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FREE LOOP SPACES AND CYCLOHEDRA

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ABSTRACT. In this note we introduce an action of the module of cyclohedra on the free loop space. We will then discuss how this action can be used in an appropriate recognition principle in the same manner as the action of Stasheff's associahedra can be used to recognize based loop spaces. We will also interpret one result of R.L. Cohen as an approximation theorem, in the spirit of Beck and May, for free loop spaces.

Recall that, while a (topological) operad is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of topological spaces with structure operations ('compositions')

$$(1) \qquad \circ_i : \mathcal{P}(k) \times \mathcal{P}(l) \to \mathcal{P}(k+l-1), \ 1 \leq i \leq k,$$

(see [7], [4]), a (right) *module* over the operad $\mathcal{P}$ is a collection $M = \{M(n)\}_{n \geq 0}$ with structure operations (called again 'compositions')

$$(2) \qquad \circ_i : M(k) \times \mathcal{P}(l) \to M(k+l-1), \ 1 \leq i \leq k,$$

satisfying appropriate axioms which are in fact 'linearizations' of that of an operad, see [4, Definition 1.3].

Example 1. Let U and V be topological spaces. The collection

$$\mathcal{E}nd_{U,V} := \{Hom(U^{\times n}, V)\}_{n \geq 0}$$

is a natural right module over the endomorphism operad $\mathcal{E}nd_U = \{Hom(U^{\times n}, U)\}_{n \geq 0}$. The structure operations are given in the expected manner as

$$(3) \quad (f \circ_i g)(u_1, \ldots, u_{k+l-1}) := f(u_1, \ldots, u_{i-1}, g(u_i, \ldots, u_{l-1+i}), u_{l+i}, \ldots, u_{k+l-1}),$$

for $f : U^{\times k} \to V \in \mathcal{E}nd_{U,V}(k)$, $g : U^{\times l} \to U \in \mathcal{E}nd_U(l)$ and $u_1, \ldots, u_{k+l-1} \in U$.

An algebra over the operad $\mathcal{P}$ (also called a $\mathcal{P}$-space) is a topological space U together with Σ_n-equivariant maps $A_n : \mathcal{P}(n) \to Hom(U^{\times n}, U)$, $n \geq 0$, which are compatible with the compositions. Thus algebras are 'representations' of operads. Representations of modules are traces, introduced in [5, Definition 2.6] as follows:

Definition 2. *Let M be a $\mathcal{P}$-module as above and U a $\mathcal{P}$-algebra. An M-trace over U is a topological space V together with Σ_n-equivariant maps $T_n : M(n) \to Hom(U^{\times n}, V)$, $n \geq 0$, which are compatible with compositions (2) and the $\mathcal{P}$-algebra structure on U, that is*

$$T_{k+l-1}(m \circ_i p) = T_k(m) \circ_i A_l(p),$$

for $m \in M(k)$, $p \in \mathcal{P}(l)$, $1 \leq i \leq k$, where $\circ_i$ in the right hand side is as in (3).

One may also say that a trace is a map $T : M \to \mathcal{E}nd_{U,V}$ of collections which is an homomorphism of modules over the homomorphism $A : \mathcal{P} \to \mathcal{E}nd_U$ of operads.

Example 3. Each space U is clearly a tautological algebra over its endomorphism operad. Similarly, each space V is a tautological $\mathcal{E}nd_{U,V}$-trace over U considered as an $\mathcal{E}nd_U$-algebra.

