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In: Stanley P. Franklin and Zdeněk Frolík and Václav Koutník (eds.): General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971. pp. 319--327.

Persistent URL: http://dml.cz/dmlcz/700563

# FUNCTIONAL MOVEMENTS IN DENDRITIC STRUCTURES

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**1. Introduction.** A study will be made of the action of a function from a topological space $X$ into itself which is influenced and to a considerable extent controlled by the structure of the space in so far as it is dendritic in character or dendritic relative to its cyclic elements. Continuity restrictions on the function are minimal and are related to peripheral continuity and connectivity requirements in the main. Invariant and nearly invariant cyclic elements will be identified when the space is a Peano continuum, thus giving nearly fixed points in the case of a dendrite; and this is done with very limited continuity restrictions on the function.

**2. Inductive Properties.** This term is used here in the sense as usually employed in connection with the Brouwer Reduction Theorem.

**Theorem 2.1.** *In a compact metric space $X$ for any property $P$, the property $S_p$ of being the topological limit of a sequence of compact sets each having property $P$ is an inductive property.*

For let $A_1 \supset A_2 \supset A_3 \supset \ldots$ be a monotone decreasing sequence of compact sets in $X$ each having property $S_p$. We have to show that $A = \bigcap A_n$ also has $S_p$.

To that end, for each $n$ let $[A_k^n]$ be a sequence of compact sets converging to $A_n$, where each $A_k^n$ has property $P$. Clearly we may suppose these sequences chosen so that for each $n$ the sets $A_k^n$ all lie inside the $1/n$-sphere $V_{1/n}(A_n)$ about $A_n$ for all $k = 1, 2, \ldots$ and each also meets $V_{1/n}(x)$ for every $x \in A_n$. It is then easy to show that the diagonal sequence $[A_n^n]$ converges to the intersection set $A$ of the sets $A_n$.

**Theorem 2.2.** *If $P$ is any finite intersection property, then in a compact metric space $X$ the property $T_p$ of being the intersection of a monotone decreasing sequence of compact sets each having property $P$ is inductive.*

That $P$ is a finite intersection property means that the intersection of any two sets having property $P$ is either empty or has property $P$. To prove that $T_p$ is inductive, let $A_1 \supset A_2 \supset A_3 \supset \ldots$ be a sequence of non-empty compact sets in $X$ each having property $T_p$ and let $A = \bigcap A_n$. We have to show that $A$ is the intersection of a monotone decreasing sequence of sets having property $P$.

To that end again for each $n$ we take a sequence $A_1^n \supset A_2^n \supset A_3^n \supset \ldots$ of sets with intersection $A_n$ and each of which has property $P$. Further we may take these sequences such that, for each $n$, $A_1^n$ (and thus $A_k^n$) lies in the $1/n$-sphere about $A_n$. Then if we define

$$B_1 = A_1^1, B_2 = A_1^1 \cdot A_2^2, B_3 = A_1^1 \cdot A_2^2 \cdot A_3^3, \ldots$$

it is apparent that $B_1 \supset B_2 \supset B_3 \supset \ldots$ and that $\bigcap B_n = A$. Also since $P$ is a finite intersection property, each of the sets $B_n$ has property $P$. Thus $B$ has $T_p$.

**3. Inward Moving $A$-sets.** A non-empty closed subset $A$ of a connected space $X$ is an *$A$-set* provided $X - A$ is the union of a collection of open sets each of which has a single point of $A$ as its boundary. In case $X$ is locally connected, this is equivalent to saying that each component of $X - A$ has a single point of $A$ as its boundary.

It is known [6] that in any connected space $X$ the intersection of any two $A$-sets is either empty or an $A$-set. In case $X$ is also locally connected and Hausdorff, the same holds for arbitrary collections of $A$-sets, i.e., the intersection is either empty or an $A$-set. Further, in a space which is connected and locally connected, every $A$-set is not only itself connected and locally connected but it meets every connected subset of the space in a set which is either empty or connected.

