

Yan S. Soibelman

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QUANTUM WEYL GROUP AND SOME ITS APPLICATIONS

Ya.S.Soibelman

1. This paper contains some results from author's talk at the school "Geometry and Physics" (January, 1990).  
 2. Let  $\mathcal{G}$  be a simple complex Lie algebra. Let us fix an invariant scalar product  $(,)$  on  $\mathcal{L}^*$  where  $\mathcal{L} \subset \mathcal{G}$  is a Cartan subalgebra. We choose a basis of simple roots  $\{\alpha_i\}_{i=1}^l$  such that  $(\alpha_i, \alpha_j) \in \mathbb{Q}$ . Let  $U_h(\mathcal{G})$  be a quantized universal enveloping algebra. I recall that  $U_h(\mathcal{G})$  is a topological Hopf algebra over  $\mathbb{C}[[\hbar]]$ . It contains 1 and generated by  $\{X_i^\pm, H_i\}_{i=1}^l$  and relations:

$$[H_i, X_j^\pm] = \pm(\alpha_i, \alpha_j) X_j^\pm, \quad [X_i^+, X_j^-] = \delta_{ij} \frac{\text{sh}(\frac{\hbar}{2} H_i)}{\text{sh}(\frac{\hbar}{2})},$$

$$\sum_{k=0}^{1-\alpha_{ij}^-} (-1)^k \binom{1-\alpha_{ij}^-}{k} q_i^{-k} q_i^{-k(1-\alpha_{ij}^-)} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-\alpha_{ij}^- - k} \quad (1)$$

where  $((\alpha_{ij}^-))$  is a Cartan matrix for  $\mathcal{G}$ ,  $q_i = e^{\frac{\hbar}{2}(\alpha_i, \alpha_i)}$ ,  $(n)_\pm = \frac{q^\pm - 1}{q^\pm - 1}$ ,  $(n)_\pm! = (1)_\pm \dots (n)_\pm$ .

$U_h(\mathcal{G})$  is a Hopf algebra with comultiplication  $\Delta$  defined by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^\pm) = X_i^\pm \otimes e^{\pm \frac{\hbar}{4} H_i} + e^{\mp \frac{\hbar}{4} H_i} \otimes X_i^\pm. \quad (2)$$

3. Let  $G$  be a simply connected simple Lie group such that  $\text{Lie } G = \mathcal{G}$ . Hopf algebra  $\mathbb{C}[G]_h$  of regular functions on quantum group  $G$  was introduced in [1] (see also [4-6]). It is the algebra of matrix elements of some finite dimensional representations of  $U_h(\mathcal{G})$ . Following [4-6] we introduce structure of Hopf  $\ast$ -algebra on  $\mathbb{C}[G]_h$  such that

$$\ell^\ast(a) = \overline{\ell(S(a)^\ast)} \quad (3)$$

where  $\ell \in \mathbb{C}[G]_h$ ,  $a \in U_h(\mathcal{G})$ ,  $S$  is antipode,  $\ast$  is an antilinear anti-

automorphism of algebra  $U_n(\mathcal{G})$  defined by  $H_i^* = H_i, (X_i^\pm)^* = X_i^\mp$ .  
 4. It is known ([4,6]) that for every Schubert cell  $X_w \subset \mathcal{G}/B$  ( $\mathcal{G}/B$  is flag manifold) there is the irreducible  $*$ -representation  $\pi_w$  of  $\mathcal{C}[\mathcal{G}]_h$  in Hilbert space. Here  $w$  is an arbitrary element of Weyl group  $W$ . Then we can define ([7] for  $\mathcal{G} = SL_2(\mathbb{C})$  and [4,5] for general case) quantum Weyl group. I recall that for every  $w \in W$  one can construct some Gelfand-Naimark-Segal state  $\bar{w} \in \mathcal{C}[\mathcal{G}]_h^*$  ([4,5]). Let  $S_i$  be a simple reflection which corresponds to  $i$ -th vertex of Dynkin diagram.

THEOREM 1 ([4]) a) If  $i \neq j$  then  $\underbrace{S_i S_j S_i \dots}_{m_{ij}} = \underbrace{S_j S_i S_j \dots}_{m_{ij}}$

where  $(S_i S_j)^{m_{ij}} = 1$  in usual Weyl group.

b)  $e^{-\frac{1}{2} \sum_{i \in I} H_i^2} S_i^2$  commutes with algebra  $U_h(\mathcal{S}\ell(2)_i)$  generated by  $X_i^\pm, H_i$ .

