Ryszard Holubowicz; Witold Mozgawa

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ON COMPACT NON-KÄHLERIAN MANIFOLDS ADMITTING AN ALMOST KÄHLER STRUCTURE

RYSZARD HOŁUBOWICZ, WITOLD MOZGAWA

ABSTRACT. In this paper we construct an infinite family of non-diffeomorphic, $(2r+2)$-dimensional, non-Kähler and compact manifolds admitting an almost Kähler structure.
KEYWORDS. almost Kähler structure, Kähler structure, almost Hermitian structure

1. Introduction. Let M be a $2n$-dimensional differentiable manifold and let $\mathfrak{X}(M)$ denote the set of all differentiable vector fields on M. We say that M admits an *almost Hermitian structure* if there exist a Riemannian metric g and a tensor J of type $(1,1)$ such that

(1) J is an almost complex structure on M, i.e.

$$J(J(X)) = -X \quad \text{for each} \quad X \in \mathfrak{X}(M)$$

(2) the metric g is J-invariant, i.e.

$$g(JX, JX) = g(X, Y) \quad \text{for each} \quad X, Y \in \mathfrak{X}(M)$$

A form F defined by

$$F(X, Y) = g(X, JY) \quad \text{for each} \quad X \in \mathfrak{X}(M)$$

is said to be the fundamental 2-form of the almost Hermitian structure (J, g).

Definition 1.1. *An almost Hermitian structure (J, g) is said called an almost Kähler structure if its fundamental form is closed , that is $dF = 0$.*

Let us now define the Nijenhuis tensor $\{J, J\}$ of J by

$$\{J, J\}(X, Y) = [JX, JY] - [X, Y] + J[JX, Y] - J[X, JY] \quad \text{for each} \quad X, Y \in \mathfrak{X}(M)$$

It is well known (cf. [KN]) that an almost complex structure is integrable if and only iff the Nijenhuis tensor $\{J, J\}$ of J vanishes identically.

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Definition 1.2. *An almost Kähler structure (J, g) on M is said to be Kählerian structure if the tensor J is integrable.*

The existence of a Kählerian structure on a manifold M imposes very restrictive conditions on the topology of M, as the following theorem shows

Theorem 1.1. *(cf. [G]) If M is a Kählerian manifold then its odd Betti numbers are even, i.e.*

$$b_{2i-1}(M) = 2p_i,$$

where $i = 1, \dots, \frac{1}{2}\dim M$.

It seems natural to pose the following problem. How to find a family of non-diffeomorphic, compact, almost Kähler manifolds with odd first Betti number? In next sections, using the so called toral bundles (cf. [HM]) we solve this problem, i.e. we construct a countable family of compact, almost Kähler manifolds with odd first Betti number. In particular, for an arbitrary natural number n, we obtain a four dimensional compact, almost Kähler manifold M_n which is not Kählerian. Moreover the manifolds M_n and M_k are not diffeomorphic if $n \neq k$. The basic idea of our constructions is a modification of that given in the Ph.D. thesis of the first author.

1. The toral bundles. Let us fix a natural number $r \geq 2$. For $i = 1, \dots, r$ and $d \in \mathbb{Z}$ $A_i(d)$ denotes a matrix

$$A_i(d) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ d & \dots & \dots & 1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}, \tag{1.1}$$

where the number d begins the (i+1)-th row. The remaining entries of this matrix except for the main diagonal and the number d are equal to zero. Let us observe that

$$[A_i(d)]^t = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ td & \dots & \dots & 1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}, \tag{1.2}$$

for $t \in \mathbb{R}$. Choosing arbitrary integers $d_1, d_2, \dots, d_r$ we get the family of matrices of the shape (1.1) which are pairwise commuting. Each of the matrices $A_i(d_i)$, $i = 1, \dots, r$ can be considered as a diffeomorphism of the $(r+1)$-dimensional torus $T^{r+1} = \mathbb{R}^{r+1}/\mathbb{Z}^{r+1}$ onto itself. We denote by $[\,] : \mathbb{R}^{r+1} \to T^{r+1}, x \mapsto [x]$ the canonical projection and we put $A_i(d_i)[x] = [A_i(d_i)(x)]$. The relationships:

$$(t_1, \dots, t_i, \dots, t_{r+1}, [x]) \sim_i (t_1, \dots, t_i + 1, \dots, t_{r+1}, A_i(d_i)[x]),$$

where $i = 1, \ldots, r+1$ and $(t_1, \ldots, t_{r+1}) \in \mathbb{R}^{r+1}, x \in \mathbb{R}^{r+1}$ give an equivalence relation $\sim$ in the product $\mathbb{R}^{r+1} \times T^{r+1}$. In the standard manner the product $\mathbb{R}^{r+1} \times T^{r+1}/\sim$ is furnished with the structure of a real (2r+2)-dimensional orientable manifold. We also obtain a fibre bundle with a typical fibre T^{r+1} over the base T^{r+1}. This manifold is denoted by $T^{2r+2}_{I,A_1(d_1),\ldots,A_r(d_r)}$ and called a toral bundle of type $(r+1, r+1)$ (cf. [HM]).

