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EXTENDING POINT-FINITE COVERS

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A space X is said to be *strongly normal* if it is normal and given a locally finite family $(F_\alpha)_{\alpha\in I}$ of closed subsets of X, there exists a locally finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in I$. Katětov [4] proved that a space is strongly normal if and only if it is collectionwise normal and countably paracompact. Considering spaces in which every point-finite family of closed sets expands to a point-finite family of open sets would not be fruitful. Note that $\{x\}_{x\in[0,1]}$ is a point-finite family of closed subsets of $[0, 1]$, and there does not exist a point-finite family $(G_x)_{x\in[0,1]}$ of open subsets of $[0, 1]$ such that $x \in G_x$ for all $x \in [0, 1]$.

A more meaningful direction for generalization is to consider a point-finite open cover $(H_\alpha)_{\alpha\in I}$ of a closed subspace F of a topological space X and to ask when there exists a point-finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$. Spaces having this property are characterized and the class of such spaces is seen to contain all perfectly normal, collectionwise normal spaces. It is also shown that the class of perfectly normal, collectionwise normal spaces has the property that every point-finite cozero set cover of a closed subspace F extends to a point-finite cozero set cover of X that is locally finite on the complement of F.

These ideas should be useful in connection with the unsolved problem of finding a broad class of spaces for which a polytope with the weak topology will be an AE. Any continuous function from X into a polytope with either the weak topology or the metric topology gives rise to a point-finite partition of unity on X that is not necessarily locally finite.

We will first define three new concepts and then use these concepts to characterize those spaces in which every point-finite open cover on a closed subspace can be extended to a point-finite cover on the whole space.

Definition 1. A topological space X satisfies (α) if given a closed subspace F of X and a pairwise disjoint family $(H_\alpha)_{\alpha\in I}$ of open subsets of F, there exists a point-finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$.

A topological space X satisfies (β) if given a discrete family $(S_\alpha)_{\alpha\in I}$ of subsets of X, there exists a point-finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $S_\alpha \subset G_\alpha$ for all $\alpha \in I$.

A topological space X satisfies (γ) if given a closed subspace F of X and $(A_n)_{n\in N}$ an increasing countable closed cover of F, there exists $(B_n)_{n\in N}$ an increasing closed cover of X such that $B_n \cap F \subset A_n$ for all $n \in N$.

A family $\mathscr{S} = (S_\alpha)_{\alpha\in I}$ of subsets of X is of order n, written ord $\mathscr{S} = n$ if every element of X is in no more than n members of $\mathscr{S}$ and there is an element of X which is in exactly n members of $\mathscr{S}$. The family $\mathscr{S}$ is of finite order if ord $\mathscr{S} = n$ for some n.

Note that every collectionwise normal space satisfies (β) and every perfectly normal space satisfies (γ). We will show later that every hereditary collectionwise normal space satisfies (α). We will now use these concepts to characterize those spaces in which every point-finite open cover of finite order on a closed subspace extends to a point-finite family on the whole space.

Theorem 2. *Let X be a topological space. The following are equivalent:*

1. *The space X satisfies (α) and (β).*
2. *Given F a closed subspace of X and $\mathscr{H} = (H_\alpha)_{\alpha\in I}$ a point-finite open cover of F of finite order, there exists a point-finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$.*

Proof. The proof of 1. $\Rightarrow$ 2. will proceed by induction on ord $\mathscr{H}$. Suppose ord $\mathscr{H} = 1$. Then $\mathscr{H}$ is a discrete collection of subsets of X. Since X satisfies (β), there exists a point-finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$.

Assume the theorem is true for all point-finite open covers with order $\leqq m$ of a closed subset of X, and let $(H_\alpha)_{\alpha\in I}$ be a point-finite open cover of F with order $m + 1$. Let $I^* = \{J \subset I \mid |J| = m + 1\}$. Let $H_J = \bigcap\{H_\alpha \mid \alpha \in J\}$ and consider the family $(H_J)_{J\in I^*}$. This is a pairwise disjoint family of open subsets of F, so by (α), there exists a point-finite family $(G_J)_{J\in I^*}$ of open subsets of X such that $H_J \subset G_J$ for all $J \in I^*$.