Now let $f$: $X \to X$ be a function where $X$ is connected and locally connected. An $A$-set $A$ in $X$ will be said to *move inward* under $f$ provided that for each $p \in A$, $f(p)$ lies in no component of $X - A$ having $p$ as its boundary. Thus for each such $p \in A, f(p) \subset A + U$, where $U$ is the union of all components of $X - A$ with boundary points in $A - p$.

**Remark 3.1.** *In any connected space, any connected set $N$ which meets each of two intersecting $A$-sets $A_1$ and $A_2$ also meets their intersection.*

For let $Q$ be an open set in $X - A_1$ containing a point $a_2$ of $N \cdot A_2$ and having a single point $q$ of $A_1$ as its boundary. (If no such $Q$ existed we would have $N \cdot A_2 \subset \subset A_1$). Then $N$ meets both $Q$ and $X - Q$ and thus contains $q$ so that $q \in N \cdot A_1$. However $A_2$ also must contain $q$. For if not, $q$ lies in an open set $R \subset X - A_2$ whose boundary is a single point $r$ of $A_2$; and this is not possible because $X - R$ would be connected and would meet both $Q$ (in $a_2$) and $X - Q$ (in $A_1 \cdot A_2$) without containing $q$. Thus $q \in N \cdot A_1 \cdot A_2$.

**Theorem 3.2.** *If each of two intersecting $A$-sets $A_1$ and $A_2$ in a connected and locally connected space $X$ moves inward under a function $f$: $X \to X$, so also does their intersection $A$.*

Proof. Suppose, on the contrary that for some component $N$ of $X - A_1 \cdot A_2$ with boundary $p$ in $A$ we have $f(p) \in N$. Then neither $N \cdot A_1$ nor $N \cdot A_2$ can be empty as otherwise $N$ would be in a component $Q$ of, say $X - A_1$, bounded by $p$ and $f(p) \in N \subset Q$. Thus by the Remark 3.1 we have $N \cdot A_1 \cdot A_2 \neq \emptyset$, which is absurd since $N \subset X - A_1 \cdot A_2$.

**4. Stabilized Cyclic Elements.** Now suppose the space $X$ is a Peano continuum (i.e., a connected and locally connected compact metric space). A *true* $A$-set of such a space is an $A$-set which is either non-degenerate or else is a cut point or an end point of $X$. Since as just shown the property $A$ of being an inward moving $A$-set is a finite intersection property, and since clearly the same holds for true $A$-sets, by § 2 the property $T_a$ of being the intersection of a monotone decreasing sequence of inward moving (true) $A$-sets in $X$ is inductive. Accordingly, by the Brouwer Reduction Theorem, any inward moving (true) $A$-set in $X$ contains a (true) $A$-set which is irreducible relative to the property $T_a$ of being such an intersection. In this connection note that any non-empty intersection of (true) $A$-sets is an (true) $A$-set. Thus we have proved the first part of

**Theorem 4.1.** *Any inward moving $A$-set $A$ in a Peano continuum $X$ under a function $f$: $X \to X$ contains a fixed point or a true $A$-set $E$ irreducible relative to the property of being the intersection of a monotone decreasing sequence of inward moving true $A$-sets. Further, if $E$ is non degenerate, it can have no cut point.*

To prove the last statement, let $A_1 \supset A_2 \supset A_3 \supset \dots$ be a sequence of inward moving $A$-sets with intersection $E$ and suppose, contrary to our conclusion, that $E$ has a cut point $p$. Since $E$ is non degenerate, $f(p) \neq p$. Let $H = Q + p$, where $Q$ is the component of $X - p$ containing $f(p)$.