We define multiplication in $\mathbb{R}^{r+1}$ as follows

$$(t, x_1, \ldots, x_r, y, z_1, \ldots, z_r) * (t', x'_1, \ldots, x'_r, y', z'_1, \ldots, z'_r) =$$
$$= (t + t', x_1 + x'_1, \ldots, A_1^{x_1} \ldots A_r^{x_r}(y', z'_1, \ldots, z'_r) + (y, z_1, \ldots, z_r)).$$

The pair $(\mathbb{R}^{2r+2}, *)$ forms a Lie group, denoted by $\mathbb{G}^{r+1,r+1}_{d_1,\ldots,d_2}$. For a uniform discrete subgroup $\Gamma = \{(t, x_1, \ldots, x_r, y, z_1, \ldots, z_r) : x_1, \ldots, x_r, y, z_1, \ldots, z_r \in \mathbb{Z}\}$ one can see that the compact orientable manifold $T^{2r+2}_{I,A_1(d_1),\ldots,A_r(d_r)}$ and the homogeneous space $\Gamma \setminus \mathbb{G}^{r+1,r+1}_{d_1,\ldots,d_2}$ are diffeomorphic (cf. [HM], [H]).

Since the fundamental group $\pi(T^{2r+2}_{I,A_1(d_1),\ldots,A_r(d_r)})$ of the toral bundle

$$T^{2r+2}_{I,A_1(d_1),\ldots,A_r(d_r)} = \Gamma \setminus \mathbb{G}^{r+1,r+1}_{d_1,\ldots,d_2}$$

is isomorphic with the group Γ (cf. [HM], [H], [C]) then by the Hurewicz theorem (cf. [BT] we get

$$H_1(T^{2r+2}_{I,A_1(d_1),\ldots,A_r(d_r)}, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma],$$

where $[\Gamma, \Gamma]$ denotes the commutator subgroup of Γ. Note that the group Γ has $2r+2$ generators $e, a_i, b, c_i, \quad i = 1, \ldots, r$ acting on $\mathbb{R}^{2r+2}$ as follows:

$$e : (t, x_1, \ldots, x_r, y, z_1, \ldots, z_r) \mapsto (t+1, x_1, \ldots, x_r, y, z_1, \ldots, z_r)$$
$$a_i : (t, x_1, \ldots, x_r, y, z_1, \ldots, z_r) \mapsto (t, x_1, \ldots, x_i + 1, \ldots, x_r, y, z_1, \ldots, z_i + d_i y, \ldots, z_r)$$
$$b : (t, x_1, \ldots, x_r, y, z_1, \ldots, z_r) \mapsto (t, x_1, \ldots, x_r, y+1, z_1, \ldots, z_r)$$
$$c_i : (t, x_1, \ldots, x_r, y, z_1, \ldots, z_r) \mapsto (t, x_1, \ldots, x_r, y, z_1, \ldots, z_i + 1, \ldots, z_r).$$

Therefore, we have the following relations

$$ea_i = a_i e, \quad eb = be, \quad ec_i = c_i e, \quad a_i c_j = c_j a_i, \quad bc_i = c_i b$$

and

$$a_i b = b a_i c_i^{d_i} \quad \text{for} \quad i, j = 1, \ldots, r.$$

The abelianization $[\Gamma, \Gamma]$ of the group Γ is a group which is isomorphic with a direct sum $\mathbb{Z}^{r+2} \oplus \mathbb{D}$, where $\mathbb{D}$ is a group generated by the elements $c_1, \ldots, c_r$ with the relations $c_i^{r_i} = 0$ for $i = 1, \ldots, r$. Thus we have prove the following

Theorem 2.1. $H_1(T^{2r+2}_{I,A_1(d_1),\ldots,A_r(d_r)}, \mathbb{Z}) = \mathbb{Z}^{r+2} \oplus \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_r}$.

Directly from this theorem we obtain

Corollary 2.1. *The first Betti number of the manifold* $T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}$ *is given by* $b_1(T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}) = r+2$.

In virtue of Theorem 1.1 we get

Theorem 2.2. *If r is odd then* $T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}$ *is a compact, non-Kählerian manifold.*

3. Some almost Kähler structure on $T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}$**.** In this section we give explicite an almost Kähler structure on $T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}$, which is a solution of the problem stated in Introduction.