Consider $A = \{x \in F \mid |K(x)| \leqq m\}$, where $K(x) = \{\alpha \in I \mid x \in H_\alpha\}$. The set A is closed in F, hence in X, since if x is a point of F not in A, then $\bigcap\{H_\alpha \mid \alpha \in K(x)\}$ is a neighborhood of x missing A. Now $(H_\alpha \cap A)_{\alpha\in I}$ is a point-finite open cover of A and has order $\leqq m$. By the induction hypothesis, there exists a point-finite family $(M_\alpha)_{\alpha\in I}$ of open subsets of X such that $H_\alpha \cap A \subset M_\alpha$ for all $\alpha \in I$.

Let $N_\alpha = M_\alpha \cup \bigcup\{G_J \mid \alpha \in J\}$. Clearly N_α is open in X and $H_\alpha \subset N_\alpha$ for all $\alpha \in I$. It remains to show that $(N_\alpha)_{\alpha\in I}$ is point-finite. If x is in X, there exists K^* a finite subset of I^* such that if $x \in G_J$, then $J \in K^*$. Similarly, there is a finite subset K of I such that if $x \in M_\alpha$, then $\alpha \in K$. Let $K' = K \cup \bigcup\{J \mid J \in K^*\}$. Then K' is a finite subset of I and if $x \in N_\alpha$, then $\alpha \in K'$. This completes the induction step.

To show that 2. $\Rightarrow$ (α), let F be a closed subset of X and let $(H_\alpha)_{\alpha\in I}$ be a pairwise disjoint family of open subsets of F. Then $(H_\alpha)_{\alpha\in I} \cup \{F\}$ is a point-finite open cover of F, so by 2., there is a point-finite open family $(G_\alpha)_{\alpha\in I}$ of X such that $H_\alpha \subset G_\alpha$

for all $\alpha \in I$. To show that X satisfies (β) let $(S_\alpha)_{\alpha \in I}$ be a discrete family of subsets of X. Then $(\text{cl } S_\alpha)_{\alpha \in I}$ is a discrete family and hence a point-finite open cover of $F = \bigcup_\alpha \text{cl } S_\alpha$. By 2., there exists a point-finite family $(G_\alpha)_{\alpha \in I}$ of open subsets of X such that $S_\alpha \subset \subset \text{cl } S_\alpha \subset G_\alpha$ for all $\alpha \in I$. This completes the proof.

Corollary 3. *Let X be a topological space. The following are equivalent*:

1. *The space X satisfies (α) and (β).*

2. *Every point-finite open cover of finite order of a closed subspace of X extends to a point-finite open cover of X.*

Proof. To show 1. $\Rightarrow$ 2., let $(H_\alpha)_{\alpha \in I}$ be a point-finite open cover of finite order of a closed subspace F of X. By Theorem 2, there exists a point-finite family $(G_\alpha)_{\alpha \in I}$ of open subsets of X such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$. For each $\alpha \in I$, let M_α be an open subset of X such that $M_\alpha \cap F = H_\alpha$. Fix $\alpha_0 \in I$ and let $N_{\alpha_0} = (M_{\alpha_0} \cap G_{\alpha_0}) \cup (X - F)$. For $\alpha \neq \alpha_0$, let $N_\alpha = M_\alpha \cap G_\alpha$. Note that $(N_\alpha)_{\alpha \in I}$ is a point-finite open cover of X and that $N_\alpha \cap F = H_\alpha$ for all $\alpha \in I$, hence it is the desired extension of $(H_\alpha)_{\alpha \in I}$.

The proof of 2. $\Rightarrow$ 1. is clear.

By making a stronger assumption about the space X we can require the extended family to be locally finite on $X - F$.

Corollary 4. *Let X be a hereditary collectionwise normal space. Then every point-finite open cover of finite order of a closed subspace F of X extends to a point-finite open cover of X that is locally finite on $X - F$.*

Proof. The proof follows the proof of Theorem 2 with some modifications. If $\mathscr{H} = (H_\alpha)_{\alpha \in I}$ is a point-finite open cover of finite order of a closed subspace F of X, we will show that there exists a point-finite family $(G_\alpha)_{\alpha \in I}$ of open subsets of X, locally finite on the complement of F, such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$. By the same construction as in Corollary 3, the family $(G_\alpha)_{\alpha \in I}$ can be modified to give a point-finite open cover of X that is locally finite on $X - F$ and that extends $\mathscr{H}$.