Example 4. Let $\mathcal{C}_1 = \{\mathcal{C}_1(n)\}_{n \geq 0}$ be the little intervals operad (= the little cubes operad in dimension 1), see [2, Example 2.49]. Let $\mathcal{J}_1 = \{\mathcal{J}_1(n)\}_{n \geq 0}$ be the collection of 'little intervals' in the circle $\mathbb{S}^1$, that is, $\mathcal{J}_1(n)$ is the space of all linear imbeddings of n unit intervals to the circle such that the images of interiors are mutually disjoint. By linear imbeddings we of course mean imbeddings $\alpha : [0,1] \to \mathbb{S}^1$ that factor as

$$[0,1] \xrightarrow{\tilde{\alpha}} \mathbb{R} \xrightarrow{\exp} \mathbb{S}^1$$

with some linear increasing map $\tilde{\alpha}$. Therefore, elements of $\mathcal{J}_1(n)$ are maps

$$(4) \qquad\qquad d : \bigsqcup_{i=1}^{n} I_i \to \mathbb{S}^1$$

from the disjoint union of n-copies of the unit interval $I = [0,1]$ to the circle such that the restrictions $\alpha_i := d|_{I_i}$ are, for $1 \leq i \leq n$, linear imbeddings and

$$\alpha_i(int(I_i)) \cap \alpha_j(int(I_j)) = \emptyset,$$

whenever $i \neq j$. Then $\mathcal{J}_1$ is a right $\mathcal{C}_1$-module, the module structure being an obvious generalization of the operadic structure of the little intervals operad.

There is a subspace $F_0(\mathbb{S}^1, n) \subset \mathcal{J}_1(n)$ consisting of *cyclically ordered* sequences of little intervals, introduced by R.L. Cohen in [3, Definition 1.2]. A map d as in (4) belongs to $F_0(\mathbb{S}^1, n)$ if and only if there is a sequence

$$\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$$

of real numbers such that $e^{i\theta_i} \in Im(\alpha_i)$, for each $1 \leq i \leq n$. The space $F_0(\mathbb{S}^1, n)$ carries a natural right action of the cyclic group $\mathbb{Z}_n$ and $\mathcal{J}_1(n)$ can be expressed as the induced representation

$$(5) \qquad\qquad \mathcal{J}_1(n) \cong Ind_{\mathbb{Z}_n}^{\Sigma_n} F_0(\mathbb{S}^1, n),$$

where $\mathbb{Z}_n \subset \Sigma_n$ is interpreted as the subgroup of cyclic permutations. This equation is crucial for the proof of Theorem 12.

Let X be a topological space. Recall that the *free loop space* $\wedge X$ is the space of continuous maps $f : \mathbb{S}^1 \to X$ with compact-open topology. Given a distinguished point $b \in X$, the *based loop space* $\Omega_b X$ or simply ΩX is the subspace of $\wedge X$ of all maps such that $f(*) = b$, where $* := \exp(0)$ is the distinguished point of $\mathbb{S}^1$.
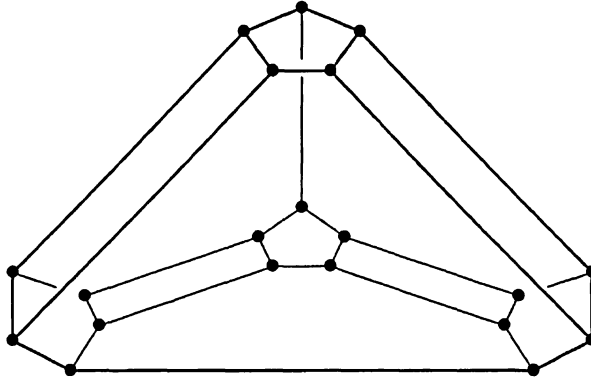
FIGURE 1. The cyclohedron W_4.

Theorem 5. *Consider ΩX as an $\mathcal{C}_1$-algebra, with the classical action of M. Boardman and R. Vogt [2, Example 2.49]. Then $\wedge X$ is a $\mathcal{J}_1$-trace over the $\mathcal{C}_1$-space ΩX.*

Proof .   A $\mathcal{J}_1$-trace on $\wedge X$ is, by Definition 2, given by assigning, to any $d \in \mathcal{J}_1(n)$ and based loops $\phi_1, \ldots, \phi_n \in \Omega X$, a free loop $T(d)(\phi_1, \ldots, \phi_n) \in \wedge X$.

