Shinji Kawaguchi; Ryoken Sokei

Some relative properties on normality and paracompactness, and their absolute embeddings

Commentationes Mathematicae Universitatis Carolinae, Vol. 46 (2005), No. 3, 475--495

Persistent URL: http://dml.cz/dmlcz/119542

Some relative properties on normality and paracompactness, and their absolute embeddings

Shinji Kawaguchi*, Ryoken Sokei

Dedicated to Professor Takao Hoshina on his 60th birthday.

Abstract. Paracompactness (= 2-paracompactness) and normality of a subspace Y in a space X defined by Arhangel'skii and Genedi [4] are fundamental in the study of relative topological properties ([2], [3]). These notions have been investigated by primary using of the notion of weak C- or weak P-embeddings, which are extension properties of functions defined in [2] or [18]. In fact, Bella and Yaschenko [8] characterized Tychonoff spaces which are normal in every larger Tychonoff space, and this result is essentially implied by their previous result in [8] on a corresponding case of weak C-embeddings. In this paper, we introduce notions of 1-normality and 1-collectionwise normality of a subspace Y in a space X, which are closely related to 1-paracompactness of Y in X. Furthermore, notions of quasi-C^*- and quasi-P-embeddings are newly defined. Concerning the result of Bella and Yaschenko above, by characterizing absolute cases of quasi-C^*- and quasi-P-embeddings, we obtain the following result: a Tychonoff space Y is 1-normal (or equivalently, 1-collectionwise normal) in every larger Tychonoff space if and only if Y is normal and almost compact. As another concern, we also prove that a Tychonoff (respectively, regular, Hausdorff) space Y is 1-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space if and only if Y is compact. Finally, we construct a Tychonoff space X and a subspace Y such that Y is 1-paracompact in X but not 1-subparacompact in X. This is a negative answer to a question of Qu and Yasui in [25].

Keywords: 1-paracompactness of Y in X, 2-paracompactness of Y in X, 1-collectionwise normality of Y in X, 2-collectionwise normality of Y in X, 1-normality of Y in X, 2-normality of Y in X, quasi-P-embedding, quasi-C-embedding, quasi-C^*-embedding, 1-metacompactness of Y in X, 1-subparacompactness of Y in X

Classification: Primary 54B10; Secondary 54B05, 54C20, 54C45, 54D15, 54D20

1. Introduction

Throughout this paper all spaces are assumed to be T_1 and the symbol γ denotes an infinite cardinal.

As central notions in the study of relative topological properties which has been posed by Arhangel'skii and Genedi [4], and also in the subsequent articles [2], [3] by Arhangel'skii, we can mention those on relative normality and relative paracompactness.

*Corresponding author.

Let X be a space and Y a subspace of X. A subspace Y is said to be *normal* (respectively, *strongly normal*) *in* X if for each disjoint closed subsets F_0, F_1 of X (respectively, of Y), there exist disjoint open subsets G_0, G_1 of X such that $F_i \cap Y \subset G_i$ for $i = 0, 1$. A subspace Y is said to be 1- (respectively, 2-) *paracompact in* X if for every open cover $\mathcal{U}$ of X, there exists a collection $\mathcal{V}$ of open subsets of X with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y. Here, $\mathcal{V}$ is said to be a *partial refinement* of $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing V. The term "2-paracompact" is often shortened to "paracompact". In the definition of 2-paracompactness of Y in X above, when we replace "open cover of X" by "collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$", Y is said to be *Aull-paracompact in* X ([3], [5]). The 1-paracompactness and Aull-paracompactness of Y in X need not imply each other ([5]), but each of them clearly implies 2-paracompactness of Y in X.

On the other hand, Hoshina and Yamazaki ([18]) introduced two relative notions called *collectionwise normality* and *strong collectionwise normality of* Y *in* X (see Section 2 for the definition), which imply normality and strong normality of Y in X, respectively. It is known that in case X is regular (more strictly, Y is strongly regular in X ([2])), 2-paracompactness of Y in X implies normality of Y in X ([4]); in fact, 2-paracompactness of Y in X implies collectionwise normality of Y in X (see Section 2). It will be also shown that in case Y is Hausdorff in X, Aull-paracompactness of Y in X implies strong collectionwise normality of Y in X (Lemma 2.6).

In view of these results, it is suggested to define suitable notions on relative normality and relative collectionwise normality which are closely related to 1-paracompactness. For this purpose, in Section 2 we define the notions of 1-normality and 1-collectionwise normality of Y in X, and prove several results including relations between these notions and those mentioned above. In particular, we show that in case of Y being strongly regular in X, 1- (respectively, 2-) paracompactness of Y in X implies 1- (respectively, 2-) collectionwise normality of Y in X (Proposition 2.5).

In Section 3, we are concerned to describe spaces that are 1-normal or 1-collectionwise normal in every larger space (that is, in every space containing Y as a subspace). Bella and Yaschenko [8] and Matveev et al. [24] proved a related theorem, where a Tychonoff space Y being normal in every larger Tychonoff (or every regular) space was characterized (Theorem 2.10). This result was implied by another theorem in [8] which describes a Tychonoff space Y being weakly C-embedded in every larger Tychonoff space (Theorem 2.8), where weak C-embedding is due to Arhangel'skii [2]. In [18], a characterization of weak C-embedding was given by extending collections of subsets.

Being motivated by this result, we define a new extension property called *quasi-C^*-embedding* for a subspace Y of a space X. We characterize Tychonoff spaces Y

which are quasi-C^*-embedded in every larger Tychonoff space (Theorem 3.11). By virtue of this result, we establish a theorem that a Tychonoff space Y is 1-normal in every larger Tychonoff space if and only if Y is normal and almost compact. We also introduce "quasi-P-embedding" and give a similar characterization for 1-collectionwise normality of Y in X (Corollary 3.12).

In Section 4, we consider 1-metacompactness defined by Kočinac [20]. Extending [16, Corollary 27] essentially, we obtain the following theorems; a Tychonoff (respectively, regular, Hausdorff) space Y is 1-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space X if and only if Y is compact (Theorems 4.2 and 4.3).

In the final section, on 1-subparacompactness defined by Qu and Yasui [25], we obtain a Tychonoff space X and a subspace Y such that Y is 1-paracompact, but not 1-subparacompact in X. This is a negative answer to a question in [25].

Other undefined notations and terminology are used as in [12].

2. Preliminaries and 1- or 2- (collectionwise) normality of a subspace in a space

Throughout this paper symbols $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{I}$ denote the set of real numbers, the set of natural numbers and the closed unit interval, respectively.

First let us recall some preliminary notions and facts. Let Y be a subspace of a space X. As is known, Y is said to be C^*- (respectively, C-) *embedded in* X if every bounded real-valued (respectively, real-valued) continuous function on Y is continuously extended over X. A subspace Y is said to be P^γ- (respectively, P-) *embedded in* X if every continuous γ-separable (respectively, continuous) pseudo-metric on Y is extended to a continuous pseudo-metric on X ([1]); a pseudo-metric d on Y is γ-*separable* if the pseudo-metric space (Y, d) has weight $\leq \gamma$. It is known that P^ω-embedding is equal to C-embedding ([1]).

By [2], Y is said to be *weakly C-embedded* in X if for every real-valued continuous function f on Y there exists a real-valued function on X which is an extension of f and continuous at each point of Y. By [18], Y is said to be *weakly* P^γ- (respectively, *weakly* P-) embedded in X if every continuous γ-separable (respectively, continuous) pseudo-metric on Y is extended to a pseudo-metric on X which is continuous at each point of $Y \times Y$. Weak P^ω-embedding is equal to weak C-embedding ([18]). A space X is γ-*collectionwise normal* if for every discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of closed subsets there exists a pairwise disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Clearly, X is collectionwise normal if X is γ-collectionwise normal for every γ.