It is easy to observe that the forms

$$\widetilde{\theta} = dt, \quad \widetilde{\alpha}_i = dx_i, \quad \widetilde{\beta} = dy, \quad \widetilde{\gamma}_i = dz_i - d_i x_i dy, \quad i = 1, \dots, r$$

create a basis for left-invariant 1-forms on $\mathbb{G}^{r+1,r+1}_{d_1,\dots,d_2}$, whereas the vector fields

$$\widetilde{X}_i = \frac{\partial}{\partial x_i}, \quad \widetilde{Y} = \frac{\partial}{\partial y} + \sum_{i=1}^{r} d_i x_i \frac{\partial}{\partial z_i}, \quad \widetilde{Z}_i = \frac{\partial}{\partial z_i}, \quad \widetilde{T} = \frac{\partial}{\partial t}, \quad i = 1, \dots, r$$

create a basis for left-invariant vector fields on $\mathbb{G}^{r+1,r+1}_{d_1,\dots,d_2}$. These forms and vector fields are Γ-invariant and give at the same time globally defined, linearly independent 1-forms $\theta, \alpha_i, \beta, \gamma_i$ and vector fields X_i, Y, Z_i, T on $T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}$, where $i = 1, \dots, r$. Putting

$$JX_i = Z_i, \quad JZ_i = -X_i, \quad JY = T, \quad JT = -Y$$

and extending these formulas by linearity we get an almost complex structure on $T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}$, such that the Riemannian metric

$$g = \sum_{i=1}^{r} (\alpha_i^2 + \gamma_i^2) + \beta^2 + \theta^2$$

is J-invariant. Since the fundamental 2-form of this structure (J, g)

$$F = \beta \wedge \theta + \sum_{i=1}^{r} \alpha_i \wedge \gamma_i$$

is closed then we obtain the following

Theorem 3.1. *If r is odd then the manifold* $T^{2r+2}_{I,A_1(d_1),\dots,A_r(d_r)}$ *is a compact, almost Kähler which does not admit any Kähler structure.*

In particular, if we put

$$B(n) = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

then four-dimensional toral bundles $T^4_{I,B(n)}$ are compact, almost Kähler, but non-Kähler manifold. As $H_1(T^4_{I,B(n)}, \mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_n$ we have

Theorem 3.2. *There exists a countable family four-dimensional non-diffeomorphic, compact, almost Kähler, non-Kähler manifolds.*

Remark 3.1. The manifold $T^4_{I,B(1)}$ is the well known Thurston example (cf. [T]).

References

[A] Abbena E., *An example of an almost Kähler manifold which is not Kählerian*, Boll. U. M. I. **6** (1984), 383–392.

[BT] Bott R., Tu L.W., *Differential forms in algebraic topology*, Springer-Verlag, New York Heidelberg Berlin, 1982.

[C] Carrière Y., *Flots riemanniens*, Asterisque **116** (1984), 31–52.

[CFL] Cordero L., Fernandez M., de Léon M., *Examples of compact non-Kähler almost Kähler manifolds*, Proc.Amer.Math.Soc. (1985), 280–286.

[CFL2] Cordero L., Fernandez M., de Léon M., *Some examples of compact coomplex manifolds with Norden metric*, Hommage to Professor Dr. Nacere Hayek Calil, La Laguna, Universidad la Laguna, 1990, pp. 85–94.

[G] Goldberg S., *Curvature and homology*, Academic Press, New York and London, 1962.

[H] Hołubowicz R., *Riemannian foliations and geometric structures on toral bundles (in Polish)*, Ph.D. Thesis, Łódź, 1987.

[HM] Hołubowicz R., Mozgawa W., *Non-isometric transversally parallelizable foliations on four-dimensional toral bundle*, Bull. Soc. Sci. Lettr. Łódź **XXXVIII,4** (1988), 1-8.

[KN] Kobayashi S., Nomizu K., *Foundation of differential geometry*, vol. 2, John Wiley & Sons, Inc., New York, 1969.

[T] Thurston W.P., *Some simple examples of symplectic manifolds*, Proc.Amer.Math.Soc. **55** (1976), 467–468.

[W] Watson B., *New examples of strictlyalmost Kähler manifolds*, Proc.Amer.Math.Soc. **88** (1983), 541–544.

Institute of Mathematics
Maria Curie–Skłodowska University
pl. M. Curie–Skłodowskiej 1
20–031 Lublin
Poland
gambit@golem.umcs.lublin.pl
mozgawa@golem.umcs.lublin.pl