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MONOTONE MAPPINGS AND CELLULARITY OF ORDERED SETS

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0. Notations and terminology.

(0-1) For any ordered set $(O, \leq)$ and any x we put $O(.,x):=\{y \mid y \in O, y \leq x\}$, $O(x,.):=\{y \mid y \in O, x < y\}:=(x,.)_O$, $O[x,.):=\{y \mid y \in O, x \leq y\}$; $O[x]:=\{y \mid y \in O, y \leq x \vee y \geq x\}$.

(0-2) $(T, \leq)$ is a ramified table or a tree iff for every $x \in T$, $(T(.,x), \leq)$ is well-ordered. For every ordinal α, $R_\alpha T:=\{x \mid x \in T, (T(.,x), \leq)$ has the order type $\alpha\}$; γT (or $\gamma(T, \leq)$) is the first ordinal α such that $R_\alpha T = \emptyset$. $(T, \leq)$ is a ramified sequence iff $\gamma T[x] = \gamma T$ for every $x \in T$.

(0-3) If in a subset S of $(O, \leq)$ the comparability relation is transitive, S is called a D-subset of $(O, \leq)$. $(O, \leq)$ is D-reflexive iff $(O, \leq)$ contains a D-subset, Od, of power $|O|$.

(0-4) An increasing mapping f of $(O, \leq)$ is almost strictly increasing iff for every $x \in O$ having at least one successor in $(O, \leq)$ there is some $y \in O$ such that $x < y$ and $fx < fy$.

(0-5) Since the separability number sC, the cellularity number cC is the same for C and its Dedekind completion we shall assume, if not stated otherwise, that $(C, \leq)$ is without gaps.

(0-6) For any system S of intervals of $(C, \leq)$ we denote by eS the system of all endpoints of members of S.

1. Theorem. Let T be a ramified sequence such that $\mathrm{cf}\, \gamma T = \omega_{\alpha+1}$; if there is an almost strictly increasing mapping f of T into an $\aleph_\alpha$-separable chain C, then T is D-reflexive (for $\alpha = 0$, v. Théorème fondamental in Đ. Kurepa (1937) p. 1035 and (1941) p. 493).

Proof. Let us consider the critical case (cf. Đ. Kurepa (1935) p. 108/9, §3) that T is a sequence of rank γ with $\tau\gamma = \omega_{\alpha+1}$ and that every level $R_\alpha T$ $(\alpha < \gamma T)$ is $\leq \aleph_\alpha$. Let f be any almost strictly increasing mapping of T into an $\aleph_\alpha$-separable chain C; let W be a subset of cardinality $\aleph_\alpha$ and everywhere dense on $(C, \leq)$. Let

(1-1) $g: T \to T$ be a mapping of T into itself such that

(1-2) $a \in T \Rightarrow a < g(a)$ and $f(a) < f(g(a))$. For any $w \in W$ let

(1-3) $T^w := \{a \mid a \in T,\ f(a) \leq w < f(gg(a))\}$.

(1-4) Lemma. $\bigcup_w T^w = T$, $(w \in W)$.

Proof. Let $a \in T$; then $C(a, gg(a))$ is an open interval of C;

this interval is non-empty (e.g. it contains the point $f(a)$; therefore the non-empty open interval $C(a,gg(a))$ of C contains at least one point w of W ; by definition of T , we have $a \in T^w$.

(1-5) Lemma. There exists some $n \in W$ such that $|T^n| = |T|$, and
(1-5-1) $f(a) \leq n < f(gg(a))$ for every $a \in T^n$.

The lemma is the consequence of Lemma (1-4). At first, the relation (1-5-1) is true by definition of T^n ; consequently, if Lemma (1-5) was false, this would mean that $|T^w| < |T|$ $(w \in W)$ what joint to $\sum_W |T^w| \geq T$ (implied by Lemma (1-4)) would mean that $|W| \geq cf|T| \geq \aleph_{\alpha+1}$ -contradicting the assumption that $|W| = \aleph_\alpha$.

(1-6) Lemma. T contains an antichain A of cardinality $cf|T|$ $(= \aleph_{\alpha+1})$.