A subspace Y is said to be *Hausdorff in* X if for every two distinct points y_1, y_2 of Y, there are disjoint open subsets U_1, U_2 of X such that $y_i \in U_i$ for $i = 0, 1$. A subspace Y is said to be *strongly regular in* X if for each $x \in X$ and each closed subset F of X with $x \notin F$, there exist disjoint open subsets U, V of X such that $x \in U$ and $F \cap Y \subset V$.

Following [18], Y is said to be *γ-collectionwise normal* (respectively, *strongly γ-collectionwise normal*) *in* X if for every discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of closed subsets of X (respectively, Y), there is a pairwise disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $E_\alpha \cap Y \subset U_\alpha$ (respectively, $E_\alpha \subset U_\alpha$) for every $\alpha < \gamma$. As is easily seen, in case $\gamma = \omega$ it is equivalent to say that Y is normal (respectively, strongly normal) in X. When Y is γ-collectionwise normal (respectively, strongly γ-collectionwise normal) in X for every γ, we say Y is *collectionwise normal* (respectively, *strongly collectionwise normal*) *in* X; we see that collectionwise normality (respectively, strongly collectionwise normality) of Y in X is equal to being $\alpha - CN$ (respectively, $\gamma - CN$) of Y in the sense of Aull [7].

Let X_Y denote the space obtained from the space X, with the topology generated by a subbase $\{U \mid U$ is open in X or $U \subset X \setminus Y\}$. Hence, points in $X \setminus Y$ are isolated and Y is closed in X_Y. Moreover, X and X_Y generate the same topology on Y ([12]). As is seen in [2], the space X_Y is often useful in discussing several relative topological properties. It is easy to see that Y is Hausdorff in X if and only if X_Y is Hausdorff. The following results given in [2], [18] are fundamental in the present paper; (a)$\Leftrightarrow$(c)$\Leftrightarrow$(e) in Lemma 2.1 have been already shown in [2].

Lemma 2.1 ([2], [18]). *For a subspace Y of a space X the following statements are equivalent.*

(a) *Y is strongly normal in X.*
(b) *Y is normal in G for every open subset G of X with $Y \subset G$.*
(c) *X_Y is normal.*
(d) *Y is normal in X_Y.*
(e) *Y is normal itself and weakly C-embedded in X.*

Lemma 2.2 ([18]). *For a subspace Y of a space X the following statements are equivalent.*

(a) *Y is strongly γ-collectionwise normal in X.*
(b) *Y is γ-collectionwise normal in G for every open subset G of X with $Y \subset G$.*
(c) *X_Y is γ-collectionwise normal.*
(d) *Y is γ-collectionwise normal in X_Y.*
(e) *Y is γ-collectionwise normal itself and weakly P^γ-embedded in X.*

We now introduce notions of 1- or 2- (collectionwise) normality of Y in X. We say that a subspace Y of a space X is 1- (respectively, 2-) *normal in* X if for each disjoint closed subsets F_0, F_1 of X there exist open subsets G_0, G_1 of X such that $F_i \cap Y \subset G_i$ for $i = 0, 1$ and $\{G_0, G_1\}$ is discrete in X (i.e. $\overline{G_0} \cap \overline{G_1} = \emptyset$) (respectively, discrete at each point of Y in X (i.e. $\overline{G_0} \cap \overline{G_1} \cap Y = \emptyset$)).

A subspace Y of a space X is 1-γ- (respectively, 2-γ-) *collectionwise normal in* X if for each discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each

$\alpha < \gamma$ and $\{G_\alpha \,|\, \alpha < \gamma\}$ is discrete in X (respectively, discrete at each point of Y in X). If Y is 1- (respectively, 2-) γ-collectionwise normal in X for every γ, Y is said to be 1- (respectively, 2-) *collectionwise normal in* X.

In the above definitions of 2-normality and 2-γ-collectionwise normality of Y in X, it is easy to see that both $\{G_1, G_2\}$ and $\{G_\alpha \,|\, \alpha < \gamma\}$ can be taken to be disjoint. Therefore, 2- (collectionwise) normality of Y in X implies (collectionwise) normality of Y in X.

As was mentioned in the introduction, we have

Proposition 2.3. *Suppose Y is strongly regular in X. If Y is 1-paracompact in X, then Y is 1-collectionwise normal in X.*

Proof: Assume Y is 1-paracompact in X. Let $\{F_\alpha \,|\, \alpha \in \Omega\}$ be a discrete collection of closed subsets of X. Since Y is strongly regular in X, for each $x \in X$ we can choose an open neighborhood U_x of x in X such that $\big|\{\alpha \in \Omega \,|\, \overline{U_x} \cap Y \cap F_\alpha \neq \emptyset\}\big| \leq 1$. Set $\mathcal{U} = \{U_x \,|\, x \in X\}$. Since $\mathcal{U}$ is an open cover of X and Y is 1-paracompact in X, there exists an open cover $\mathcal{V}$ of X such that $\mathcal{V}$ refines $\mathcal{U}$ and is locally finite at each point of Y in X. We put for $\alpha \in \Omega$

$$G_\alpha = X \setminus \overline{\bigcup\{V \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\}}.$$

Since $Y \cap \overline{\bigcup\{V \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\}} = Y \cap (\bigcup\{\overline{V} \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\})$, we have $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha \in \Omega$. Note that for $V \in \mathcal{V}$, $\overline{V} \cap Y \cap F_\alpha \neq \emptyset$ if $G_\alpha \cap V \neq \emptyset$. Hence $\{G_\alpha \,|\, \alpha \in \Omega\}$ is discrete in X. Thus, Y is 1-collectionwise normal in X. This completes the proof. □

Proposition 2.4.[†] *Suppose Y is strongly regular in X. If Y is 2-paracompact in X, then Y is 2-collectionwise normal in X.*

Proof: Assume Y is 2-paracompact in X. Let $\{F_\alpha \,|\, \alpha \in \Omega\}$ and $\mathcal{U}$ be the same as in the proof of Proposition 2.3. Take a collection $\mathcal{V}$ of open subsets of X such that $\mathcal{V}$ partially refines $\mathcal{U}$, $\mathcal{V}$ is locally finite at each point of Y in X and $Y \subset \bigcup \mathcal{V}$. Put

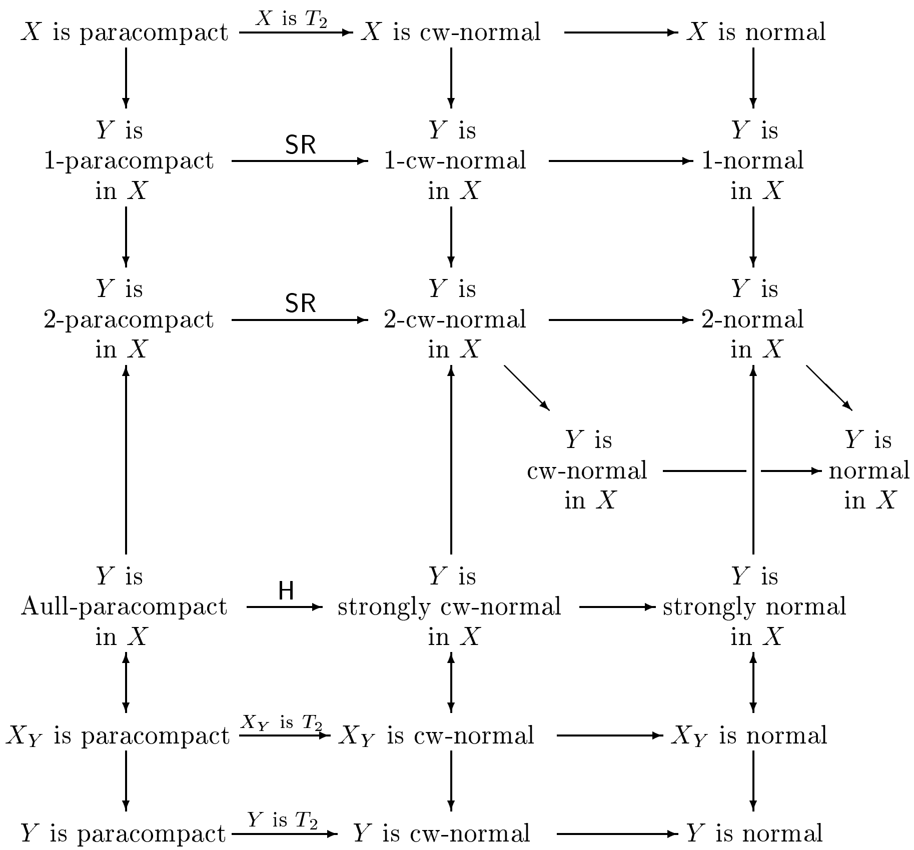
$$G_\alpha = \mathrm{St}(F_\alpha \cap Y, \mathcal{V}) \setminus \overline{\bigcup\{V \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\}}$$

for each $\alpha \in \Omega$. Then $\{G_\alpha \,|\, \alpha \in \Omega\}$ is the desired collection. Hence Y is 2-collectionwise normal in X, which completes the proof. □

