STRUCTURE OF CONNECTED LOCALLY COMPACT GROUPS

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0. Let G denote a connected locally compact topological group. By the theorem of YAMABE the group G is the projective limit of a family G_i , $i \in I$ of Lie groups : $G = \varprojlim G_i$. Denote by L_i the Lie algebra of the Lie group G_i ; the inverse spectrum of the Lie groups $(G_i , g_i^{i'})_I$ induces an inverse spectrum of the corresponding Lie algebras $(L_i , dg_i^{i'})_I$, the homomorphisms $g_i^{i'}$ and $dg_i^{i'}$ being connected by the exponential mappings : $\exp_i dg_i^{i'} = g_i^{i'} \exp_{i'}$. The topological Lie algebra L of the group G is by definition the projective limit of the family L_i , $i \in I$ of finite dimensional Lie algebras : $L = \varprojlim L_i$. The lift $\exp$ of the exponential mappings $\exp_i$, $i \in I$ is a continuous mapping of L into G .

Proposition 1 . The algebraic subgroup G_0 of G generated by $\exp L$ is dense in G .

Proposition 2 . There exists a compact totally disconnected subgroup Δ of the center Z of G such that $G = G_0 \cdot \Delta$.

1. Assume L to be finite dimensional. In this cas it is possible to strengthen the topology of G_0 induced by G in such a way, that G_0 becomes a Lie group with corresponding Lie algebra L . By proposition 2 exists a continuous epimorphism f which maps the direct product $G_0 \times \Delta$ onto G , and by well known theorems on Lie groups the kernel of f is discrete and a subgroup of the center of $G_0 \times \Delta$. Let $\tilde{G}$ denote the universal covering group of the Lie group G_0 , i.e. the unique simply connected Lie group defined by L , we get

Theorem A_1 . Let G denote a finite dimensional connected locally compact topological group. There exists a unique simply connected Lie group $\tilde{G}$, a compact totally disconnected abelian group Δ and a discrete subgroup D of the center of $\tilde{G} \times \Delta$ such that

$$G \cong \tilde{G} \times \Delta / D .$$

The topological group Δ can be chosen as a subgroup of the center Z of G .

Since $\Delta \subset Z$, there exists a continuous and open epimorphism which maps the direct product $G_o \times Z$ resp. $G \times Z$ onto G .

Corollary . A finite dimensional connected locally compact topological group is a Lie group if and only if its center is a Lie group .

2. Consider the general case. It is possible to strengthen the topology of G_o induced by G in such a way, that G_o becomes a connected locally connected complete topological group and the exponential mapping from L into G_o is locally onto - exp maps a neighborhood of zero in L onto a neigborhood of the identity of G_o in the strengthened topology. The topological group G_o is in general not locally compact but a projective limit of Lie groups : $G_o = \varprojlim G_{oj}$. The inverse spectrum $(G_{oj} , g_{oj}^{j'})_J$ induces an inverse spectrum of the corresponding universal covering groups G_{oj} of the Lie groups G_{oj} , $j \in J$ $(\tilde{G}_{oj} , \tilde{g}_{oj}^{j'})_J$, the homomorphisms $g_{oj}^{j'}$ and $\tilde{g}_{oj}^{j'}$ being connected by the covering epimorphisms $\tilde{f}_{oj}$ from $\tilde{G}_{oj}$ onto G_{oj} : $\tilde{f}_{oj} \tilde{g}_{oj}^{j'} = g_{oj}^{j'} \tilde{f}_{oj'}$. The projective limit $\tilde{G} = \varprojlim \tilde{G}_{oj}$ equals the projective limit of the inverse spectrum of the universal covering groups of the Lie groups G_i which occur in the representation of G as a projective limit of Lie groups : $\tilde{G} = \varprojlim \tilde{G}_i$. The topological group $\tilde{G}$ is the universal covering group of G_o as well as of G in the sense of LASHOF [6] . It must be noticed that the group $\tilde{G}$ in general does not cover G , the lift $\tilde{f}$ of the covering epimorphisms $\tilde{f}_i$, $i \in I$ from $\tilde{G}_i$ onto G_i is a continuous homomorphism of $\tilde{G}$ into G .

