2\pi) \le \sigma_2(0). \tag{3.25}$$

On the contrary, suppose that

$$u(0) + u'(0) - u'(2\pi) > \sigma_2(0). \tag{3.26}$$

Then from (3.21) we get

$$u(0) = u(2\pi) = \alpha(0, u(0) + u'(0) - u'(2\pi)) = \sigma_2(0) = \sigma_2(2\pi) \tag{3.27}$$

and using (3.26)

$$u'(0) > u'(2\pi). \tag{3.28}$$

On the other side we proved

$$u(t) \le \sigma_2(t) \quad t \in [0, 2\pi]$$

and with (3.27) and (3.28) this yields

$$\sigma_2'(0) \ge u'(0) > u'(2\pi) \ge \sigma_2'(2\pi)$$

which contradicts to (1.11). Similarly we will prove that

$$\sigma_1(0) \le u(0) + u'(0) - u'(2\pi).$$

With respect to (3.21) and (3.12) we have $u'(0) = u'(2\pi)$. Thus, we have proved that every solution of (3.17) with $\lambda = 1$ is a solution of (2.21), (1.2), (1.3) which satisfies (3.24). Hence $u \in \Omega$. Since $F_\mu = \widetilde{F}_\mu$ on $\overline{\Omega}$ and $x \ne F_\mu x$ for $x \in \partial\Omega$, we use the excision property of the topological degree and get

$$d(I - F_\mu, \Omega) = d(I - \widetilde{F}_\mu, \Omega) = d(I - \widetilde{F}_\mu, K(\rho)) = 1. \qquad \square$$

**Corollary 3.3** *Let the assumptions of Theorem 3.2 be satisfied. Then the problem (1.1)–(1.3) has a solution $u$, which fulfills (3.24).*

**Lemma 3.4** *Let $\sigma_1, \sigma_2$ be strict lower and upper functions such that*

$$\sigma_2(t) < \sigma_1(t) \text{ for each } t \in [0, 2\pi]. \tag{3.29}$$

*Moreover, let $p, q \in L[0, 2\pi]$ be positive a.e. on $[0, 2\pi]$ such that for a.e. $t \in [0, 2\pi]$ and all $x, y \in R$*

$$|f(t, x, y) - b(t)x| < p(t) + q(t)|y|. \tag{3.30}$$

*Then for every solution* $u \in \widetilde{AC}^1[0,2\pi]$ *of (1.1)–(1.3) which fulfills*

$$\sigma_2(t_u) < u(t_u) < \sigma_1(t_u) \quad \textit{for some } t_u \in [0,2\pi] \tag{3.31}$$

*the estimate*

$$\|u'\|_{\widetilde{C}} < N_1, \qquad \|u\|_{\widetilde{C}} < N_2, \tag{3.32}$$

*where*

$$N_1 = (2 + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty)e^{\|p\|_1+\|q\|_1+\|a\|_1}, \quad N_2 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 2\pi N_1,$$

*is valid.*

**Proof** At first, we prove the existence of such $r \in (0,\infty)$ that the condition

$$|u'(s_u)| < r \tag{3.33}$$

is valid for some $s_u \in [0,2\pi]$. Denote $v_i(t) = (-1)^i(\sigma_i(t) - u(t))$, $i = 1,2$

i) Let

$$v_i'(s_u) = 0 \tag{3.34}$$

for some $s_u \in (0,t_1) \cup (t_1,2\pi)$. Then

$$|u'(s_u)| = |\sigma_i'(s_u)| \le \|\sigma_i'\|_\infty.$$

ii) Assume that (3.34) is not valid. Then from (1.3),(1.8) and (1.11) we have $v_i'(t) < 0$ for $t \in (0,t_1)$ and $v_i'(t) > 0$ for $t \in (t_1,2\pi)$ i.e.