Proof. Again, we can assume that T^n is a nice ramified sequence of hight $\gamma := \gamma T^n = \omega_\sigma$ and such that $|R_\alpha T^n| < \aleph_\sigma$ $(\alpha < \gamma T^n)$.
Let τ be the least initial ordinal cofinal to $\gamma T := \omega_\sigma$. Let us define a τ -sequence

(1-6-1) $a_0, a_1, \ldots, a_i, \ldots$ $(i < \tau)$ of pairwise distinct points of T^n such that the numbers

(1-6-2) α_i, β_i defined by $a_i \in R_{\alpha_i} T^n$, $g(a_i) \in R_{\beta_i} T^n$ satisfy

(1-6-3) $\alpha_0 < \alpha_1 < \ldots < \alpha_i < \ldots \longrightarrow \gamma T^n$ $(i < \tau)$
(1-6-4) $\beta_i < \alpha_{i+1}$ $(i < \tau)$.

The existence of (1-6-1) satisfying (1-6-2), (1-6-3) is obvious by induction argument, because T was assumed to be a nice sequence. Further, the numbers (1-6-3) satisfy the following relations:

(1-6-5) $\alpha_i < \beta_i < \alpha_{i+1}$ $(i < \tau)$.

Let us prove that the points

(1-6-6) $g(a_0), g(a_1), \ldots, g(a_i)$ $(i < \tau)$

constitute an antichain in T , i.e. that

(1-6-7) $i < j < \tau \Rightarrow g(a_i) \parallel g(a_j)$.

At first, in virtue of (1-6-5) we infer that $\beta_i < \alpha_{i+1} \leq \alpha_j < \beta_j$, thus $\beta_i < \beta_j$; consequently, $\neg [g(a_j) = g(a_i)]$. On the other hand, if $g(a_i) < g(a_j)$, then since also $a_j < g(a_j)$ (v.(1-2)) these relations would imply that the points a_j, $g(a_i)$ as predecessors of the same point $g(a_j) \in T$, would be comparable: either

(1-6-8) $a_j \leq g(a_i)$ or
(1-6-9) $g(a_i) < a_j$. The case (1-6-8) does not hold, because (1-6-8) would imply $\alpha_j \leq \beta_i$, contrarily to $\beta_i < \alpha_j$. On the other side if (1-6-9) holds, then

$f(g(a_i)) \leq f(a_j)$; this relation joint to $f(a_j) \leq n < f(g^2(a_j))$

(cf. the definition of T^w) would yield

$f(g(a_i)) \leq n$, contrarily to the assumption $f(a_i) \leq n < f(g^2(a_i))$ for every ordinal $i < \tau$. Thus we established an antichain $\subset T$ of cardinality $|\tau(\gamma T)|$.

(1-7-4) Lemma. T^n is D-reflexive. Since, by hypothesis $\gamma[a,\cdot)_{Tn} = \gamma T^n$ for every $a \in T^n$, it is sufficient to consider any τ-sequence of cardinals k_i $(i < \tau)$ such that $\sum_i k_i = |T^n| = |T|$ and for any $i < \tau$ to consider in the upper cone $T^n[g(a_i),\cdot)$ a chain L_i of cardinality $\geq k_i$ (the existence of L_i is obvious because $\gamma T^n[a] = \gamma T^n$ $(a \in T^n)$, T^n being a ramified sequence). Then the union of the sets L_i is the requested D-subset of T of the cardinality $|T|$. Q.E.D.

2. Theorem. Any ramified decreasing table $(T, \supset)$ of intervals of $(C, \leq)$ such that $\overline{eT} = C$ satisfies $|T| = s(C, \leq)$. (cf. Đ. Kurepa (1935) p. 120 L. 3; also J. Novák (1952)b Th. 1.).

Proof. One has

(2-1) $|eT| = |T| = \sup_{i < \gamma T} |R_i T| \cdot |T| = \sup\{mT, |\gamma T|\}$, where $mT :=$ $= \sup_\xi |R_\xi T|$, $(\xi < \gamma T)$. (Cf. Đ. Kurepa (1935) p. 74 § 10.)
On the other hand, eT being everywhere dense on $(C, \leq)$ one has $eT =$ $= sC$ .Consequently, $|T| = sC$ and in virtue of (2-1):
$\sup\{mT, |\gamma T|\} \geq sC$. In other words,

(2-2) $\overline{eT} = (C, \leq)$ implies $\sup\{mT, |\gamma T|\} = sC$.