We will indicate the modifications in the proof of Theorem 2 that are necessary. If ord $\mathscr{H} = 1$, then $\mathscr{H}$ is a discrete family of subsets of X. Since X is collectionwise normal, there exists a discrete family $(G_\alpha)_{\alpha \in I}$ of open subsets of X such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$.

In completing the induction step, note that $\mathscr{K} = (H_J)_{J \in I^*}$ is a pairwise disjoint family of open subsets of F. If $B = \{x \in F \mid \mathscr{K} \text{ is not discrete at } x\}$, then B is closed in X, is contained in F, and $\mathscr{K}$ is a discrete family of subsets of $X - B$. Since X is hereditary collectionwise normal, there exists a discrete (in $X - B$) family $(G_J)_{J \in I^*}$ of open subsets of $X - B$ such that $H_J \subset G_J$ for all $J \in I^*$. Note that each G_J is open in X and $(G_J)_{J \in I^*}$ is discrete on the complement of F. By the induction hypothesis the family $(M_\alpha)_{\alpha \in I}$ can be assumed to be locally finite on $X - F$. Then it is easy to show that $(N_\alpha)_{\alpha \in I}$ is also locally finite on $X - F$. This completes the proof.

From Corollary 4 and Theorem 2 it is clear that every hereditary collectionwise normal space satisfies (α). The next theorem will characterize the extendability of point-finite open covers that do not necessarily have finite order.

Theorem 5. *Let X be a topological space. The following are equivalent:*

1. *The space X satisfies (α), (β) and (γ).*
2. *Given F a closed subspace of X and $(H_\alpha)_{\alpha\in I}$ a point-finite open cover of F, there exists a point-finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $H_\alpha \subset G_\alpha$ for all $\alpha \in I$.*

Proof. To show 1. $\Rightarrow$ 2., let $(H_\alpha)_{\alpha\in I}$ be a point-finite open cover of F, a closed subspace of X. For each $n \in N$, let $A_n = \{x \in F \mid |K(x)| \leqq n\}$. Then $(A_n)_{n\in N}$ is an increasing closed cover of F, so by (γ), there is an increasing closed cover $(B_n)_{n\in N}$ of X such that $B_n \cap F \subset A_n$ for all $n \in N$. The family $(H_\alpha \cap B_n \cap F)_{\alpha\in I}$ is a point-finite open cover of order n of $B_n \cap F$. By 1. $\Rightarrow$ 2. of Theorem 2, there exists $(G_\alpha^n)_{\alpha\in I}$, a point-finite family of open subsets of X such that $H_\alpha \cap B_n \cap F \subset G_\alpha^n$ for all $\alpha \in I$.

Let $G_\alpha = G_\alpha^1 \cup \bigcup\{G_\alpha^n \cap (X - B_{n-1}) \mid n \geqq 2\}$. Each G_α is clearly open. If x is an element of H_α, let n_0 be the first n such that $x \in B_n$. If $n_0 = 1$, then $x \in H_\alpha \cap B_1 \cap \cap F$, hence $x \in G_\alpha^1$. If $n_0 > 1$, then $x \notin B_{n_0 - 1}$. But $x \in H_\alpha \cap B_{n_0} \cap F$, so $x \in G_\alpha^{n_0}$, therefore $x \in G_\alpha$. Hence $H_\alpha \subset G_\alpha$ for all $\alpha \in I$.

It remains to show that $(G_\alpha)_{\alpha\in I}$ is point-finite. Let x be an element of X. There exists a finite subset N^* of N such that $x \in X - B_n$ if and only if $n \in N^*$. Given $n \in N$, there exists a finite subset K_n of I such that if $x \in G_\alpha^n$, then $\alpha \in K_n$. Let $K = K_1 \cup \cup \bigcup\{K_{n+1} \mid n \in N^*\}$. Then K is a finite subset of I. If $x \in G_\alpha^1$, then $\alpha \in K_1 \subset K$, and if $x \in G_\alpha^n \cap (X - B_{n-1})$ for some $n \geqq 2$, then $n - 1 \in N^*$, hence $\alpha \in K_n \subseteq K$.