$$v_i'(t_1) \le 0 \text{ and } v_i'(t_1+) \ge 0.$$

On the other hand,

$$v_i'(t_1+) \le (-1)^i[M(\sigma_i'(t_1)) - M(u'(t_1))] \le 0$$

and hence $v_i'(t_1+) = 0$. Then $|u'(t_1+)| = |\sigma_i'(t_1+)|$ and there exists $s_u \in (t_1,2\pi)$, that

$$|u'(s_u)| \le \|\sigma_i'\|_\infty + 1.$$

The condition (3.33) is proved for $r = \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1$. Now, suppose that (3.30) is valid. Then for a.e. $t \in [0,2\pi]$ we get

$$|u''(t) + a(t)u'(t)| = |f(t,u(t),u'(t)) - b(t)u(t)| \le p(t) + q(t)|u'(t)|,$$

$$|u''(t)| \le p(t) + (q(t) + |a(t)|)|u'(t)|,$$

$$-p(t) - q(t) - |a(t)| \le \frac{u''(t)u'(t)}{1+u'^2(t)} \le p(t) + q(t) + |a(t)|. \tag{3.35}$$

We integrate the left inequality of (3.35) on $(t, s_u)$ for $t \in (t_1, s_u)$ and the right inequality of (3.35) on $(s_u, t)$ for $t \in (s_u, 2\pi]$ and using (1.3) we can extend that for $t \in [0, t_1)$. Thus we have for each $t \in [0, 2\pi]$

$$1 + u'^2(t) \le (1 + u'^2(s_u))e^{2(\|p\|_1+\|q\|_1+\|a\|_1)} \le (1 + |u'(s_u)|)^2 e^{2(\|p\|_1+\|q\|_1+\|a\|_1)},$$

$$|u'(t)| \le (1 + |u'(s_u)|)e^{\|p\|_1+\|q\|_1+\|a\|_1} < N_1.$$

Moreover for each $t \in [0, t_1) \cup (t_1, 2\pi]$

$$|u(t)| \le |u(t_u)| + \left| \int_{t_u}^{t} u'(\tau)d\tau \right| < \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 2\pi N_1 = N_2,$$

is valid and then we get (3.32). □

**Remark 3.5** Let $p, q \in L[0, 2\pi]$ be positive a.e. on $[0, 2\pi]$ such that for a.e. $t \in [0, 2\pi]$ and all $x, y \in R$ (3.30) is satisfied. Then

$$|f_\mu(t, x, y) - \mu x| < p(t) + q(t)|y| \tag{3.36}$$

is valid for a.e. $t \in [0, 2\pi]$ and each $x, y \in R$ where $f_\mu(t, x, y)$ is given by (2.22).

**Theorem 3.6** *Let $\sigma_1$ and $\sigma_2$ be respectively strict lower and upper functions of (1.1)–(1.3) which fulfill (3.29), let M(0) = 0 and let there exist $p, q \in L[0, 2\pi]$ positive a.e. on $[0, 2\pi]$ such that (3.30) is satisfied. Then for any $\mu \in (-\infty, 0)$*

$$d[I - F_\mu, \Omega_2] = -1, \tag{3.37}$$

*where $F_\mu$ is defined by (2.24),*

$$\begin{aligned} \Omega_2 = \{x \in \widetilde{C}^1[0, 2\pi];\ \|x\|_\infty < \widetilde{N}_2,\ \|x'\|_\infty < \widetilde{N}_1, \\ \sigma_2(t_x) < x(t_x) < \sigma_1(t_x) \textit{ for some } t_x \in [0, 2\pi]\}, \end{aligned} \tag{3.38}$$

$$\widetilde{N}_1 = (1 + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty)e^{2(\|a\|_1+\|q\|_1+3\|p\|_1)}$$

*and*

$$\widetilde{N}_2 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 2\pi\widetilde{N}_1.$$