Let  $w_0$  be the element of maximal length,  $R \in U_h(\mathcal{G})^{\otimes 2}$  be the universal quantum R-matrix ([1], § 13). Let us choose an orthonormal basis  $\{I_\kappa\}$  in  $\mathcal{G}$  and let  $\bar{w}_i' = e^{-\frac{1}{4} \sum_{\kappa} I_\kappa^2} \bar{w}_i$ .

THEOREM 2.

$$\Delta(\bar{w}_i') = R^{-1} (\bar{w}_i' \otimes \bar{w}_i')$$

In the case  $\mathcal{G} = \mathcal{S}\ell(n)$  theorem 2 is proved in [5].

Let  $w_0 = S_{i_1} \dots S_{i_N}$  be a reduced expansions. It is known that set  $\mathcal{D} = \{d_1, \dots, S_{i_1} \dots S_{i_{p-1}} d_p\}$  coincides with the set  $\Delta_+$  of positive roots. Then we define a total order on  $\Delta_+ = \mathcal{D}$  (read  $\mathcal{D}$  from right to left). Let  $\alpha = S_{i_1} S_{i_2} \dots S_{i_{p-1}} (\alpha_{i_p})$ . We define

$$E_\alpha, F_\alpha \in U_h(\mathcal{G}), E_\alpha = \pi_{i_1} \dots \pi_{i_{p-1}} (E_{i_p}), F_\alpha = \pi_{i_1} \dots \pi_{i_{p-1}} (F_{i_p}),$$

$$E_i = X_i^+ \exp(-\frac{1}{2} H_i), F_i = X_i^- \exp(\frac{1}{2} H_i),$$

$$\pi_i(x) = \bar{S}_i x \bar{S}_i^{-1}.$$

THEOREM 3.

$$R = \prod_{\alpha \in \Delta_+} \exp_{q_\alpha} ( (1 - q_\alpha^{-2}) E_\alpha \otimes F_\alpha ) e^{\frac{1}{2} \sum_{\kappa} I_\kappa \otimes I_\kappa}$$

where  $q_\alpha = e^{\frac{1}{4} \langle \alpha, \alpha \rangle}$ ,  $\exp_\epsilon(x) = \sum_{n \geq 0} \frac{x^n}{(n)_\epsilon!}$

and product is taken according to our order.

In the case  $\mathcal{G} = \mathcal{S}\ell(n)$  theorem 3 is proved in [3].

Another application of quantum Weyl group is connected with Hecke algebras. Let  $\mathcal{G}$  be a simply-laced Lie algebra such that  $(\alpha_i, \alpha_i) = 2$ . I recall that Hecke algebra  $H_\mathcal{G}(W)$  has

generators  $\{\mathbb{T}_i\}_{i=1}^2$  with relations:

$$\underbrace{\mathbb{T}_i \mathbb{T}_j \mathbb{T}_i \dots}_{m_{ij}} = \underbrace{\mathbb{T}_j \mathbb{T}_i \mathbb{T}_j \dots}_{m_{ij}}, \quad (\mathbb{T}_i - q^{-2})(\mathbb{T}_i + 1) = 0$$

We consider a simple  $U_h(\mathfrak{g})$ -module  $L(\Lambda)$  with the highest weight  $\Lambda$  such that  $\Lambda(H_i) = (\alpha_{max}, \alpha_i)$  where  $\alpha_{max}$  is a maximal root. Therefore  $L(\Lambda)$  is a quantum analogue of the adjoint representations of  $\mathfrak{g}$ . Let  $L(\Lambda)_0 = \{x \in L(\Lambda) \mid \alpha x = 0 \ \forall \alpha \in \mathfrak{f}\}$ .

THEOREM 4. For every  $i \in \{1, 2\}$  we have:

- a)  $\bar{S}_i(L(\Lambda)_0) \subset L(\Lambda)_0$ ;
- b)  $(\bar{S}_i - q^{-2})(\bar{S}_i + 1)|_{L(\Lambda)_0} = 0$

where  $q = e^{h/2}$ .

Therefore we can say that quantum Weyl group acts on  $L(\Lambda)_0$  as Hecke algebra. This result is also obtained by G. Lusztig ([2]).

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AUTHOR: Ya.S. SOIBELMAN, CHAIR OF ALGEBRA, MECH.-MATHEMATICAL DEPARTMENT, ROSTOV UNIVERSITY, ROSTOV-ON-DON, ENGELSA 105, 344006, USSR.