It is readily seen that $H$ is an inward moving $A$-set under $f$; and thus, by § 3, so also is $H \cdot A_n$ for every $n$. Since $H \cdot A_1 \supset H \cdot A_2 \supset \dots$ it follows that $\bigcap H \cdot A_n = H \cdot \bigcap A_n =$ $= H \cdot E$ is an $A$-set having property $T_a$. This is impossible because $H \cdot E$ is a proper subset of $E$. Note: $E$ cannot lie in $H = Q + p$ because $Q \cdot E$ is connected.

Now since $X$ itself is an inward moving true $A$-set under any $f$: $X \to X$, we get

**Theorem 4.2.** *If $X$ is a Peano continuum, then for each function $f$: $X \to X$ there exists a cyclic element $E$ of $X$ such that for each $\varepsilon > 0$ there is an inward moving $A$-set $A_\varepsilon$ in $X$ with $E \subset A_\varepsilon \subset V_\varepsilon(E)$.*

This results at once from Theorem 4.1 together with the facts (i) any non-degenerate $A$-set in $X$ with no cut point is a true cyclic element, and (ii) any single point of $X$ which is contained in arbitrarily small true $A$-sets in $X$ is either a cut point or an end point of $X$ and thus is itself a (degenerate) cyclic element of $X$.

**Theorem 4.3.** *Let $f$: $X \to X$ be any function where $X$ is a Peano continuum. Either some true cyclic element $E$ of $X$ moves inward under $f$ or else some cut point or end point $p$ of $X$ is the intersection of a monotone decreasing sequence of inward moving $A$-sets.*

Proof. Suppose no true cyclic element moves inward. Let $E = \bigcap_1^\infty A_n$ where $A_1 \supset A_2 \supset A_3 \supset \ldots$ is a sequence of inward moving true $A$-sets chosen so that $E$ is irreducible relative to the property of being such an intersection. Then if $E$ is not a cut point or an end point, it is a true cyclic element and thus it does not move inward. Hence for some $p \in E$, $f(p)$ lies in a component $Q$ of $X - E$ bounded by $p$. Then since, for each $n$, both $Q + p$ and $A_n$ move inward, so also does $A_n \cdot (Q + p) = B_n$. Clearly $p = \bigcap_1^\infty B_n$.

## 5. Connectedness Preserving Functions.

**Theorem 5.1.** *Suppose the function $f: X \to X$ preserves connectedness, where $X$ is a connected and locally connected space, and let $A$ be any inward moving $A$-set in $X$. Then either some point of $A$ is fixed under $f$ or $A \cdot f(A)$ is non-degenerate.*

Proof. Suppose no point of $A$ is fixed. Take $p \in A$. Let $q = f(p)$ in case $f(p) \in A$ and $q = \partial(Q)$ where $Q$ is the component of $X - A$ containing $f(p)$ otherwise. Similarly let $r = f(q)$ or the boundary of the component $R$ of $X - A$ containing $f(q)$ according as $f(q)$ is or is not in $A$. Then in any case $f(A)$ contains $q$ because if $f(p) \in Q$, $f(q)$ is not in $Q$ and $f(A)$ is connected. Similarly $f(A)$ must contain $r$. Finally $r \neq q$ because, in the first case, no point of $A$ is fixed and in the second, $f(r)$ is not in $R$.

**Theorem 5.2.** *Let $f: X \to X$ preserve connectedness where $X$ is a Peano continuum. Then either some inward moving true cyclic element of $X$ meets its image in a non-degenerate set or some cut point or end point of $X$ is the intersection of a monotone decreasing sequence of $A$-sets each moving inward and each meeting its image.*

This is a direct consequence of Theorems 5.1 and 4.3. In case $f$ is continuous it is clear that any such cut point or end point would be fixed under $f$. Thus we have

**Corollary 5.3.** *Under any mapping of a Peano continuum $X$ into itself either some cut point or end point is fixed or else some inward moving true cyclic elements meet its image in a non-degenerate continuum* [see ref. 7]. *In particular, every dendrite has the fixed point property for mappings* [1, 5].