By proposition 2 exists a continuous epimorphism h_o which maps the direct product $G_o \times \Delta$ onto G and since the Lie algebras of all these groups coincide, the kernel of h_o is a totally disconnected subgroup, which is contained in the center of $G_o \times \Delta$.

We cite the following

Proposition 3 (GLUSHKOV [5]) The topological Lie algebra of a locally compact group is topologically isomorphic to a direct sum of finite dimensional Lie algebras

$$L \cong L' \oplus R^{\mathfrak{m}} \oplus \sum_{m \in M} \oplus L_m \ .$$

L' denotes an arbitrary finite dimensional Lie algebra, $\mathfrak{m}$ a cardinal number, and L_m , $m \in M$ a family of compact non abelian Lie algebras.

Taking the unique simply connected Lie group to any finite dimensional Lie algebra which occurs in the direct sum of proposition 3 we get the topological isomorphism

$$\tilde{G} \cong \tilde{H} \times R^{\mathfrak{m}} \times \prod_{m \in M} \tilde{K}_m \ .$$

$\tilde{H}$ denotes a simply connected Lie group not necessarily compact, while all groups $\tilde{K}_m$, $m \in M$ are compact simply connected non abelian Lie groups.

The following theorem is a generalization of a result of PONTRJAGIN [7] .

Theorem A_2 . Let G denote a connected locally compact topological group. There exists a unique simply connected locally connected topological group $\tilde{G}$, a compact totally disconnected abelian group Δ , and a totally disconnected subgroup D of the center of $\tilde{G} \times \Delta$ such that

$$G \cong \tilde{G} \times \Delta / D \ .$$

The group $\tilde{G}$ may be represented as a direct product of a connected simply connected non abelian Lie group $\tilde{H}$, a cardinal number $\mathfrak{m}$ of copies of the additive group of the reals, and a family $\tilde{K}_m$, $m \in M$ of connected simply connected compact non abelian Lie groups

$$\tilde{G} \cong \tilde{H} \times R^{\mathfrak{m}} \times \prod_{m \in M} \tilde{K}_m \ .$$

The topological group Δ can be chosen as a compact subgroup of the center Z of G .

As in the finite dimensional case using the inclusion $\Delta \subseteq Z$ we get the following

Corollary . A connected locally compact topological group is locally connected if its center is locally connected.

3. The following theorem states necessary and sufficient conditions for the compact totally disconnected component Δ to vanish in the above description of the group G under the

assumption that the center Z of G is metrizable.

Theorem B . Let G denote a connected locally compact topological group with the property that the center Z is metrizable. The following conditions are equivalent

(1) G is locally connected

(2) G is arcwise connected

(3) G is an L-group - any finite dimensional quotient group of G is a Lie group

(4) the universal covering group $\tilde{G}$ covers G - the covering map $\tilde{f}$ from $\tilde{G}$ into G is an open epimorphism

(5) the exponential mapping from L into G is locally onto - exp maps a neighborhood of zero onto a neighborhood of the identity .

Corollary . A connected locally connected locally compact topological group with a metrizable center is the quotient group of a direct product of Lie groups by a totally disconnected central subgroup.

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\begin{array}{ccccccccc}
L & \xrightarrow{\exp} & G & \xleftarrow{\tilde f} & \tilde G & = & \tilde G & \xrightarrow{\tilde f_o} & G_o \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_{i'} & \xrightarrow{\exp_{i'}} & G_{i'} & \xleftarrow{\tilde f_{i'}} & \tilde G_{i'} & & \tilde G_{oj'} & \xrightarrow{\tilde f_{oj'}} & G_{oj'} \\
\downarrow{\scriptstyle dg_i^{i'}} & & \downarrow{\scriptstyle g_i^{i'}} & & \downarrow{\scriptstyle \tilde g_i^{i'}} & & \downarrow{\scriptstyle \tilde g_{oj}^{j'}} & & \downarrow{\scriptstyle g_{oj}^{j'}} \\
L_i & \xrightarrow{\exp_i} & G_i & \xleftarrow{\tilde f_i} & \tilde G_i & & \tilde G_{oj} & \xrightarrow{\tilde f_{oj}} & G_{oj} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
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