These propositions and definitions above admit the following result; for brevity "cw-normal" means collectionwise normal. Moreover, the symbols "H" and "SR" mean the assumptions that "Y is Hausdorff in X" and "Y is strongly regular in X", respectively. "T_2" means "Hausdorff".

[†] This was independently proved by E. Grabner, G. Grabner, K. Miyazaki and J. Tartir, assuming that all spaces are Hausdorff.

Proposition 2.5. *For a subspace Y of a space X the following implications hold.*



Proof: The implication "Y is 1- (respectively, 2-) paracompact in $X \Rightarrow Y$ is 1- (respectively, 2-) collectionwise normal in X" is Proposition 2.3 (respectively, Proposition 2.4).

The facts "Y is Aull-paracompact in $X \Leftrightarrow X_Y$ is paracompact", "Y is strongly collectionwise normal in $X \Leftrightarrow X_Y$ is collectionwise normal" and "Y is strongly normal in $X \Leftrightarrow X_Y$ is normal" were proved in [29], [18] and [2], respectively; the last two equivalences are available to prove immediately "Y is strongly collectionwise normal in $X \Rightarrow Y$ is 2-collectionwise normal in X" and "Y is strongly normal in $X \Rightarrow Y$ is 2-normal in X", respectively.

For the implication "Y is Aull-paracompact in $X \Rightarrow Y$ is strongly collectionwise normal", see Proposition 2.7 below.

Other implications are obvious. □

The reverse implications in Proposition 2.5 will be discussed later (see Remark 3.5).

Corresponding to Lemmas 2.1 and 2.2 we have the following lemma: (a)$\Leftrightarrow$(c) was recently obtained in [29], and (c)$\Leftrightarrow$(e) for Y being Hausdorff in X was proved in [18, Lemma 4.6]. Other equivalences are easy to prove.

Lemma 2.6. *For a subspace Y of a space X, the following statements from* (a) *to* (d) *are equivalent. If Y is Hausdorff in X, these are equivalent to* (e).

(a) *Y is Aull-paracompact in X.*
(b) *Y is 2-paracompact in G for every open subset G of X with $Y \subset G$.*
(c) *X_Y is paracompact.*
(d) *Y is 2-paracompact in X_Y.*
(e) *Y is paracompact itself and weakly P-embedded in X.*

Combining Lemmas 2.2 and 2.6, we have

Proposition 2.7. *Suppose Y is Hausdorff in X. If Y is Aull-paracompact in X, then Y is strongly collectionwise normal in X.*

A space X is *almost compact* if for every pair of disjoint zero-sets Z_0, Z_1 in X, either Z_0 or Z_1 is compact. Note that a Tychonoff space X is almost compact if and only if $|\beta X \setminus X| \leq 1$, where βX is the Stone-Čech compactification of X. As was mentioned in the introduction, Bella and Yaschenko [8] proved the following theorem.

Theorem 2.8 ([8])**.** *For a Tychonoff space Y, the following statements are equivalent.*

(a) *Y is weakly C-embedded in every larger Tychonoff* (*or equivalently, regular*) *space.*
(b) *Y is weakly C-embedded in every larger Tychonoff* (*or equivalently, regular*) *space containing Y as a closed subspace.*
(c) *Y is either Lindelöf or almost compact.*

Theorem 2.8 was improved to the following.

Theorem 2.9 ([18])**.** *For a Tychonoff space Y, the following statements are equivalent.*

(a) *Y is weakly P^γ-embedded in every larger Tychonoff* (*or equivalently, regular*) *space.*
(b) *Y is weakly P^γ-embedded in every larger Tychonoff* (*or equivalently, regular*) *space containing Y as a closed subspace.*
(c) *Y is either Lindelöf or almost compact.*

Using Theorem 2.8, Bella and Yaschenko [8] further proved the following theorem, which was independently proved by Matveev et al. [24]. As was pointed out in [18], with Lemma 2.1, Theorem 2.10 directly follows from Theorem 2.8 for the case when Y is Tychonoff. For the case when Y is regular, since each of conditions from (a) to (d) induces normality of Y itself, Theorem 2.10 also follows from Lemma 2.1 and Theorem 2.8.

Theorem 2.10 ([8], [24]). *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is strongly normal in every larger Tychonoff (respectively, regular) space.*
(b) *Y is normal in every larger Tychonoff (respectively, regular) space.*
(c) *Y is normal in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(d) *Y is either Lindelöf or normal and almost compact.*

Similarly, Theorem 2.9 and Lemma 2.2 imply the following theorem. Notice that spaces satisfying (d) are collectionwise normal.

Theorem 2.11 ([18]). *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is strongly collectionwise normal in every larger Tychonoff (respectively, regular) space.*
(b) *Y is collectionwise normal in every larger Tychonoff (respectively, regular) space.*
(c) *Y is collectionwise normal in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(d) *Y is either Lindelöf or normal and almost compact.*

Remark 2.12. Combining Proposition 2.5 and Theorems 2.10, 2.11, it is clear that "strongly normal" (respectively, "strongly collectionwise normal") can be replaced by "2-normal" (respectively, "2-collectionwise normal") in Theorem 2.10 (respectively, Theorem 2.11).

Moreover, Theorem 2.9 and Lemma 2.6 imply the following theorem: (b)⇔(c)⇔(d) was actually obtained by Arhangel'skii and Genedi [4] and Gordienko [15] (see also [2, Theorems 52 and 53] or [3, Theorem 7.10]), (a)⇔(d) was pointed out by Yamazaki in [29]. Note that almost compact paracompact Hausdorff spaces are Lindelöf.

Theorem 2.13 ([4], [15], [29]). *For a Tychonoff space Y, the following statements are equivalent.*

(a) *Y is Aull-paracompact in every larger Tychonoff (or equivalently, regular) space.*
(b) *Y is 2-paracompact in every larger Tychonoff (or equivalently, regular) space.*

(c) *Y is 2-paracompact in every larger Tychonoff (or equivalently, regular) space containing Y as a closed subspace.*
(d) *Y is Lindelöf.*

Remark 2.14. Yamazaki [28] showed that the following are equivalent for a Hausdorff space Y:

(a) Y is weakly C-embedded (or equivalently, weakly P-embedded) in every larger Hausdorff space.
(b) Y is weakly C-embedded (or equivalently, weakly P-embedded) in every larger Hausdorff space containing Y as a closed subspace.
(c) Y is either compact or every continuous real-valued function on Y is constant.

Hence, applying Lemmas 2.1 and 2.2, if we replace all "Tychonoff" in Theorems 2.10, 2.11 and 2.13 by "Hausdorff", the conditions (d) of each theorems are replaced by "Y is compact" (see also [28], [29]).