The fixed point property for dendritics is also a direct consequence of

**Corollary 5.4.** *If $f$ is any connectedness preserving function of a Peano continuum $X$ into itself, then for each $\varepsilon > 0$ some point of $X$ moves a distance $< \Delta + \varepsilon$ where $\Delta$ is the supremum of the diameters of the true cyclic elements of $X$.*

**6. Functions with Certain Continuous Restrictions.** For any connected space $X$ and any pair of points $a$, $b$ in $X$, $E(a, b)$ denotes the set of all points of $X$ each of which separates $a$ and $b$ in $X$ and $K(a, b) = a + b + E(a, b)$. It is known [6] that in case $X$ is also locally connected and satisfies a weakened Hausdorff type axiom, then $K(a, b)$ is always closed and compact.

**Theorem 6.1.** *Given $f: X \to X$ where $X$ is a connected and locally connected Hausdorff space. If $f \mid K(a, b)$ is continuous for each $a, b \in X$, any point $q$ of $X$ which lies in an "arbitrarily small" inward moving $A$-set is a fixed point.*

For suppose $f(q) = r \neq q$. Take a region $R$ about $r$ with $\bar{R} \subset X - q$. Since $f \mid K(q, r)$ is continuous, by hypothesis there exists an inward moving $A$-set $A$ with $q \in A \subset X - \bar{R}$ and such that

$$f[A \cdot K(q, r)] \subset R .$$

Then the boundary point $x$ of the component $Q$ of $X - A$ containing $R$ belongs to $A \cdot K(q, r)$ but $f(x) \in Q$. This contradicts the fact that $A$ moves inward under $f$.

**Theorem 6.2.** *Any function $f: D \to D$ of a dendrite into itself whose restriction to each simple arc in $D$ is continuous has a fixed point.*

We recall that a dendrite is a Peano continuum containing no simple closed curve. It follows by Theorem 4.3 that some point $q$ of $D$ lies in an arbitrarily small inward moving $A$-set in $D$. Thus by Theorem 6.1 any such point $q$ must be fixed under $f$.

Examples are easily constructed of mappings of a dendrite $D$ into itself which have continuous restrictions on all arcs of $D$ but which are not continuous. Thus Theorem 6.2 is a substantial extension of the classical result that any dendrite has the fixed point property for (continuous) mappings.

Now let $X$ be a connected and locally connected Hausdorff space. For any $A$-set $A$ in $X$ let $r: X \to A$ denote the unique retraction of $X$ onto $A$ obtained by mapping each $x \in A$ into itself and each $x$ not in $A$ into the boundary point of the component of $X - A$ containing $x$. This function is continuous and is monotone in the sense that $r^{-1}(y)$ is connected for each $Y \in A$; and as such it is uniquely determined by these two properties.

**Remark 6.3.** *If $f: X \to X$ is any function, then for any inward moving $A$-set $A$ in $X$, any point which is fixed under $rf: A \to A$ is fixed under $f$.*

For if $x$ is any point of $A$ with $f(x) \neq x$, $r f(x)$ is either $f(x)$ or the boundary point of the component of $X - A$ containing $f(x)$. In either case $r f(x) \neq x$, since $A$ moves inward under $f$.

**Definition.** *We will call a class $\mathscr{F}$ of functions of a Peano continuum $X$ into itself r-invariant provided that for each $f \in \mathscr{F}$ we have $rf \in \mathscr{F}$ and $rf \mid E \in \mathscr{F}$ for each monotone retraction $r$ of $X$ into a true cyclic element $E$ of $X$.*

We note that the class of continuous functions and also the class of connectedness preserving functions is $r$-invariant, as is also the class of functions with continuous restrictions to sets $K(a, b)$ in $X$ and other classes to be identified later.

