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COVERING DIMENSIONS AND PARTITIONS OF UNITY

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Various covering dimensions are studied in topology. They are defined by means of the order of covers. In this note, another characterization of covering dimensions is given. Their values can be derived from a certain quantitative characteristic of partitions of unity, which is closely related to the Lebesgue number of uniform covers.

A partition of unity on a set $X$ is a family $\{f_a\}$ of non-negative real-valued functions on $X$ such that $\sum_a f_a(x) = 1$ for each $x$. Given such a partition of unity, put

$$\lambda\{f_a\} = \inf_{x \in X} \{\sup_a f_a(x)\} .$$

We say that the partition is subordinated to a cover $\mathcal{G}$, if the family $\{\operatorname{coz} f_a\}$ refines $\mathcal{G}$; recall that $\operatorname{coz} f = \{x \in X \mid f(x) \neq 0\}$. Partitions of unity with finite index sets are said to be finite. If $X$ is a topological space and $\{f_a\}$ is a partition of unity on $X$, then all functions $f_a$ are supposed to be continuous. Let $X$ be a uniform space. A partition $\{f_a\}$ of unity on $X$ is said to be uniform if all $f_a$ are uniformly continuous; it is said to be equiuniform if $\{f_a\}$ is a uniformly equicontinuous family of functions.

Let us start with the uniform covering dimensions (for the definitions see e.g. [3]).

<u>Theorem 1.</u> Let $X$ be a uniform space, $\Omega$ a basis for uniform covers of $X$. For any uniform cover $\mathcal{G}$ of $X$ and a cardinal number $\gamma > \operatorname{card} \mathcal{G}$, put $c(\mathcal{G}) = \sup \lambda\{f_a\}$ where the supremum is taken over all equiuniform partitions $\{f_a\}$ of unity on $X$ which are subordinated to $\mathcal{G}$, and $c'(\mathcal{G}) = \sup \lambda\{f_a\}$ where the supremum is taken only over those partitions for which the cardinality of their index set is less than $\gamma$. Then $c'(\mathcal{G}) = c(\mathcal{G})$ for each $\mathcal{G}$ and

$$\inf_{\mathcal{G} \in \Omega} c(\mathcal{G}) = \frac{1}{\Delta d\, X + 1} .$$

Remark. The formula above is to be understood with the usual conventions, like $1/\infty = 0$ etc. If $X$ is void one has to be careful giving sense the used symbols.

The proof of Theorem 1 will be omitted here. It is, in fact, identical with the proof of Theorem 1 in [2]. We present here only a sequence of lemmas, without proofs, which correspond to the main steps of the proof. Notice that the order of a collection $\mathcal{C}$ of sets is understood as the smallest number, denoted by $\operatorname{ord} \mathcal{C}$, such that the intersection of more than $\operatorname{ord} \mathcal{C}$ members of $\mathcal{C}$ is void.

Lemma 1. Let $\{U_a\}$ be a uniform cover of a uniform space $X$, $\operatorname{ord}\{U_a\}$ finite. Then there exists an equiuniform partition $\{f_a\}$ of unity on $X$ such that $\operatorname{coz} f_a < U_a$ for each $a$.

Lemma 2. Let $\{f_a\}$ be a partition of unity, $\operatorname{ord}\{\operatorname{coz} f_a\} \leq m$. Then $\lambda\{f_a\} \geq 1/m$.

Lemma 3. Let $\{f_a\}$ be an equiuniform partition of unity on a uniform space $X$ and let $\lambda\{f_a\} > 1/m$. Then there exists a uniform cover $\{V_a\}$ of $X$ such that $V_a < \operatorname{coz} f_a$ for each $a$ and $\operatorname{ord}\{V_a\} < m$.

Lemma 4. Let $\mathcal{G}$ be a uniform cover of a uniform space $X$. Let $m$ (finite or infinite) be the minimum of the orders of all uniform covers of $X$ which refine $\mathcal{G}$. Then (the notation from Theorem 1)

$$c'(\mathcal{G}) = c(\mathcal{G}) = 1/m .$$

To prove Lemma 4, it suffices to show that

$$1/m \leq c'(\mathcal{G}) \leq c(\mathcal{G}) \leq 1/m .$$

The second inequality is obvious. The first one follows from Lemmas 1 and 2, the third one follows from Lemma 3. Lemma 4 and the definition of the dimension immediately imply Theorem 1. The following theorem follows from Lemma 4 applied to finite covers only.