3. Quasi-C*-, C- and P$^\gamma$-embeddings

Let us introduce extension properties called quasi-C^*-, C- and P-embeddings, which will play basic roles in study of 1- (collectionwise) normality.

Let X be a space and $\mathcal{E} = \{E_\alpha \mid \alpha \in \Omega\}$ a collection of subsets of X. Then $\mathcal{E}$ is said to be *uniformly discrete* in X if there exist a collection $\{Z_\alpha \mid \alpha \in \Omega\}$ of zero-sets of X and a discrete collection $\{G_\alpha \mid \alpha \in \Omega\}$ of cozero-sets of X such that $E_\alpha \subset Z_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$ ([9]).

Let us now define that a subspace Y of a space X is *quasi-C^*-embedded in X* if for each pair Z_0, Z_1 of disjoint zero-sets of Y, there exist open subsets G_0, G_1 of X such that $\{G_0, G_1\}$ is discrete in X and $Z_i \subset G_i$ for $i = 0, 1$.

A subspace Y of a space X is said to be *quasi-P^γ-embedded in X* if for each uniformly discrete collection $\{Z_\alpha \mid \alpha < \gamma\}$ of zero-sets of Y, there exists a discrete collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $Z_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. A subspace Y is *quasi-P-embedded in X* if Y is quasi-P^γ-embedded in X for every γ. In case $\gamma = \omega$, quasi-P^ω-embedding is called *quasi-C-embedding*.

Definitions of quasi-C^*-embedding and quasi-P^γ-embedding should be compared with the following results obtained in [9], [18] and [19].

Lemma 3.1 ([9]). *A subspace Y of a space X is P^γ-embedded in X if and only if every uniformly discrete collection of subsets of Y of cardinality $\leq \gamma$ is also uniformly discrete in X.*

Lemma 3.2 ([18]). *A subspace Y of a space X is weakly C-embedded in X if and only if for each pair Z_0, Z_1 of disjoint zero-sets of Y, there exist disjoint open subsets G_0, G_1 of X such that $Z_i \subset G_i$ for $i = 0, 1$.*

Lemma 3.3 ([19]). *A subspace Y of a space X is weakly P^γ-embedded in X if and only if for each uniformly discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of zero-sets of Y there exists a pairwise disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$.*

By Lemmas 3.1, 3.2 and 3.3, we have the following implications.

$$\begin{array}{ccccc}
P & \longrightarrow & C & \longrightarrow & C^* \\
\downarrow & & \downarrow & & \downarrow \\
\text{quasi-}P & \longrightarrow & \text{quasi-}C & \longrightarrow & \text{quasi-}C^* \\
\downarrow & & & & \downarrow \\
\text{weak } P & \longrightarrow & \longrightarrow & \longrightarrow & \text{weak } C
\end{array}$$

The following examples show that none of reverse implications above is true.

Example 3.4. (1) Let X be the Tychonoff plank $(\omega_1 + 1) \times (\omega + 1) \setminus \{\langle \omega_1, \omega \rangle\}$ and Y the right edge $\{\langle \omega_1, n \rangle \mid n < \omega\}$ of X. Then Y is weakly P-embedded, closed, but not quasi-C^*-embedded in X.

(2) Let $X_i = \big((\omega_1 + 1) \times (\omega_1 + 1) \setminus \{\langle \omega_1, \omega_1 \rangle\}\big) \times \{i\}$ for $i = 0, 1$. Let X be the space obtained from $X_0 \oplus X_1$ by identifying two points $\langle \omega_1, \alpha, 0 \rangle$ and $\langle \omega_1, \alpha, 1 \rangle$ for each $\alpha < \omega_1$, and let $q : X_0 \oplus X_1 \to X$ be the resulting quotient map. Let $Y = q(\omega_1 \times \{\omega_1\} \times \{0, 1\})$.

To prove that Y is quasi-P-embedded in X, let $\{E_\alpha \mid \alpha \in \Omega\}$ be a uniformly discrete collection of zero-sets of Y. Let $E^i_\alpha = \{t \in \omega_1 \mid \langle t, \omega_1, i \rangle \in E_\alpha\}$ for each $\alpha \in \Omega$ and $i = 0, 1$. Since Y is countably compact, we may assume Ω is finite. Since ω_1 is normal, for $i = 0, 1$, there is a discrete collection $\{V^i_\alpha \mid \alpha \in \Omega\}$ of open subsets of ω_1 such that $E^i_\alpha \subset V^i_\alpha$ for every $\alpha \in \Omega$ and $i = 0, 1$. Set for every $\alpha \in \Omega$,

$$U_\alpha = q\big(\textstyle\bigcup_{i=0,1}((V^i_\alpha \times (\omega_1 + 1) \times \{i\}) \cap \{\langle \beta_1, \beta_2, i \rangle \mid \beta_1 < \beta_2 \le \omega_1\})\big).$$

It is easy to see that $\{U_\alpha \mid \alpha \in \Omega\}$ is a discrete collection of open subsets of X satisfying $E_\alpha \subset U_\alpha$ for every $\alpha \in \Omega$. This shows Y is quasi-P-embedded in X.

On the other hand, Y is not C^*-embedded in X.

(3) Let $\Lambda = \beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$. It is well-known that the subspace $\mathbb{N}$ of the space Λ is C^*-embedded closed, but not C-embedded in Λ (see [13]). In fact, $\mathbb{N}$ is not quasi-C-embedded in Λ.

(4) Bing's example G [10] gives a normal space X containing a closed discrete subset $F = \{f_\alpha \mid \alpha < \omega_1\}$ which admits no disjoint collection $\{U_\alpha \mid \alpha < \omega_1\}$ of open subsets of X such that $f_\alpha \in U_\alpha$ for every $\alpha < \omega_1$. Hence, F is C-embedded, but not weakly P-embedded in X ([18]).

Remark 3.5. Using Example 3.4, let us return to discuss reverse implications given in Proposition 2.5. First, observe Lemmas 2.1, 2.2 and also Proposition 3.6 below. A non-normal subspace Y of a paracompact Hausdorff space X is 1-paracompact, but not strongly normal in X.

In (1) of Example 3.4, Y is Aull-paracompact, but not 1-normal in X. Hence, Y is 2-paracompact, but not 1-collectionwise normal in X. In (3) of Example 3.4, $\mathbb{N}$ is 1-normal but not 1-ω-collectionwise normal in Λ. In (4) of Example 3.4, F is 1-normal and strongly normal, but not collectionwise normal in X. Examples for other reverse implications are easy to see.

Yamazaki [30] constructed a T_1-space X and a subspace Y such that Y is normal in X, but not 2-normal in X. We do not know similar examples under higher separation axioms. Furthermore, it is unknown whether 2-normality implies 2-ω-collectionwise normality, or whether collectionwise normality implies 2-collectionwise normality.

Proposition 3.6. *For a subspace Y of a space X, the following statements hold. If Y is closed in X, each of them reverses.*

(a) *If Y is itself γ-collectionwise normal and quasi-P^γ-embedded in X, then Y is 1-γ-collectionwise normal in X.*
(b) *If Y is itself normal and quasi-C^*-embedded in X, then Y is 1-normal in X.*

The proof of Proposition 3.6 is easy and omitted. Trivially, the reverse implications need not be true unless Y is closed.

In [6], Aull defined that a subspace Y of a space X is *α-paracompact in X* if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of X such that $Y \subset \bigcup \mathcal{V}$, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite in X. Note that α-paracompactness of Y in X implies Aull-paracompactness of Y in X, and the converse does not necessarily hold.

Related to α-paracompactness, let us recall the following results in [21] and [22, Theorem 1.3].