**Theorem 6.4.** *Let $\mathscr{F}$ be an r-invariant class of functions of a Peano continuum $X$ into itself. Suppose each true cyclic element $E$ of $X$ has the fixed point property relative to restrictions to $E$ of functions $f \in \mathscr{F}$ which map $E$ into itself. Then any function $f$ of $\mathscr{F}$ which is continuous on its restrictions to sets $K(a, b)$ in $X$ has a fixed point.*

This is a direct consequence of Theorems 4.3 and 6.1 together with Remark 6.3. For Theorems 4.3 and 6.1 give either a fixed cut point or end point or else an inward moving true cyclic element $E$. In the latter case, we have $rf \mid E \in \mathscr{F}$ for the monotone retraction $r$ of $X$ onto $E$. Thus $rf \mid E$ has a fixed point $x$; and by Remark 6.3 $x$ is also fixed under $f$.

**Corollary 6.5.** *The fixed point property for continuous functions is cyclicly extensible (and reducible)* [1].

**7. Connectivities on Dendrites.** A function $f\colon X \to Y$ is a *connectivity* provided its graph function $g\colon X \to X \times Y$ defined by

$$g(x) = [x, f(x)] \in X \times Y$$

preserves connectedness. Thus the graph $\Gamma(f \mid C)$ of every restriction of $f$ to a connected set $C$ in $X$ is a connected set. It is known [3, 4] that the $n$-cell has the fixed point property for connectivities for all $n \geqq 0$. We proceed now to extend this result to dendrites with countably many end points.

**Definition.** If $f\colon D \to D$ is a function where $D$ is a dendrite, and $a \in D$, then a point $x \in D$ is said to *move toward* $a$ under $f$ provided $f(x)$ lies in the component of $D - x$ which contains $a$.

**Theorem 7.1.** *Let $f\colon D \to D$ be a connectivity where $D$ is a dendrite. If each of two points $a$ and $b$ of $D$ moves toward the other under $f$, then uncountably many points on the arc $ab$ of $D$ move toward $a$ and uncountably many toward $b$ under $f$.*

Let $a_1$ be an interior point of the arc which the arcs $ab$ and $af(a)$ have in common and let $U$ and $Q$ be the components of $D - a_1$ containing $a$ and $b$ respectively. Then $Q \supset f(a)$ and $U \times Q$ is a neighbourhood of $[a, f(a)]$ in $D \times D$. Since the graph $\Gamma$ of $f \mid ab$ is a non-degenerate connected subset of $D \times D$ containing $[a, f(a)]$, it follows that $\Gamma . (U \times Q)$ must be uncountable. This means that $[x, f(x)] \in U \times Q$, i.e., $f(x) \in Q$, for uncountably many $x \in abU$. Any such $x \in ab$ moves toward $b$ under $f$. Similarly, uncountably many points of $ab$ move toward $a$ under $f$.

For any dendrite $D$ let $B(D)$ be the least subdendrite of $D$ containing all branch points of $D$, setting $B(D) = D$ in case $D$ is an arc or a point. For any ordinal $\alpha$ set $D_\alpha = B_\alpha(D) = B(D_{\alpha-1})$ if $\alpha - 1$ exists, and $D_\alpha = \bigcap_{\beta<\alpha} D_\beta$ in case $\alpha$ is a limit ordinal. Now in case the end points of $D$ are countable, we have $D_{\alpha+1} \neq D_\alpha$ so long as $D_\alpha$ has branch points. Accordingly, for a given dendrite $D$ with countably many end points there exists a least $\alpha < \Omega$ such that $D_\alpha$ has no branch points and thus is an arc or a single point. We set $\alpha = \alpha(D)$ and call this the *branching order* of $D$.

Now let $f$: $D' \to D'$ be a connectivity where $D'$ is a dendrite with countably many end points. Let $D$ be a subdendrite of $D'$ which moves inward under $f$ and has the least possible branching index $\alpha(D)$.

(*) *For each* $\beta \leqq \alpha(D)$, $D_\beta = B_\beta(D)$ *moves inward under* $f$.