**Proof** Let $\varrho > \widetilde{N}_2$. Denote

$$\widetilde{f}(t, x, y) = \begin{cases} f(t, x, y) + p(t) & \text{for } x \ge \varrho \\ f(t, x, y) + \frac{x - \widetilde{N}_2}{\varrho - \widetilde{N}_2} p(t) & \text{for } \widetilde{N}_2 < x < \varrho \\ f(t, x, y) & \text{for } -\widetilde{N}_2 \le x \le \widetilde{N}_2 \\ f(t, x, y) - \frac{x + \widetilde{N}_2}{-\varrho + \widetilde{N}_2} p(t) & \text{for } -\varrho < x < -\widetilde{N}_2 \\ f(t, x, y) - p(t) & \text{for } x \le -\varrho \end{cases}. \tag{3.39}$$

Then for a.e. $t \in [0, 2\pi]$ and each $x, y \in R$

$$|\widetilde{f}(t, x, y) - b(t)x| \le |f(t, x, y) - b(t)x| + p(t) \le 2p(t) + q(t)|y|.$$

In view to (3.30)

$$-p(t) < f(t,x,0) - b(t)x < p(t)$$

is valid for a.e. $t \in [0, 2\pi]$ and each $x \in \mathrm{R}$ and hence

$$\widetilde{f}(t,\varrho,0) - b(t)\varrho = f(t,\varrho,0) - b(t)\varrho + p(t) > 0,$$

$$\widetilde{f}(t,-\varrho,0) + b(t)\varrho = f(t,-\varrho,0) + b(t)\varrho - p(t) < 0.$$

Consider a problem (1.3),

$$x'' = \overline{f}(t,x,x'), \tag{3.40}$$

$$x(t_1+) = \widetilde{J}(x(t_1)), \quad x'(t_1+) = M(x'(t_1)), \tag{3.41}$$

where

$$\overline{f}(t,x,y) = \widetilde{f}(t,x,y) - a(t)y - b(t)x, \tag{3.42}$$

$$\widetilde{J}(x) = \begin{cases} x & \text{for } x \le -\varrho, \\ -\frac{x+\widetilde{N}_2}{-\varrho+\widetilde{N}_2}\varrho + \left[1 - \frac{x+\widetilde{N}_2}{-\varrho+\widetilde{N}_2}\right] J(-\widetilde{N}_2) & \text{for } -\varrho < x < -\widetilde{N}_2, \\ J(x) & \text{for } -\widetilde{N}_2 \le x \le \widetilde{N}_2, \\ \frac{x-\widetilde{N}_2}{\varrho-\widetilde{N}_2}\varrho + \left[1 - \frac{x-\widetilde{N}_2}{\varrho-\widetilde{N}_2}\right] J(\widetilde{N}_2) & \text{for } \widetilde{N}_2 < x < \varrho, \\ x & \text{for } x \ge \varrho. \end{cases} \tag{3.43}$$

We can see that $\widetilde{J}$ is a continuous and increasing on R and

$$\widetilde{J}(\varrho) = \varrho, \quad \widetilde{J}(-\varrho) = -\varrho.$$

Moreover $\sigma_1, \sigma_2$ are strict lower and strict upper functions of (1.3), (3.40), (3.41). For a.e. $t \in J$ and each $x, y \in \mathrm{R}$ define a function

$$\overline{h}(t,x,y) = \begin{cases} \overline{f}(t,-\varrho,y) - \omega_1(t, \frac{-\varrho-x}{1-\varrho-x}) & \text{for } x < -\varrho \\ \overline{f}(t,x,y) & \text{for } -\varrho < x < \varrho \,, \\ \overline{f}(t,\varrho,y) + \omega_2(t, \frac{x-\varrho}{1+x-\varrho}) & \text{for } x > \varrho \end{cases} \tag{3.44}$$

where

$$\omega_i(t,\varepsilon) = \sup_{y \in [-\varepsilon,\varepsilon]} \{|\overline{f}(t,(-1)^i\varrho,y) - \overline{f}(t,(-1)^i\varrho,0)|\}, \quad i = 1,2 \tag{3.45}$$