Theorem 3.7 ([21]). *A Hausdorff (respectively, regular, Tychonoff) space Y is α-paracompact in every larger Hausdorff (respectively, regular, Tychonoff) space containing Y as a closed subspace if and only if Y is compact.*

Theorem 3.8 ([22]). *For a closed subspace Y of a regular space X, Y is 1-paracompact in X if and only if Y is α-paracompact in X.*

Theorems 3.7 and 3.8 immediately induce the following:

Corollary 3.9. *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is 1-paracompact in every larger Tychonoff (respectively, regular) space.*

(b) *Y is α-paracompact in every larger Tychonoff (respectively, regular) space.*
(c) *Y is 1-paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(d) *Y is α-paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(e) *Y is compact.*

Lemma 3.10. *Let Y be an almost compact Tychonoff space. If Y is contained in a regular space X, then Y is quasi-P-embedded in X.*

PROOF: Let $\{Z_\alpha \mid \alpha \in \Omega\}$ be a uniformly discrete collection of zero-sets of Y. We may assume that $\Omega = \{0,1\}$ and Z_0 is compact since Y is almost compact. Since X is regular, there exist open subsets G, H of X such that $Z_0 \subset H \subset \overline{H} \subset G$ and $\overline{G} \cap Z_1 = \emptyset$. Then $\{H, X \setminus \overline{G}\}$ is discrete in X. □

Theorem 3.11. *For a Tychonoff space Y, the following statements are equivalent.*

(a) *Y is quasi-P-embedded in every larger Tychonoff space.*
(a′) *Y is quasi-P-embedded in every larger Tychonoff space containing Y as a closed subspace.*
(b) *Y is quasi-C-embedded in every larger Tychonoff space.*
(b′) *Y is quasi-C-embedded in every larger Tychonoff space containing Y as a closed subspace.*
(c) *Y is quasi-C^*-embedded in every larger Tychonoff space.*
(c′) *Y is quasi-C^*-embedded in every larger Tychonoff space containing Y as a closed subspace.*
(d) *Y is almost compact.*

In the above conditions from (a) *to* (c′)*, "Tychonoff" can be replaced by "regular".*

PROOF: Assume that Y is Tychonoff. Then, the implications (a)⇒(a′)⇒(b′)⇒(c′) and (a)⇒(b)⇒(c)⇒(c′) are obvious. To complete the proof, we show the implications (c′)⇒(c)⇒(d)⇒(a).

(c′)⇒(c). Let X be a Tychonoff space containing Y as a subspace, and let Z_0, Z_1 be disjoint zero-sets of Y. Let $D(2^{|X|})$ denote the discrete space of cardinality $2^{|X|}$, and $A(2^{|X|})$ the one-point compactification of $D(2^{|X|})$. Put $W = \big(X \times A(2^{|X|})\big) \setminus \big((X \setminus Y) \times \{\infty\}\big)$, where ∞ is the non-isolated point of $A(2^{|X|})$. Then Y is homeomorphic to $Y \times \{\infty\}$ which is a closed subset of W. By (c′), there exist open subsets $G_0, G_1 \subset W$ with $\{G_0, G_1\}$ discrete such that $Z_i \subset G_i$ for $i = 0, 1$. For every $a \in Z_0$ and $b \in Z_1$, take finite subsets E_a, F_b of $D(2^{|X|})$ such that

$$\{a\} \times (D(2^{|X|}) \setminus E_a) \subset G_0 \ \text{ and } \ \{b\} \times (D(2^{|X|}) \setminus F_b) \subset G_1.$$

Since the cardinality of $(\bigcup_{a\in Z_0} E_a) \cup (\bigcup_{b\in Z_1} F_b)$ is less than $2^{|X|}$, we can take $\alpha^* \in D(2^{|X|}) \setminus (\bigcup_{a\in Z_0} E_a \cup \bigcup_{b\in Z_1} F_b)$ so that $Z_i \times \{\alpha^*\} \subset (X \times \{\alpha^*\}) \cap G_i$ for $i = 0, 1$. Then, the collection $\{(X \times \{\alpha^*\}) \cap G_0,\ (X \times \{\alpha^*\}) \cap G_1\}$ is discrete in W. Define $G'_i = \{x \in X \mid \langle x, \alpha^*\rangle \in G_i\}$ for $i = 0, 1$. Then, $\{G'_0, G'_1\}$ is a discrete collection of open subsets in X and $Z_i \subset G'_i$ for $i = 0, 1$.

(c)⇒(d). Assume that Y is not almost compact. There exist disjoint zero sets Z_0, Z_1 in Y such that neither Z_0 nor Z_1 is compact. Pick $x_i \in \overline{Z_i}^{\beta Y} \setminus Z_i$ for $i = 0, 1$. Let X be the space obtained from βY by identifying two points x_0 and x_1, and let $q : \beta Y \to X$ be the resulting quotient map. Then X is compact Hausdorff and Y is obviously a subspace of X. By assumption (c), there exist open subsets G_0, G_1 of X such that $\overline{G_0}^X \cap \overline{G_1}^X = \emptyset$ and $Z_i = q(Z_i) \subset G_i$ for $i = 0, 1$. We may assume $q(x_0) = q(x_1) \notin \overline{G_0}^X$. Then $X \setminus \overline{G_0}^X$ is an open neighborhood of $q(x_0)$. Hence, $x_0 \notin \overline{Z_0}^{\beta Y}$, a contradiction.

(d)⇒(a) follows from Lemma 3.10.

In the case Y is embedded in every larger regular space, it suffices to show (d)⇒(a), and this is obvious. □

By Proposition 3.6 and Theorem 3.11, we have

Corollary 3.12. *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

- (a) *Y is 1-collectionwise normal in every larger Tychonoff (respectively, regular) space.*
- (a′) *Y is 1-collectionwise normal in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
- (b) *Y is 1-normal in every larger Tychonoff (respectively, regular) space.*
- (b′) *Y is 1-normal in every Tychonoff (respectively, regular) space containing Y as a closed subspace.*
- (c) *Y is normal and almost compact.*

In Corollary 3.12, (b′)⇔(c) also follows from the following result in [24].

Theorem 3.13 ([24]). *For a Tychonoff space Y, the following statements are equivalent.*

- (a) *Any disjoint closed subsets of Y are completely separated in every larger Tychonoff space.*
- (b) *For any larger Tychonoff space X containing Y as a closed subspace and any disjoint closed subsets F_0, F_1 of Y, there exist disjoint open subsets G_0, G_1 of X such that $F_i \subset G_i$ for $i = 0, 1$ and $\overline{G_0}^X \cap \overline{G_1}^X = \emptyset$.*
- (c) *Y is normal and almost compact.*

For the Hausdorff case, we have the following.

Theorem 3.14. *For a Hausdorff space Y, the following statements are equivalent.*

(a) *Y is quasi-C^*-embedded in every larger Hausdorff space.*
(b) *Y is quasi-C^*-embedded in every larger Hausdorff space containing Y as a closed subspace.*
(c) *Every continuous real-valued function on Y is constant.*

In (a) *and* (b), *"quasi-C^*-embedded" can be replaced by "quasi-P-embedded" or "quasi-C-embedded".*

Proof: We only prove (b)⇒(c). Suppose that there exists a continuous real-valued function f on Y such that f is not constant. Then we choose distinct points y_0, y_1 in Y such that $f(y_0) \neq f(y_1)$.