Now, we have the following two lemmas:

(2-3) Lemma. $s(C, \leq) = mT$ for every T

(2-4) Lemma. $s(C, \leq) = |\gamma T|$ for every T .

The lemmas (2-3), (2-4) imply

(2-5) $s(C, \leq) \geq \sup\{mT, |\gamma T|\}$ for every T , in particular for every T satisfying $\overline{eT} = C$. The relations (2-2), (2-5) yield

(2-6) $s(C, \leq) = \sup\{mT, |\gamma T|\}$ for every $(T, \supset)$ such that $\overline{eT} = C$; therefore also $s(C, \leq) = |T|$ for every $(T, \supset)$ satisfying $\overline{eT} = C$.
Theorem 2 is completely proved.

(2-7) We have still to prove Lemmas (2-3), (2-4). The first one being obvious, let us prove the second one. Now, Lemma (2-4) is obvious if $mT > |\gamma T|$ or if $mT = |\gamma T|$ and γT is not an initial ordinal number. Therefore let us consider the following case

(2-8) $mT \leq \aleph_\alpha$, $\gamma T = \omega_\alpha$.

(2-9) One has not $s(C, \leq) < |\gamma T|$.

In opposite case, there would be a subset M of $(C, \leq)$ such that $\overline{M} = C$, $|M| < \aleph_\alpha$; now, let $x \in M$; there would be an index $i(x) < \gamma T$ and some

X such that

(2-10) $x \in X \in R_{i(x)}T$ (in the opposite case, there would be some $X_j \in R_j T$ such that $x \in X_j$ for every $j < \omega_\alpha$ what would imply that $(x_j)_{j<\omega_\alpha}$ would be a strictly decreasing ω_α-sequence of intervals of $(C, \leq)$, in contradiction with the assumption $s(C, \leq) < \aleph_\alpha$).

The relation (2-10) being established, we have the following two cases:

(2-11) First case: $\gamma T (= \omega_\alpha)$ is regular. Since by hypothesis $|M| < \aleph_\alpha$ and $i(X) < \omega_\alpha$ for every $x \in M$, then the ordinal $\beta := \sup i(x)$ $(x \in M)$ would be $< \omega_\alpha$ - impossibility, because no interval $I \in R_\beta T$ contains any point of M, which is supposed to be everywhere dense in $(C, \leq)$.

(2-12) Second case: $\gamma T (= \omega_\alpha)$ is singular. This case is not possible either because by assumption $|M| < \aleph_\alpha$ there would be some ordinal i such that $|M| < |i|$ and $i < \omega_\alpha$; therefore, for any $B \in R_i T$ the system $(\cdot, B)_T$ of all members u of T such that $u \supset B$ would be a strictly decreasing i-sequence of intervals of $(C, \leq)$, in contradiction with $|i| > s(C, \leq)$.

Consequently, the relation (2-9) is not possible which proves that Lemma (2-4) is true.

3. <u>Theorem</u>. Every totally ordered infinite set $(C, \leq)$ satisfies $s(C, \leq) = \sup_T |T|$, T being a ramified table of decreasing non overlapping subintervals of $(C, \leq)$. (Cf. Đ. Kurepa (1935) p. 120 § 12. C. 3; also J. Novák (1952) b Th. 1.)

<u>Proof</u>. In order to prove Theorem 3 let us prove the following

(3-1) <u>Lemma</u>. If $(C, \leq)$ is any ordered chain and D any disjoint system of non-void intervals, there is a disjoint system D^+ of disjoint intervals of $(C, \leq)$ such that $D^+ \supset D$ and $\bigcup D^+$ is everywhere dense in $(C, \leq)$.