for $\varepsilon > 0$. $\omega_i$ is positive and nondecreasing with the second variable and with respect to (3.42) and (3.30) we have

$$\omega_i(t,\varepsilon) = \sup_{y \in [-\varepsilon,\varepsilon]} \{|\widetilde{f}(t,(-1)^i\varrho,y) - \widetilde{f}(t,(-1)^i\varrho,0) - a(t)y|\},$$

$$\omega_i(t,\varepsilon) \le 4p(t) + (q(t) + |a(t)|)|y| \tag{3.46}$$

for a.e. $t \in [0, 2\pi]$ and each $y \in [-\varepsilon, \varepsilon]$. Now, consider the problem (1.3), (3.41)

$$x'' = \overline{h}(t, x, x'). \tag{3.47}$$

Choose $\eta > 0$ and put $\sigma_3(t) = -\varrho - \eta$, $\sigma_4(t) = \varrho + \eta$ for $t \in [0, 2\pi]$. Then for a.e. $t \in [0, 2\pi]$

$$\overline{h}(t, \varrho + \eta, 0) = \overline{f}(t, \varrho, 0) + \omega_2\Big(t, \frac{\eta}{1+\eta}\Big) = f(t, \varrho, 0) - b(t)\varrho + \omega_2\Big(t, \frac{\eta}{1+\eta}\Big) > 0.$$

For $\varepsilon = \frac{\eta/2}{1+\eta/2}$ and for $x \in [\varrho+\eta-\varepsilon, \varrho+\eta]$ , $y \in [-\varepsilon, \varepsilon]$ we obtain $x \in (\varrho+\eta/2, \varrho+\eta]$ and $|y| < \frac{x-\varrho}{1+x-\varrho}$ i.e. $\omega_2(t, |y|) \le \omega_2(t, \frac{x-\varrho}{1+x-\varrho})$.

Hence, in view of (3.44), we get

$$\begin{aligned} \overline{h}(t, x, y) &= \overline{f}(t, \varrho, y) + \omega_2\Big(t, \frac{x-\varrho}{1+x-\varrho}\Big) \\ &\ge \overline{f}(t, \varrho, 0) - |\overline{f}(t, \varrho, y) - \overline{f}(t, \varrho, 0)| + \omega_2\Big(t, \frac{x-\varrho}{1+x-\varrho}\Big) > 0. \end{aligned}$$

Thus $\sigma_4$ is a strict upper function of (3.47), (3.41), (1.3). Similarly we can prove, that $\sigma_3$ is a strict lower function of (3.47), (3.41), (1.3). Now, we choose an arbitrary $\mu \in (-\infty, 0)$ and rewrite the equation to the form

$$x'' + a(t)x' + \mu x = h_\mu(t, x, x'), \tag{3.48}$$

$$h_\mu(t, x, y) = h(t, x, y) + (\mu - b(t))x, \tag{3.49}$$

$$h(t, x, y) = \overline{h}(t, x, y) + a(t)y + b(t)x. \tag{3.50}$$

Then $\sigma_1, \sigma_3$ are strict lower and $\sigma_2, \sigma_4$ strict upper functions of (3.48), (3.41), (1.2) such that

$$\sigma_3(t) < \sigma_2(t) < \sigma_1(t) < \sigma_4(t) \text{ for all } t \in [0, 2\pi]. \tag{3.51}$$

Denote

$$\begin{aligned} \widetilde{\Omega} &= \{x \in \widetilde{C}^1[0, 2\pi] : \|x\|_\infty < \varrho + \eta, \|x'\|_\infty < \widetilde{N}_1\}, \\ \Delta_1 &= \{x \in \widetilde{\Omega} : x(t) > \sigma_1(t) \text{ for } t \in [0, 2\pi]\}, \\ \Delta_2 &= \{x \in \widetilde{\Omega} : x(t) < \sigma_2(t) \text{ for } t \in [0, 2\pi]\}. \end{aligned}$$