Let $X_i = \big((\omega_1+1)\times(\omega+1)\setminus\{(\omega_1,\omega)\}\big)\times\{i\}$ and $q_i : X_i \to X_i/(\{\omega_1\}\times\omega\times\{i\})$ be the quotient map obtained by collapsing $\{\omega_1\}\times\omega\times\{i\}$ to one point, for $i = 0, 1$. Let X be the space obtained from $\big(X_0/(\{\omega_1\} \times \omega \times \{0\})\big) \oplus \big(X_1/(\{\omega_1\} \times \omega \times \{1\})\big) \oplus Y$ by identifying two points $\langle\alpha,\omega,0\rangle$ and $\langle\alpha,\omega,1\rangle$ for each $\alpha < \omega_1$ and by identifying two points $q_i(\{\omega_1\} \times \omega \times \{i\})$ and y_i for $i = 0, 1$. Then X is Hausdorff and Y is closed in X. Put $Z_i = f^{-1}(f(y_i))$ for $i = 0, 1$. Then Z_0, Z_1 are disjoint zero-sets of Y. By (b), there exist open subsets $G_0, G_1 \subset X$ with $\{G_0, G_1\}$ discrete such that $Z_i \subset G_i$ for $i = 0, 1$. But, by the construction of X, $\overline{G_0}^X \cap \overline{G_1}^X \neq \emptyset$, a contradiction. □

By Theorem 3.14 and Proposition 3.6, we have

Corollary 3.15. *For a Hausdorff space Y, the following statements are equivalent.*

(a) *Y is* 1*-collectionwise normal* (*or equivalently,* 1*-normal*) *in every larger Hausdorff space.*
(b) *Y is* 1*-collectionwise normal* (*or equivalently,* 1*-normal*) *in every larger Hausdorff space containing Y as a closed subspace.*
(c) $|Y| \leq 1$.

Finally we consider a condition under which 2-paracompactness implies 1-paracompactness. We say a subspace Y of a space X is T_4- (respectively, T_3-) *embedded in* X if for every closed subset F of X disjoint from Y (respectively, $z \in X\setminus Y$), F (respectively, z) and Y are separated by disjoint open subsets of X. The idea of these notions already appeared in Aull [6].

The following result refines Theorem 3.8; the implication "(b) ⇒ Y is T_4-embedded in X" is due to Aull [6, Theorem 6]. By using this fact, Lupiañez and Outerelo [22, Lemma 1.2 and Theorem 1.3] essentially proved the implications (a)⇒(c)⇒(b)⇒(c)⇒(a).

Proposition 3.16. *Let X be a space and Y a subspace. Then the following statements are equivalent.*

(a) *Y is 1-paracompact in X and T_3-embedded in X.*
(b) *Y is α-paracompact in X and for every $y \in Y$ and every closed subset F of X with $F \cap Y = \emptyset$, there exists an open subset U of X such that $y \in U \subset \overline{U}^X \subset X \setminus F$.*
(c) *Y is 2-paracompact in X and T_4-embedded in X.*

Corollary 3.17. *A closed subspace Y of a regular space X is 1-paracompact in X if and only if Y is 2-paracompact in X and T_4-embedded in X.*

Proposition 3.18. *For a Tychonoff space Y, the following statements are equivalent.*

(a) *Y is T_4-embedded in every larger Tychonoff space.*
(b) *Y is T_4-embedded in every larger Tychonoff space containing Y as a closed subspace.*
(c) *Y is compact.*

In (a) *and* (b), *"Tychonoff" can be replaced by "regular".*

PROOF: We only prove (b)⇒(c).

Assume that Y is T_4-embedded in every larger Tychonoff space containing Y as a closed subspace and Y is not compact. Then pick $x_0 \in \beta Y \setminus Y$. Let $D(2^{|\beta Y|})$ denote the discrete space of cardinality $2^{|\beta Y|}$, and $A(2^{|\beta Y|})$ the one-point compactification of $D(2^{|\beta Y|})$. Put $X = (\beta Y \times A(2^{|\beta Y|})) \setminus ((\beta Y \setminus Y) \times \{\infty\})$, where ∞ is the non-isolated point of $A(2^{|\beta Y|})$. Then Y is homeomorphic to $Y \times \{\infty\}$ which is a closed subset of X. Put $F = \{x_0\} \times D(2^{|\beta Y|})$. Then F is a closed subset of X disjoint from Y. By (b), there exist disjoint open subsets G_0, G_1 of X such that $Y \subset G_0$ and $F \subset G_1$. For each $y \in Y$, take a finite subset E_y of $D(2^{|\beta Y|})$ such that $\{y\} \times (D(2^{|\beta Y|}) \setminus E_y) \subset G_0$. Since the cardinality of $\bigcup_{y \in Y} E_y$ is not greater than $2^{|\beta Y|}$, we can take $\alpha^* \in D(2^{|\beta Y|}) \setminus \bigcup_{y \in Y} E_y$ so that $Y \times \{\alpha^*\} \subset (\beta Y \times \{\alpha^*\}) \cap G_0$ and $\langle x_0, \alpha^* \rangle \in (\beta Y \times \{\alpha^*\}) \cap G_1$. Then, $(\beta Y \times \{\alpha^*\}) \cap G_0$ and $(\beta Y \times \{\alpha^*\}) \cap G_1$ are disjoint. Define $G'_i = \{x \in \beta Y \mid \langle x, \alpha^* \rangle \in G_i\}$ for $i = 0, 1$. Then, G'_0, G'_1 are disjoint open subsets of βY with $Y \subset G'_0$ and $x_0 \in G'_1$. This is a contradiction. □

Theorem 2.13, Propositions 3.17 and 3.18 give an alternative proof of Corollary 3.9.

In case Y is Hausdorff, we have

Proposition 3.19. *For a Hausdorff space Y, the following statements are equivalent.*

(a) *Y is T_4-embedded in every larger Hausdorff space.*

(b) *Y is T_4-embedded in every larger Hausdorff space containing Y as a closed subspace.*
(c) *$Y = \emptyset$.*

Proof: We only prove (b)⇒(c). Assume $Y \neq \emptyset$. Take $y \in Y$ and let $T = (\omega_1+1)\times(\omega+1)\setminus\{(\omega_1,\omega)\}$. Consider the quotient space Z obtained from $T\oplus Y$ by identifying the right edge $\{\omega_1\}\times\omega$ of T and y to one point. Let $q : T\oplus Y \to Z$ be the natural quotient map. Then, Z is Hausdorff and Y is homeomorphic to $q(Y)$ which is a closed subset of Z. But, Y is not T_4-embedded in Z. Hence (b) implies (c). □

A similar proof provides the following proposition, and this should be compared with Theorem 3.7 and Corollary 3.9.

Proposition 3.20. *For a Hausdorff space Y, the following statements are equivalent.*

(a) *Y is 1-paracompact in every larger Hausdorff space.*
(b) *Y is 1-paracompact in every larger Hausdorff space containing Y as a closed subspace.*
(c) *$Y = \emptyset$.*

4. On 1-metacompactness of a subspace in a space

In this section, we prove an extension of Corollary 3.9.

A subspace Y of a space X is said to be 1-*metacompact in* X if for every open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is point-finite at every $y \in Y$ ([20]).

A space X satisfies the *discrete finite chain condition* (DFCC, for short) if every discrete collection of non-empty open subsets of X is finite (see [23], for example). Recall that a Tychonoff space X is pseudocompact if and only if X satisfies the DFCC. It is also known that a Tychonoff space X is compact if and only if X is pseudocompact and metacompact ([26], [27]). Furthermore, a regular space X is compact if and only if X satisfies the DFCC and is metacompact ([26]).

According to [2], in [4], Arhangel'skii and Genedi remarked the following fact: *Let Y be a countable dense subset of a regular space X. Then Y is* 1*-metacompact (or equivalently,* 1*-paracompact) in X if and only if X is Lindelöf.*

The following result is based on this fact.

Lemma 4.1. *Let Z be an arbitrary separable space and Y an arbitrary non-DFCC space.*

Let $D = \{d_n \,|\, n \in \mathbb{N}\}$ be a countable dense subset of Z, $\{U_n \,|\, n \in \mathbb{N}\}$ a countable discrete collection of non-empty open subsets of Y and $A = \{y_n \,|\, n \in \mathbb{N}\}$

a countable subset of Y such that $y_n \in U_n$ for each $n \in \mathbb{N}$. Let X be the quotient space obtained from $Y \oplus Z$ by identifying y_n with d_n for each $n \in \mathbb{N}$.

If Y is 1-metacompact in X, then Z is Lindelöf.

Moreover, if Y and Z are Tychonoff (respectively, regular), then X is also Tychonoff (respectively, regular).