Theorem 2. Let $X$ be a uniform space. Let $\Omega_0$ be the collection of all finite uniform covers of $X$. For $\mathcal{G}$ in $\Omega_0$ let $c(\mathcal{G})$, $c_0(\mathcal{G})$ be the supremum of $\lambda\{f_a\}$ over all equiuniform, finite uniform, respectively, partitions $\{f_a\}$ of unity on $X$, which are subordinated to $\mathcal{G}$. Then

$$\inf_{\mathcal{G}\in\Omega_0} c(\mathcal{G}) = \inf_{\mathcal{G}\in\Omega_0} c_0(\mathcal{G}) = \frac{1}{\delta d\, X + 1} .$$

Now, let us characterize the topological covering dimension.

Theorem 3. Let $X$ be a normal topological space. Let $\Omega$ be the collection of all locally finite open covers of $X$ and $\Omega_0$ the collection of all finite open covers of $X$. For $\mathcal{G}$ in $\Omega$ let $c(\mathcal{G})$, $c_0(\mathcal{G})$ be the supremum of $\lambda\{f_a\}$ over all, all finite, respectively, partitions $\{f_a\}$ of unity on $X$ which are subordinated to $\mathcal{G}$. Then

$$\inf_{\mathcal{G}\in\Omega} c(\mathcal{G}) = \inf_{\mathcal{G}\in\Omega_0} c(\mathcal{G}) = \inf_{\mathcal{G}\in\Omega_0} c_0(\mathcal{G}) = \frac{1}{\dim X + 1} .$$

Theorem 3 can be proved directly in a similar way as Theorem 1 (see also [2]). However, it follows from Theorems 1 and 2 applied to $X$ endowed with the fine uniformity. We need only to know that any partition of unity is equiuniform. This is implied by the following lemma.

Lemma 5. Let $\{f_a\}$ be a family of continuous non-negative real-valued functions on a fine uniform space $X$ such that $g$ defined by $g(x) = \sum_a f_a(x)$ is a continuous function on $X$. Then $\{f_a\}$ is uniformly equicontinuous.

Proof. It suffices to prove that $\varphi$ defined by

$$\varphi(x,y) = \sum_a |f_a(x) - f_a(y)|$$

is a continuous pseudometric. Since $\varphi(x,y) \leqq \sum_a f_a(x) + \sum_a f_a(y) = g(x) + g(y)$, the value of $\varphi$ is always finite. The other properties of pseudometric being clear, it remains to prove the continuity of $\varphi$. Let $y \in X$, $\varepsilon > 0$. Choose a finite $K$ with $g(y) - \sum_{a\in K} f_a(y) < \varepsilon$. As $g$ and $f_a$ are continuous there is a neighbourhood $U$ of $y$ such that, for $x \in U$, $g(x) - \sum_{a\in K} f_a(x) < \varepsilon$ and $\sum_{a\in K} |f_a(x) - f_a(y)| < \varepsilon$. Hence $\varphi(x,y) \leqq \sum_{a\in K} |f_a(x) - f_a(y)| + \sum_{a\notin K} (f_a(x) + f_a(y)) < 3\varepsilon$.

A similar characterization of the covering dimension of non-normal spaces is also possible. However, let us consider, instead, another general concept of covering dimension including various special cases which was introduced and extensively studied by M. G. Charalambous.

Let $\mathcal{F}$ denote a uniformly closed ring of bounded real-valued functions on a set $X$ containing all constant functions. One may suppose that $X$ is a space and the functions are continuous but it is not essential here. The subsets of the form $\operatorname{coz} f$ for $f$ in $\mathcal{F}$ are called $\mathcal{F}$-open, their complements are called $\mathcal{F}$-closed. Finite intersections and countable unions of $\mathcal{F}$-open sets are also $\mathcal{F}$-open, etc. (see [1] for details). A cover consisting of $\mathcal{F}$-open sets is said to be $\mathcal{F}$-open. Further, $\mathcal{F}$-$\dim X \leqq n$ means each finite $\mathcal{F}$-open cover can be refined by a finite $\mathcal{F}$-open cover with the order at most $n + 1$. Of course, $\mathcal{F}$-dim $X$ is the least $n$ with this property, etc. This dimension can be also characterized by means of partitions of unity. We shall prove two lemmas which enable to get the desired result as a consequence of Theorem 1. Let $X$, $\mathcal{F}$ keep their meaning and let $I$ stand for the unit interval of the reals. A function $f : X \to I$ is called an $\mathcal{F}$-function if $f^{-1}[G]$ is $\mathcal{F}$-open for each open subset $G$ of $I$. A partition of unity on $X$ consisting of $\mathcal{F}$-functions will be called an $\mathcal{F}$-partition of unity.