For suppose not. Let $\beta$ be the least ordinal so that $D_\beta$ does not move inward. Then for some component $Q'$ of $D' - D_\beta$ with boundary point $b$ we have $f(b) \in Q'$. Note that $Q = Q' \cdot D \neq \emptyset$. Then if $a$ is an end point of $D$ in $Q$, $a$ and $b$ move toward each other in $D'$. Thus by Theorem 7.1 there is a non-branch point $b_1$ of $D'$ interior to the arc $ab$ of $D$ which moves toward $a$ under $f$. Then if $K$ is the closure of the component of $Q - b_1$ containing $a$, $K$ moves inward under $f$. Further since $K \subset Q \subset D - D_\beta$, the branching index $\alpha(K)$ is $< \beta \leqq \alpha(D)$. This contradicts the minimal character of $\alpha(D)$ in $D'$. Thus (*) is proved.

Now since (*) holds for $\beta = \alpha = \alpha(D)$, we have that $D_\alpha$ moves inward under $f$. Since $D_\alpha$ is a simple arc or a single point, it follows that some point of $D_\alpha$ is fixed under $f$. Thus we have proved

**Theorem 7.2.** *Any dendrite with countably many end points has the fixed point property for connectivities.*

Note 1. For a connectivity $f$: $D \to D$ ($D$ a dendrite), any point of $D$ which moves inward is a fixed point. Also any simple arc $ab$ in $D$ which moves inward contains a fixed point of $D$. For if $r$: $D \to ab$ is the monotone retraction, $rf$: $ab \to ab$ is a connectivity and thus $r f(p) = p$ for some $p \in ab$. However $r f(p)$ is the boundary of the component of $D - p$ containing $f(p)$ when $f(p)$ is not in $ab$; and thus $r f(p)$ can be $p$ only in case $f(p) \in ab$, in which case $r f(p) = f(p) = p$.

Note 2. Theorem 7.2 includes the result that a dendrite with a reducible set of branch points has this same fixed point property. For if the branch points form a reducible set, the end points must be countable. To see this note that if the set $H$ of end points is uncountable, it contains a perfect set $P$; and since every end point which is a limit point of end points is also a limit point of branch points, it follows that the first derived set (and thus every derived set) of the set of branch points contains the set $P$.

Note 3. Every continuum of cut points of any connected separable metric space is a dendrite with countably many end points.

**8. Partially Continuous Function Types.** A function $f\colon X \to Y$ is *peripherally continuous* at $x \in X$ provided that if $U$ and $V$ are open sets about $x$ and $f(x)$ respectively, there exists an open set $W$ with $x \in W \subset U$ and $f(\partial W) \subset V$, where $\partial W$ is the boundary of $W$. It is known [2, 8] that on many domain spaces $X$, including the $n$-cells for $n > 1$, the peripherally continuous functions coincide with the connectivities. However this is not the case on the 1-cell and other dendritic type structures.

Here we consider briefly the following classes of functions:

(a) Connectedness preserving.
(b) Connectivities.
(c) Peripherally continuous functions.

It is clear that all three of these types are *hereditary* in that any restriction of any one of them to a sub-domain is a function of the same type. Also it is easily seen that each of these classes is $r$-invariant in the sense described in § 6. Thus the fixed point Theorem 6.4 applies to each of these classes.

Now let $X$ be a Peano continuum. If $f: X \to Y$ is a function where $Y$ is a completely normal space, as noted above if $f$ is of type (a), (b) or (c) on $X$ the same holds in particular for $f \mid E$ where $E$ is any true cyclic element of $X$. Thus we have cyclic reducibility of the properties involved in defining these types. Cyclic extensibility of these properties does not always hold. However, under the additional condition that $f$ have continuous restrictions to all sets $K(a, b)$ in $X$, it is not difficult to see that cyclic extensibility holds in each case. That is, *if $f \mid E$ is of type (a), (b) or (c) for each true cyclic element $E$ of $X$ and if $f \mid K(a, b)$ is continuous for each pair $a$, $b \in X$, then $f$ is of type (a), (b) or (c) respectively.*

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