The proof is obvious: if $B := \bigcup D$ is everywhere dense, we set $D^+ := D$. If B is not everywhere dense, we have the complement $K(D) := C \setminus B$ and the partition $p(K)$ of $K(D)$ into maximal convex subsets X of $(C, \leq)$ satisfying $\operatorname{int} X \neq \emptyset$. For every $X \in pK(D)$, let $\pi(X)$ be any partition of X into disjoint non empty intervals; then we define

$$D^+ := D \cup \bigcup \pi(X) \quad (X \in p\ K(D)).$$

One proves readily that D^+ satisfies the conditions in (3-1).

(3-2) Let us now prove Theorem 3: T being as in 3 let us determine a table T^+ of intervals of $(C, \leq)$ such that $T^+ \supset T$ and the set $e(T^+)$ of end points of members of T^+ is everywhere dense in $(C, \leq)$. To start with, let $T = \bigcup_i R_i T$ $(i < \gamma T)$ be the disjoint partition of

$(T,\supset)$ into rows or levels of $(T,\supset)$. We put $T'_o := (R_oT)^+$, $T'_1 = (R_1T \cup (T'_o \setminus R_oT))^+$ (cf. (3-1)). Let $0 < j < \gamma T$ and assume that the ramified table

(3-3) $\bigcup_i T'_i \quad (i < j)$

is defined and that γ (3-3) = j , $R_i((3\text{-}3)) = T'_i \quad (i < j)$. Let us define T'_j as well. If j is a limit ordinal, we define T'_j to consist of all members of T_j and of all sets of the form $\operatorname{int} \bigcap X_i \quad (i < j)$, where $X_o \supset X_1 \supset \ldots \supset X_i \supset \ldots$ is a strictly decreasing sequence of convex parts of $(C,\leq)$ such that $X_i \in T'_i \quad (i < j)$ and for some $i < j$ one has $X_i \in T'_i \setminus T_i$. If $j^- < j$, we define

$$T'_j := T_j \cup (T'_{j^-} \setminus T_{j^-}) .$$

(3-4) Let us define $V := \bigcup_i T'_i \quad (i < \gamma T)$.
Then obviously, $V \supset T$.

(3-5) If the set eV of endpoints of V is everywhere dense, then $|eV| = s(C,\leq)$ and since $|eV| = |V| \geq |T|$, the theorem would be proved. If the set eV is not everywhere dense in $(C,\leq)$ we extend V and define T^+ as follows:
Let us consider the set MV of all maximal chains of $(V,\supset)$; for every $X \in MV$ let $i(X) := \operatorname{int} \bigcap y \quad (y \in X)$. Then i(X) is a convex subset of $(C,\leq)$; for every i(X) having at least 2 points, let t(iX) be a complete ramified table of subintervals of i(X) (thus in particular et(i(X)) is everywhere dense in i(X)) ; finally, we define

(3-6) $T^+ := V \cup \bigcup_X t(i(X)) , \ (X \in MV)$.

Then obviously, $T^+ \supset V \supset T$ and $\overline{eT^+} = (C,\leq) = s(C,\leq)$.

(3-7) <u>Corollary</u>. Every ordered chain $(C,\leq)$ satisfies

(3-7-1) $s(C,\leq) = \sup_T\{mT, |\gamma T|\}$, (cf.(2-1)).

(3-7-2) $s(C,\leq) = \sup_T\{p_sT, |\gamma T|\}$,

(3-7-3) $s(C,\leq) = \sup\{c(C,\leq), \sup_T |\gamma T|\}$,

T running over the system of all ramified decreasing tables of convex subsets of $(C,\leq)$, and where for any partially ordered set $(E,\leq)$ we put $p_s(E,\leq) := \sup|I|$, I running through the system of all antichains (independent or free sets) $\subset (E,\leq)$ (cf. Đ. Kurepa (1937) p. 1196/7 relation fondamentale; v. also (1939) p. 62, (1959) p. 205); s in p_sE is the initial character of slavic words svobodno or slobodno (=free).

4. <u>Theorem</u>. Let α be any ordinal number, and $(C,\leq)$ be any ordered chain of celullarity $\aleph_\alpha$, i.e. $c(C,\leq) = \aleph_\alpha$; then

(4-1) $s(C,\leq) = c(C,\leq) \Leftrightarrow$ for every ramified table $(T,\supset)$ of intervals of $(C,\leq)$ there is an isotone mapping $i: T \to I(\omega_\alpha)$ and an ordinal $\beta < \omega_{\alpha+1}$ such that for every $x \in iT$ one has $\gamma_i^{-1}\{x\} \leq \beta$.