In view to (3.44)–(3.48) there exist functions $\widetilde{p}, \widetilde{q} \in L[0, 2\pi]$ positive a.e. on $[0, 2\pi]$ such that for a.e. $t \in [0, 2\pi]$ and each $(x, y) \in [-\varrho - \eta, \varrho + \eta] \times \mathrm{R}$ the inequality

$$|h_\mu(t, x, y)| \le \widetilde{p}(t) + \widetilde{q}(t)|y|$$

is fulfilled. Then by Theorem 3.2 we obtain

$$d[I - H_\mu, \widetilde{\Omega}] = 1, \; d[I - H_\mu, \Delta_1] = 1 \text{ and } d[I - H_\mu, \Delta_2] = 1,$$

where

$$(H_\mu x)(t) = x(0) + x'(0) - x'(2\pi)$$
$$+ \int_0^{2\pi} g_\mu(t,s)\big[h_\mu(s,x(s),x'(s)) - \big(x(0)+x'(0)-x'(2\pi)\big)\mu\big]\,ds$$
$$+ \widetilde{g}_\mu(t,t_1)[\widetilde{J}(x(t_1)) - x(t_1)] + g_\mu(t,t_1)[M(x'(t_1)) - x'(t_1)].$$

Denote

$$\Delta = \widetilde{\Omega}\backslash\overline{(\Delta_1 \cup \Delta_2)}. \tag{3.52}$$

Then, from the additivity of the Leray–Schauder topological degree, we have

$$d[I - H_\mu, \Delta] = -1.$$

Thus there is a solution $u$ of the problem

$$(I - H_\mu)x = 0 \tag{3.53}$$

which for some $t_u \in [0, 2\pi]$ satisfies (3.31). Moreover from (3.39), (3.42), (3.44), (3.46), (3.50) we can see that $u$ is a solution of the equation

$$x'' + a(t)x' + b(t)x = h(t, x, x'),$$

and we have for a.e. $t \in [0, 2\pi]$

$$|u''| = |\overline{h}(t,u,u')| \le |\overline{f}(t,u,u')| + (|a(t)| + q(t))|u'| + 4p(t)$$

$$\le |\widetilde{f}(t,u,u') - a(t)u' + b(t)u| + (|a(t)| + q(t))|u'| + 4p(t) \le 2(|a(t)| + q(t))|u'| + 6p(t),$$

$$\left|\frac{u''u'}{1+u'^2}\right| \le 2(|a(t)| + q(t)) + 6p(t).$$

Integrating this inequality on $(t_u, t)$ we get for each $t \in [0, 2\pi]$

$$1 + u'^2(t) \le (1 + u'^2(t_u))e^{4(\|a\|_1 + \|q\|_1 + 3\|p\|_1)}$$

$$|u'(t)| \le (1 + |u'(t_u)|)e^{2(\|a\|_1 + \|q\|_1 + 3\|p\|_1)} < \widetilde{N}_1.$$

Then we have $\|u'\|_\infty < \widetilde{N}_1$, $\|u\|_\infty < \widetilde{N}_2$ for every solution $u \in \Delta$ of (3.48) and from the excision property of the degree we have

$$d[I - H_\mu, \Delta] = d[I - H_\mu, \Omega_2] = -1$$

Finally, from (3.42)–(3.44) and (3.52) $H_\mu = F_\mu$ for $x \in \overline{\Omega}_2$ follows and so

$$d[I - H_\mu, \Omega_2] = d[I - F_\mu, \Omega_2] = -1. \qquad \square$$

**Corollary 3.7** *Let the assumptions of Theorem 3.6 be satisfied. Then the problem (1.1)–(1.3) has a solution $u$, which fulfills*

$$\sigma_2(t_u) < u(t_u) < \sigma_1(t_u)$$

*for some $t_u \in [0, 2\pi]$.*

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