This can be done as follows. Let $d : \bigsqcup_1^n I_n \to \mathbb{S}^1 \in \mathcal{J}_1(n)$ be as in (4) and denote again by α_i the restriction $d|_{I_i}$. Then put $T_n(d)(\phi_1, \ldots, \phi_n) : \mathbb{S}^1 \to X$ to be $\phi_i \circ \alpha_i^{-1}$ on $Im(\alpha_i)$, $1 \le i \le n$, and b on $\mathbb{S}^1 \setminus Im(d)$. It is easy to verify that this indeed defines a trace.   □

Example 6. Let $K = \{K_n\}_{n \ge 0}$ be Stasheff's operad of associahedra [8]. It is well-known that K_n can be interpreted as a compactification of the configuration space of n distinct points in the unit interval I. Let C_n be a similar compactification of the space of n distinct points in the circle $\mathbb{S}^1$. The space C_n decomposes as $\Sigma_{n-1} \times \mathbb{S}^1 \times W_n$, where Σ_{n-1} is understood here as the quotient of Σ_n modulo cyclic permutations. The space W_n is an $n-1$ dimensional convex polytope called, by J. Stasheff [9], the *cyclohedron*. The space W_1 is the point, W_2 is the interval and W_3 is the hexagon. The polyhedron W_4 is depicted in Figure 1.

By a slight abuse of terminology we will call C_n the cyclohedron as well.

It is known [5, Theorem 2.12] that $C = \{C_n\}_{n \ge 0}$ forms a natural right K-module. This structure was analyzed in detail in [5]. Philosophically, the couple (C, K) is a minimal cofibrant model of $(\mathcal{J}_1, \mathcal{C}_1)$ in an appropriate closed model category. Therefore the following theorem is not surprising.

Theorem 7. *Consider ΩX as a K-algebra (= an A_∞-space) as in [8]. Then $\wedge X$ is a C-trace over the K-algebra ΩX.*

Proof .   The trace structure can be constructed in the same manner as the operadic action of the operad K on the based loop space ΩX [8] and it is in fact a consequence of the contractibility of the component of the identity in the space of endomorphisms of the circle that preserve the base point.

A more conceptual proof follows form the above mentioned fact that the couple (C, K) is a cofibrant model of $(\mathcal{J}_1, \mathcal{C}_1)$ which is more or less clear from the description

of the cyclohedron given in [5]. Here 'cofibrant' means that the cyclohedron is build up inductively by attaching cells so the usual obstruction theory for extending maps applies — we do not refer to any fancy closed model category structure.

To be precise, let $\rho : K \to \mathcal{C}_1$ be a weak homotopy equivalence of operads. The existence of such a map is folklore and follows from the cofibrancy of the operad K. The map ρ induces a right K-module structure on $\mathcal{J}_1$. Then it follows from the definition of the cyclohedron and its cellular structure that there exist a weak homotopy equivalence of right K-modules $\lambda : C \to \mathcal{J}_1$.

Let $T : \mathcal{J}_1 \to \mathcal{E}nd_{\Omega X, \wedge X}$ be the trace structure of Theorem 5. Then $T \circ \lambda : C \to \mathcal{E}nd_{\Omega X, \wedge X}$ makes $\wedge X$ a C-trace over the K-algebra ΩX. $\qquad \square$

Example 8. Let us describe explicitly first components of the structure of Theorem 7. The first piece $T_1 : C_1 \times \Omega X \cong \mathbb{S}^1 \times \Omega X \to \wedge X$ of the trace is given by the reparametrization

$$(6) \qquad T_1(\alpha, \phi)(u) := \phi(u + \alpha), \ \alpha \in \mathbb{S}^1 \cong \mathbb{R}/\mathbb{N}, \ u \in [0, 1].$$

To describe a possible choice of the second piece, $T_2 : C_2 \times \Omega X \times \Omega X \to \wedge X$, introduce coordinates $(\alpha, t) \in \mathbb{R}/\mathbb{N} \times I$ of $C_2 \cong \mathbb{S}^1 \times I$. T_2 is then given by

$$T_2(\alpha, t, \phi, \psi)(u) := (\phi * \psi)(u + \tfrac{t}{2} + \alpha), \ u \in [0, 1],$$

where $\phi * \psi$ is the standard product (loop composition) in ΩX. The Σ_2-invariance of T_2 follows from the obvious, but very charming, equation

$$(\phi * \psi)(u + \tfrac{1}{2}) = (\psi * \phi)(u), \ u \in [0, 1],$$

which holds in $\wedge X$ and should be interpreted as a kind of commutativity for the $*$-product.