Proof.

(4-2) Necessity. Since, by hypothesis $(4\text{-}1)_1$, $s(C,\leq) = \aleph_\alpha$, one has necessarily $\gamma(T,\supset) < \omega_{\alpha+1}$ (cf. (3-7)); therefore, it is sufficient to consider the constant mapping $i(x) = 0$ for every $x \in T$ to see that one has an isotone mapping of $(T,\supset)$ into $I(\omega_\alpha)$ with properties requested in $(4\text{-}1)_2$.

(4-3) Sufficiency. Let now $(T,\supset)$ be any ramified table of intervals of $(C,\leq)$ such that eT is everywhere dense in $(C,\leq)$; in virtue of Theorem 2 we have

(4-4) $s(C,\leq) = |T|$.

(4-5) Again, $|T| = \sup\{mT, |\gamma T|\}$; therefore, if $mT \geq |\gamma T|$, then the last supremum equals mT, and consequently $|T| = mT$; therefore (4-4) yields $s(C,\leq) = mT$; this relation joint to $s(C,\leq) = c(C,\leq) = \aleph_\alpha = mT$ would imply the requested equality $(4\text{-}1)_1$. Therefore let us still consider the case that

(4-6) $mT < |\gamma T|$.

We claim that

(4-7) $|T| = \aleph_\alpha$ $(= c(C,\leq))$, which jointly to (4-4) implies the requested equality $(4\text{-}1)_1$. In the opposite case, either $|T| < \aleph_\alpha$ or $|T| > \aleph_\alpha$. The relation $|T| < \aleph_\alpha$ is not possible, because one has $\aleph_\alpha = c(C,\leq) = s(C,\leq) = |T|$, and thus $\aleph_\alpha \leq |T|$. Consequently, there would be

(4-8) $|T| > \aleph_\alpha$, and by (4-6)

(4-9) $\gamma T \geq \omega_{\alpha+1}$.

The relation $\gamma T > \omega_{\alpha+1}$ is impossible (in the opposite case, any $x \in R_{\omega_{\alpha+1}}(T,\supset)$ would yield the corresponding $\omega_{\alpha+1}$-sequence of strictly decreasing intervals of $(C,\leq)$, contradicting the condition $c(C,\leq) = \aleph_\alpha$). Consequently, necessarily $\gamma T = \omega_{\alpha+1}$ and every chain in $(T,\supset)$ is $\leq \aleph_\alpha$.

(4-10). Now, let us consider the mapping i and the ordinal β occuring in $(4\text{-}1)_2$. Since

$T = \bigcup_y i^{-1}\{y\}$, $(y \in iT \subset I(\omega_\alpha))$, $|T| = \aleph_{\alpha+1}$, $|i^{-1}\{y\}| \leq \aleph_\alpha$,

we infer that some $y \in iT$ satisfies $|i^{-1}\{y\}| = \aleph_{\alpha+1}$ $(= |T|)$.

The set $(X,\supset)$, where $X := i^{-1}\{y\}$, would be a subtree of the tree $(T,\supset)$ of cardinality $\aleph_{\alpha+1}$ and of a rank $\gamma X \leq \beta$, where $\beta < \omega_{\alpha+1}$; thus $|\gamma X| < |X|$; now we have the following

(4-11) Lemma. Every infinite tree X satisfying $|X| > |\gamma X|$ is D-reflexive. (v. Đ. Kurepa (1935) p. 108/9 § 3, Th. 2). Therefore, X would contain a D-subset Y of cardinality $\aleph_{\alpha+1}$; the disjoint partition $Y = \bigcup Y[y,.)$ $(y \in R_0 Y)$ would be in contradiction with the fact that $|R_0 Y| \leq c(C,\leq) = \aleph_\alpha$ and $|Y(y,.)_y| \leq \aleph_\alpha$, every $Y(y,.)$

$(y \in Y)$ being a chain in $(Y, \supset)$.

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