Proof: Let $q : Y \oplus Z \to X$ be the quotient map. Since Y is homeomorphic to $q(Y)$, Y is viewed as a subspace of X. Put $p_n = q(y_n) = q(d_n)$ for each $n \in \mathbb{N}$, and let $P = \{p_n \,|\, n \in \mathbb{N}\}$. Note that A is a closed discrete subset of Y.

Suppose Y is 1-metacompact in X. Let $\mathcal{G}$ be an open cover of Z. For each $G \in \mathcal{G}$, put $V_G = q\left(G \cup \bigcup\{U_n \,|\, d_n \in G\}\right)$. Define $\mathcal{V} = \{V_G \,|\, G \in \mathcal{G}\} \cup \{q(Y \setminus A)\}$. Then $\mathcal{V}$ is an open cover of X. Since Y is 1-metacompact in X, $\mathcal{V}$ has an open refinement $\mathcal{W}$ which is point-finite at every $x \in q(Y)$. For each $W \in \mathcal{W}$ with $W \cap q(Z) \neq \emptyset$, W must contain some element of P since D is dense in Z. Then, $\mathcal{U}$ has a countable subcover because $\mathcal{W}$ is point-finite at p_n for every $n \in \mathbb{N}$. Hence, Z is Lindelöf.

Let us prove that if Y and Z are Tychonoff, then X is also Tychonoff. Assume Y and Z are Tychonoff. Let $x \in X$ and U be a neighborhood of x in X.

If $x \in q(Z)$, pick $z \in q^{-1}(x) \cap Z$. Then, there exists a neighborhood W of z in Z such that $q(W) \subset U$. For every $m \in \mathbb{N}$ with $d_m \in W$, there exists a neighborhood V_m of y_m in Y such that $q(V_m) \subset U$. Here, we may assume that $V_m \subset U_m$. Define $G = q\left(\bigcup\{V_m \,|\, d_m \in W\} \cup W\right)$. Since Z is Tychonoff, there exists a continuous function $f : Z \to \mathbb{I}$ such that $f(z) = 1$ and $f(Z \setminus W) \subset \{0\}$. Because Y is also Tychonoff, for each $m \in \mathbb{N}$ with $d_m \in W$, there exists a continuous function $g_m : Y \to \mathbb{I}$ such that $g_m(y_m) = f(d_m)$ and $g_m(Y \setminus V_m) \subset \{0\}$. Then, define a function $h : X \to \mathbb{I}$ by

$$h(a) = \begin{cases} \sum_{d_m \in W} g_m(q^{-1}(a)), & \text{if } a \in q(Y) \setminus P, \\ f(q^{-1}(a)), & \text{if } a \in q(Z) \setminus P, \\ f(d_n), & \text{if } a = p_n \in P. \end{cases}$$

Then, h is continuous and $h(x) = 1$, $h(X \setminus G) \subset \{0\}$.

In case $x \notin q(Z)$, that is $x \in q(Y) \setminus P$, since A is closed in Y, it is easy to find a continuous function $g : X \to \mathbb{I}$ such that $g(x) = 1$ and $g(X \setminus U) \subset \{0\}$. Therefore, X is Tychonoff.

If Y and Z are regular, we can similarly prove that X is regular. □

Theorem 4.2. *A Tychonoff (respectively, regular) space Y is 1-metacompact in every larger Tychonoff (respectively, regular) space if and only if Y is compact.*

Proof: The "if" part is obvious.

To prove the "only if" part, assume that Y is 1-metacompact in every larger Tychonoff (respectively, regular) space but not compact. Since Y itself is metacompact, Y does not satisfy the DFCC. Thus, there exists a countable discrete collection $\{U_n \mid n \in \mathbb{N}\}$ of non-empty open subsets of Y. For each $n \in \mathbb{N}$, take $y_n \in U_n$, and let $A = \{y_n \mid n \in \mathbb{N}\}$. Then A is a discrete closed subset of Y.

Let Z be an arbitrary separable, non-Lindelöf Tychonoff (respectively, regular) space, for example, Z is the Ψ-space in [13, 5I]. Let $D = \{d_n \mid n \in \mathbb{N}\}$ be a countable dense subset of Z. Now, let X be the space constructed in Lemma 4.1. Then, X is Tychonoff (respectively, regular) and Y is not 1-metacompact in X by Lemma 4.1. □

Theorem 4.2 extends the following result due to E. Grabner et al. [16]: *A normal space Y is 1-metacompact in every larger regular space if and only if Y is compact.*

To prove the Hausdorff case, we need the following well-known fact: *A Hausdorff space X is compact if and only if X is countably compact and metacompact.*

Theorem 4.3. *Let Y be a Hausdorff space. Then Y is 1-metacompact in every larger Hausdorff space if and only if Y is compact.*

Proof: The "if" part is obvious.

To prove "only if" part, assume that Y is 1-metacompact in every larger Hausdorff space but not compact. Then, Y is not countably compact. Hence, there exists a countable discrete closed subset $A = \{a_n \mid n \in \mathbb{N}\}$ of Y. Let Z be an arbitrary separable, non-Lindelöf Hausdorff space with a countable dense subset $D = \{d_n \mid n \in \mathbb{N}\}$. Define X be the quotient space obtained from $Y \oplus Z$ by identifying a_n with d_n for each $n \in \mathbb{N}$.

Then, it is easy to see that X is Hausdorff. Define $U_n = (Y \setminus A) \cup \{y_n\}$ for each $n \in \mathbb{N}$. Then, similarly to the proof of Lemma 4.1, Y is not 1-metacompact in X. □

Remark 4.4. In contrast with (a)$\Leftrightarrow$(c) in Corollary 3.9, consider an analogous statement that a Tychonoff (respectively, regular, Hausdorff) space Y is 1-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space containing Y as a closed subspace. This means, however, nothing but that Y is metacompact.

5. On 1-subparacompactness of a subspace in a space

It was defined in [25] that a subspace Y of a space X is 1-*subparacompact in* X if for every open cover $\mathcal{U}$ of X, there exists a σ-discrete collection $\mathcal{P}$ of closed subsets of X with $Y \subset \bigcup \mathcal{P}$ such that $\mathcal{P}$ is a partial refinement of $\mathcal{U}$.

In [25], Qu and Yasui asked a question as follows: *Let X be a regular space and Y a subspace of X. Is it true that if Y is 1-paracompact in X, then Y is 1-subparacompact in X?*

The following theorem is a negative answer to this question.

Theorem 5.1. *There exist a Tychonoff space X and a subspace Y of X such that Y is 1-paracompact in X but not 1-subparacompact in X.*

Proof: Let X be the set $(\omega_2+1)\times(\omega_1+1)\setminus\{\langle\omega_2,\omega_1\rangle\}$. For $\alpha\in\omega_1$ and $\beta\in\omega_2$, define $G_\alpha=(\omega_2+1)\times\{\alpha\}$ and $H_\beta=\{\beta\}\times(\omega_1+1)$, respectively. Define a topology on X as follows. For $\alpha\in\omega_1$, a neighborhood base at $\langle\omega_2,\alpha\rangle$ is the family of all sets of the form $G_\alpha\setminus E$, where E is a finite subset of $\omega_2\times\{\alpha\}$. For $\beta\in\omega_2$, a neighborhood base at $\langle\beta,\omega_1\rangle$ is the family of all sets of the form $H_\beta\setminus F$, where F is a finite subset of $\{\beta\}\times\omega_1$. All other points of X are isolated in X. The construction of X is based on a example in [11]. Let $Y=\omega_2\times\omega_1\subset X$.