We believe that an appropriate form of the following *recognition principle* might be true:

Theorem 9. *A couple (A, B) has the homotopy type of $(\wedge X, \Omega X)$ if and only if B is an A_∞-space and A is a C-trace over the A_∞-space B.*

Let $\wedge^b X \subset \wedge X$ be the subspace of paths $f : \mathbb{S}^1 \to X$ passing through the distinguished point b, that is, $b \in Im(f)$. The above recognition principle seems to be related to $\wedge^b X$ rather than to $\wedge X$. Therefore the following, if fact very surprising, proposition shared with me by Saša Voronov, is important.

Proposition 10 (A. Voronov). *Let X be a connected space. Then the inclusion $\wedge^c X \hookrightarrow \wedge X$ is, for each $c \in X$, a weak homotopy equivalence.*

Proof due to A. Voronov. It is easy to see that the map $ev : \wedge^c X \to X$ given by the evaluation at $* \in \mathbb{S}^1$ is a Hurewicz fibration. The fiber $ev^{-1}(b)$ over a point $b \in X$ is the subspace $\Omega^c_b X \subset \Omega_b X$ of loops based at b and passing through c. Observe that $\Omega^c_c X = \Omega_c X$. Consider the following map of fibrations:

$$\begin{array}{ccc}
\Omega_c X = \Omega_c^c X & \longrightarrow & \Omega_c X \\
\downarrow & & \downarrow \\
\wedge^c X & \xrightarrow{\;inclusion\;} & \wedge X \\
ev \downarrow & & \downarrow ev \\
X & \longrightarrow & X
\end{array}$$

The proposition follows from the five lemma applied to the induced map of the long exact homotopy sequences of these fibrations.   □

The following corollary, also due to A. Voronov, immediately follows from the arguments used in the above proof. This result is so nice and surprising that we did not resist the temptation to state it here.

Corollary 11 (A. Voronov). *The space $\Omega_b^c X$ of based loops passing through a fixed $c \in X$ is homotopy equivalent to the ordinary based loop space $\Omega_b X$.*

Proof .   The corollary follows from the fact that $ev : \wedge^b X \to X$ is a Hurewicz fibration, but, once one believes that the statement is true, it is not difficult write homotopy equivalences explicitly.   □

The conceptual meaning of Theorem 9 is the following. While the recognition principle for based loop spaces (see, for example, [2, Theorem 6.27]) says that the path multiplication, suitably axiomatized, recognizes based loop spaces, Theorem 9 says that reparametrization (6), suitably axiomatized, recognizes free loop spaces. The cofibrancy of the couple (C, K) is very important here, without this assumption one would not get an 'iff' statement.

As argued in [6, §4], an *approximation theorem* in the spirit of J. Beck and P. May [1], [7] should describe, for any topological space X, the homotopy type of the free C-trace on the A_∞-space $\Omega S X$ which itself has, by the classical approximation theorem [7, Corollary 6.2], homotopy type of the free A_∞-space on X. Here, as usual, SX denotes the reduced suspension.