Lemma 6. The collection of all finite $\mathcal{F}$-open covers of $X$ is a base of a uniformity on $X$.

Proof. If $\mathcal{G}$, $\mathcal{H}$ are finite $\mathcal{F}$-open covers of $X$ then the collection of all $G \cap H$ where $G \in \mathcal{G}$, $H \in \mathcal{H}$, is a finite $\mathcal{F}$-open cover, too. Let $\mathcal{G}$ be a finite $\mathcal{F}$-open cover and let us search for a star-refinement of $\mathcal{G}$. Let $\mathcal{G} = \{G_i \mid i \in M\}$ where $M$ is a finite non-void set. Notice that $\mathcal{F}$ is also a lattice. Therefore we may suppose $G_i = \{x \in X \mid g_i(x) > 0\}$ where $g_i \in \mathcal{F}$ is non-negative. Put

$$H_K = \{x \in X \mid i \in K,\ j \in M \setminus K \Rightarrow g_i(x) > g_j(x)\} \quad \text{for} \quad \emptyset \neq K \subsetneqq M,$$
$$H_M = \{x \in X \mid g_i(x) > 0 \quad \text{for each} \quad i \in M\}.$$

Each $H_K$ is $\mathcal{F}$-open, as it is the intersection of finitely many sets of the form $\{x \in X \mid g(x) > 0\}$ for $g$ in $\mathcal{F}$. We assert that the collection $\{H_K \mid \emptyset \neq K \subset M\}$ is the desired star-refining cover. If $x \in X$ and $L = \{i \in M \mid g_i(x) > 0\}$, then $L \neq \emptyset$ and $x \in H_L$; thus it is a cover. For $x \in X$, choose $h \in M$ such that $g_h(x) \geqq g_i(x)$ for any $i \in M$. Then $x \in H_K$ implies $h \in K$, hence $H_K \subset G_h$, so that the star of $x$ is a subset of $G_h$.

Lemma 7. Let $X$ be endowed with the uniformity described in Lemma 6. Then $f : X \to I$ is uniformly continuous if and only if $f$ is an $\mathcal{F}$-function.

Proof. If $f$ is an $\mathcal{F}$-function and $\mathcal{K}$ is a uniform cover of $I$ then $\mathcal{K}$ can be refined by a finite open cover and the inverse images under $f$ form a finite $\mathcal{F}$-open cover of $X$, hence $f$ is uniformly continuous. Let $f$ be uniformly continuous and let $G \subset I$ be open. Choose $V_n$ open in $I$ such that $\bigcap_{n=1}^{\infty} V_n = I \setminus G$. Then, for each $n$, $\{G, V_n\}$ is an open cover of $I$; therefore $\{f^{-1}[G], f^{-1}[V_n]\}$ is a uniform cover of $X$. It can be refined by a finite $\mathcal{F}$-open cover. We may suppose this cover is $\{H_n, W_n\}$ where $H_n \subset f^{-1}[G]$, $W_n \subset f^{-1}[V_n]$. Now, $f^{-1}[G] = \bigcup_{n=1}^{\infty} H_n$, hence $f^{-1}[G]$ is $\mathcal{F}$-open.

Lemmas 6 and 7 and Theorem 1 or 2 imply the following theorem.

Theorem 4. Let $\mathcal{F}$ be a uniformly closed ring of bounded real-valued functions on a set $X$ containing all constant functions. Let $\Omega_o$ be the collection of all finite $\mathcal{F}$-open covers of $X$. For $\mathcal{G}$ in $\Omega_o$ let $c_o(\mathcal{G})$ be the supremum of $\lambda\{f_a\}$ over all finite $\mathcal{F}$-partitions $\{f_a\}$ of unity on $X$ which are subordinated to $\mathcal{G}$. Then

$$\inf_{\mathcal{G} \in \Omega_o} c_o(\mathcal{G}) = \frac{1}{\mathcal{F}\text{-dim } X + 1} .$$

References

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