By Theorem 2, we already have that 2. $\Rightarrow$ (α) and (β), hence it remains to show that X satisfies (γ). Given an increasing closed cover $(A_n)_{n\in N}$ of a closed subset F of X, note that $\{F\} \cup (F - A_n)_{n\in N}$ is a point-finite open cover of F. By 2., there exists $(G_n)_{n\in N}$, a point-finite family of open subsets of X such that $F - A_n \subset G_n$ for all $n \in N$. Let $H_n = \bigcup\{G_m \mid m \geqq n\}$, and let $B_n = X - H_n$. It is easy to show that $(B_n)_{n\in N}$ is an increasing closed cover of X such that $B_n \cap F \subset A_n$ for all $n \in N$. Hence X satisfies (α), (β) and (γ), and the proof is completed.

Theorem 6. *Let X be a collectionwise normal, perfectly normal space. Then every point-finite open cover of a closed subspace F of X extends to a point-finite open cover of X that is locally finite on $X - F$.*

Proof. Again, only the necessary modifications of the proof of Theorem 5 will be indicated. Let $(H_\alpha)_{\alpha\in I}$ be a point-finite open cover of F. Since X is perfectly normal, $X - F = CZ(f)$, where $0 \leqq f(x) \leqq 1$, for all $x \in X$. If we let $B_n = \{x \in X \mid f(x) \geqq \geqq 1/n\} \cup A_n$, then $(B_n)_{n\in N}$ is an increasing closed cover of X such that $B_n \cap F \subset A_n$ for all $n \in N$. This choice of $(B_n)_{n\in N}$ should be substituted throughout the proof of Theorem 5. By a result of Hodel [3], every perfectly normal, collectionwise normal space is hereditary collectionwise normal. By virtue of Corollary 4, for each n, the family $(G_\alpha^n)_{\alpha \in I}$ may be assumed to be locally finite on $X - F$. We will show that $(X - B_n)_{n \in N}$ is locally finite on $X - F$. From there it is easy to show that the family $(G_\alpha)_{\alpha \in I}$ constructed in the proof of Theorem 5 is locally finite on $X - F$. If $x \in X - F$, there is a natural number p such that $1/p < f(x)$. If $y \in \{x \in X \mid 1/p < f(x)\} \cap \cap (X - B_n)$, then $1/p < f(y) < 1/n$, hence $n < p$. Hence $(X - B_n)_{n \in N}$ is locally finite on $X - F$. This completes the proof.

Our final result concerns the extension of a point-finite cozero set cover of a closed subspace to a point-finite cozero set cover on the whole space. We need an idea gleaned from Hanner [2].

Lemma 7. *Let F be a closed subspace of a normal space X and let f be a continuous function on F into $[0, 1]$ such that $CZ(f)$ is contained in some open set G of X. Then there is a continuous extension f^* of f to X such that $CZ(f^*)$ is also contained in G.*

Proof. Let g be any continuous extension of f to X with values in $[0, 1]$. Define a function h on $F \cup (X - G)$ as follows: If $x \in F$, then $h(x) = f(x)$, and if $x \in X - G$, then $h(x) = 0$. It is easy to see that h is well-defined and continuous on $F \cup (X - G)$, so h has an extension h^* to X, with values in $[0, 1]$. Then $f^* = g \wedge h^*$ is the desired extension of f.

Theorem 8. *Let X be a collectionwise normal, perfectly normal space. Then every point-finite cozero set cover of a closed subspace F of X extends to a point-finite cozero set cover of X that is locally finite on $X - F$.*

Proof. Let $\mathscr{H} = (CZ(f_\alpha))_{\alpha \in I}$ be a point-finite cozero set cover of F. By Theorem 6, there exists a point-finite open cover $(G_\alpha)_{\alpha \in I}$ of X that extends $\mathscr{H}$ and is locally finite on $X - F$. By Lemma 7, for each $\alpha \in I$ there exists an extension g_α of f_α to X such that $CZ(g_\alpha) \subset G_\alpha$. Since X is perfectly normal, $X - F = CZ(g)$ where $0 \leqq g(x) \leqq 1$ for all $x \in X$. Fix $\alpha_0 \in I$ and let $f_{\alpha_0}^* = g \vee g_{\alpha_0}$. For $\alpha \neq \alpha_0$, let $f_\alpha^* = g_\alpha$. Then $(CZ(f_\alpha^*))_{\alpha \in I}$ is the desired extension of $\mathscr{H}$.

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Professor J. C. Smith has informed the author that condition (α) implies condition (β). Therefore, the statement of Theorem 2 can be improved by deleting condition (β).

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