Let us prove that Y is 1-paracompact in X. To prove this, let $\mathcal{U}$ be any open cover of X. For every $\alpha\in\omega_1$, take a finite subset E_α of $\omega_2\times\{\alpha\}$ such that $\langle\omega_2,\alpha\rangle\in G_\alpha\setminus E_\alpha$ and $G_\alpha\setminus E_\alpha$ is contained some $U\in\mathcal{U}$. Similarly, for every $\beta\in\omega_2$, take a finite subset F_β of $\{\beta\}\times\omega_1$ such that $\langle\beta,\omega_1\rangle\in H_\beta\setminus F_\beta$ and $H_\beta\setminus F_\beta$ is contained some $U\in\mathcal{U}$. Then, the collection

$$\{G_\alpha\setminus E_\alpha\,|\,\alpha\in\omega_1\}\cup\{H_\beta\setminus F_\beta\,|\,\beta\in\omega_2\}\cup\{\{\langle\beta,\alpha\rangle\}\,|\,\alpha\in\omega_1,\beta\in\omega_2\}$$

is an open refinement of $\mathcal{U}$ which is locally finite at every $y\in Y$ in X. Hence, Y is 1-paracompact in X.

Now, we shall show that Y is not 1-subparacompact in X. Let $\mathcal{U}=\{G_\alpha\,|\,\alpha\in\omega_1\}\cup\{H_\beta\,|\,\beta\in\omega_2\}$. Then, $\mathcal{U}$ is an open cover of X. Assume that there is a collection $\mathcal{P}=\bigcup_{n\in\mathbb{N}}\mathcal{P}_n$ of closed subsets of X with $\bigcup\mathcal{P}\supset Y$ such that $\mathcal{P}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{P}_n$ is discrete in X for every $n\in\mathbb{N}$. For any n, let

$$\mathcal{P}_n^0=\{P\in\mathcal{P}_n\,|\,(\exists\beta\in\omega_2)\ P\subset H_\beta\},\ \mathcal{P}_n^1=\{P\in\mathcal{P}_n\,|\,(\exists\alpha\in\omega_1)\ P\subset G_\alpha\}\setminus\mathcal{P}_n^0.$$

Note that $\mathcal{P}_n=\mathcal{P}_n^0\cup\mathcal{P}_n^1$. For each $n\in\mathbb{N}$ and each $\alpha\in\omega_1$, $\{P\in\mathcal{P}_n^0\,|\,P\cap G_\alpha\neq\emptyset\}$ is finite. Since $|P\cap G_\alpha|\le 1$ for each $P\in\mathcal{P}_n^0$, $(\bigcup\mathcal{P}_n^0)\cap G_\alpha$ is finite. Therefore, $(\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{P}_n^0)\cap G_\alpha$ is countable, and hence $V_\alpha^0=\{\lambda\in\omega_2\,|\,\langle\lambda,\alpha\rangle\in(\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{P}_n^0)\cap G_\alpha\}$ is also countable. Therefore, we can take $\beta^*\in\omega_2\setminus\bigcup_{\alpha\in\omega_1}V_\alpha^0$. Since $H_{\beta^*}\cap Y\subset\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{P}_n^1$, we can take $n\in\mathbb{N}$ so that $(\bigcup\mathcal{P}_n^1)\cap H_{\beta^*}$ is uncountable.

On the other hand, $\{P\in\mathcal{P}_n^1\,|\,P\cap H_{\beta^*}\neq\emptyset\}$ is finite and $|P\cap H_{\beta^*}|\le 1$ for each $P\in\mathcal{P}_n^1$. This is a contradiction. Hence, Y is not 1-subparacompact in X. This completes the proof. □

Acknowledgment. The authors are grateful to the referee for his/her careful reading of the manuscript. The authors would like to express their gratitude to Professors T. Hoshina and K. Yamazaki for their encouragement.

References

[1] Alò R.A., Shapiro H.L., *Normal Topological Spaces*, Cambridge Univ. Press, Cambridge, 1974.

[2] Arhangel'skii A.V., *Relative topological properties and relative topological spaces*, Topology Appl. **70** (1996), 87–99.

[3] Arhangel'skii A.V., *From classic topological invariants to relative topological properties*, Sci. Math. Japon. **55** (2002), 153–201.

[4] Arhangel'skii A.V., Genedi H.M.M., *Beginnings of the theory of relative topological properties*, in: General Topology. Spaces and Mappings, MGU, Moscow, 1989, pp. 3–48.

[5] Arhangel'skii A.V., Gordienko I.Ju., *Relative symmetrizability and metrizability*, Comment. Math. Univ. Carolinae **37** (1996), 757–774.

[6] Aull C.E., *Paracompact subsets*, Proc. the Second Prague Topological Symposium, Prague, 1966, pp. 45–51.

[7] Aull C.E., *Collectionwise normal subsets*, J. London Math. Soc. **1** (1969), 155–162.

[8] Bella A., Yaschenko I.V., *Lindelöf property and absolute embeddings*, Proc. Amer. Math. Soc. **127** (1999), 907–913.

[9] Blair R.L., *A cardinal generalization of z-embedding*, in: Rings of continuous functions, Lecture Notes in Pure and Appl. Math., Vol. 95, Marcel Dekker, New York, 1985, pp. 7–78.

[10] Bing R.H., *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186.

[11] Burke D.K., *Covering Properties*, in: K. Kunen and J.E. Vaughan, Eds., Handbook of the Set-Theoretic Topology, North-Holand, Amsterdam, 1984, pp. 347–422.

[12] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[13] Gillman L., Jerison M., *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[14] Gordienko I.Ju., *On relative properties of paracompactness and normality type*, Moscow Univ. Math. Bull. **46** (1991), 31–32.

[15] Gordienko I.Ju., *A characterization of relative Lindelöf property by relative paracompactness*, in: General Topology. Spaces, mappings and functors, MUG, Moscow, 1992, pp. 40–44.

[16] Grabner E.M., Grabner G.C., Miyazaki K., *On properties of relative metacompactness and paracompactness type*, Topology Proc. **25** (2000), 145–177.

[17] Hoshina T., *Extensions of mappings II*, in: K. Morita and J. Nagata, Eds., Topics in General Topology, North-Holland, Amsterdam, 1989, pp. 41–80.

[18] Hoshina T., Yamazaki K., *Weak C-embedding and P-embedding, and product spaces*, Topology Appl., **125** (2002), 233–247.

[19] Hoshina T., Yamazaki K., *A characterization of weak P-embedding*, preprint.

[20] Kočinac L.D., *Some relative topological properties*, Math. Vesnik **44** (1992), 33–44.

[21] Lupiañez F.G., *On covering properties*, Math. Nachr. **141** (1989), 37–43.

[22] Lupiañez F.G, Outerelo E., *Paracompactness and closed subsets*, Tsukuba J. Math. **13** (1989), 483–493.

[23] Matveev M.V., *A survey on star covering properties*, Topology Atlas, preprint no. **330**, 1998.

[24] Matveev M.V., Pavlov O.I., Tartir J., *On relatively normal spaces, relatively regular spaces, and on relative property* (a), Topology Appl. **93** (1999), 121–129.

[25] Qu Z., Yasui Y., *Relatively subparacompact spaces*, Sci. Math. Japon. **54** (2001), 281–287.

[26] Scott B.M., *Pseudocompact, metacompact spaces are compact*, Topology Proc. **4** (1979), 577–587.

[27] Watson S., *Pseudocompact metacompact spaces are compact*, Proc. Amer. Math. Soc. **81** (1981), 151–152.

[28] Yamazaki K., *Absolute weak C-embedding in Hausdorff spaces*, Topology Appl. **131** (2003), 273–279.

[29] Yamazaki K., *Aull-paracompactness and strong star-normality of subspaces in topological spaces*, Comment. Math. Univ. Carolinae **45** (2004), 743–747.
[30] Yamazaki K., *A question on relative normality*, unpublished.

Doctoral Program in Mathematics, Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

E-mail: shinji@math.tsukuba.ac.jp

Tokyo Gakugei University Senior Highschool, Setagaya, Tokyo 154-0002, Japan

E-mail: ryoken@gakugei-hs.setagaya.tokyo.jp

(*Received* May 17, 2004, *revised* March 25, 2005)