Let us inspect free traces more closely. Let $\mathcal{P}$ be an operad and M a right $\mathcal{P}$-module as in Definition 2. For a $\mathcal{P}$-algebra A, the *free M-trace* $T_M(A)$ on A is the quotient of the disjoint union

$$\bigsqcup_{n \geq 0} M(n) \times_{\Sigma_n} A^{\times n}$$

modulo relations

$$M(k) \times_{\Sigma_k} A^{\times k} \ni m \times (x_1, \ldots, x_{i-1}, p(x_i, \ldots, x_{i+l-1}), x_{i+l}, \ldots, x_{k+l-1}) \sim$$
$$\sim m \circ_i p \times (x_1, \ldots, x_{k+l-1}) \in M(k+l-1) \times_{\Sigma_{k+l-1}} A^{\times k+l-1},$$

for $p \in \mathcal{P}(l)$, $m \in M(k)$ and $1 \leq i \leq k$. It is easy to see that $T_M(A)$ is an M-trace over the $\mathcal{P}$-algebra A and that it has the obvious universal property.

In the special case when $A = \mathcal{P}X$ is the free $\mathcal{P}$-algebra on a space X, the above construction can be simplified to a formula which is very close to [7, Construction 2.4]. Namely, $T_M(\mathcal{P}X)$ is the quotient of the disjoint union

$$\bigsqcup_{n \geq 0} M(n) \times_{\Sigma_n} X^{\times n}$$

modulo relations

$$M(n) \times_{\Sigma_n} X^{\times n} \ni m \times (x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) \sim$$
$$\sim m \circ_i * \times (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in M(n-1) \times_{\Sigma_{n-1}} X^{\times n-1},$$

where $b \in X$ and $* \in \mathcal{P}(0)$ are the distinguished points. Let us formulate and prove the approximation theorem. Its proof will be based on a result by R.L. Cohen.

Theorem 12 (Approximation theorem for free loop spaces). *Let X be a connected space. Then there exists the following diagram of weak homotopy equivalences:*

(7)
$$
\begin{array}{ccc}
T_C(KX) & \xrightarrow{\ \sim\ } T_{\vartheta_1}(\mathcal{C}_1 X) & \xrightarrow{\ \sim\ } \wedge SX \\
 & \downarrow \sim & \\
 & T_{\vartheta_1}(\Omega SX) &
\end{array}
$$

Proof .   The vertical homotopy equivalence is induced by the approximation map $\alpha_1 : \mathcal{C}_1 X \to \Omega SX$ constructed in [7, Corollary 6.2].

The left horizontal arrow in (7) is induced by the weak equivalences $\rho : K \to \mathcal{C}_1$ and $\lambda : C \to \vartheta_1$ introduced in the proof of Theorem 7. The existence of the right horizontal arrow follows from the following identification of $T_{\vartheta_1}(\mathcal{C}_1 X)$ with the space $L(X)$ constructed by R.L. Cohen in [3, Definition 1.3].

It follows from the description of the free trace on the free algebra given above that $T_{\vartheta_1}(\mathcal{C}_1 X)$ is the quotient of the disjoint union

(8)
$$\bigsqcup_{n \geq 0} \vartheta_1(n) \times_{\Sigma_n} X^{\times n}$$

modulo the relation

$$\vartheta_1(n) \times_{\Sigma_n} X^{\times n} \ni (d_1, \ldots, d_n) \times_{\Sigma_n} (x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) \sim$$
$$\sim (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) \times_{\Sigma_n} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \vartheta_1(n) \times_{\Sigma_{n-1}} X^{\times n-1},$$

It follows from representation (5) that we may replace (8) by the disjoint union

$$\bigsqcup_{n \geq 0} F_0(\mathbb{S}^1, n) \times_{\mathbb{Z}_n} X^{\times n}$$

and modify the relations accordingly. We immediately recognize the space $L(X)$ of [3, Definition 1.3]. The right horizontal arrow of (7) is then identified with the map $h : L(X) \to \wedge SX$ of [3, Theorem 1.5]. This finishes the proof. $\square$

As proved by Jim Stasheff [8], the action of the associahedra induces an A_∞-structure on the chain complex $C_*(\Omega X)$ of the based loop space. Similarly, the action of the cyclohedra on the free loop space induces a structure of an (algebraic) strongly homotopy trace on $C_*(\wedge X)$. Axioms of this structure are given more or less explicitly in [5].

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