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ON TOPOLOGICAL ENTROPY

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In this communication we introduce an abstract scheme including the topological entropy (see [1]) as well as the Kolmogoroff-Sinaj's entropy (see [2], [3]) and also some other invariants.

Let P be a set with a reflexive and transitive relation $\leqq$. Assume that on the set P an associative binary operation $\vee$ is defined such that $A \vee B \geqq A$ and $A \vee B \geqq B$ for every $A, B \in P$. Further let $T: P \to P$ and $H: P \to \langle 0, \infty)$ be any functions satisfying the following conditions:

1. $H(\bigvee_{i=0}^{k} T^i(A)) \leqq H(\bigvee_{i=0}^{j} T^i(A)) + H(\bigvee_{i=j+1}^{k} T^i(A))$.
2. $T(A \vee B) = T(A) \vee T(B)$.
3. $H(T(A)) \leqq H(A)$.

Lemma. *Under these assumptions* $\lim 1/n\, H(\bigvee_{i=0}^{n-1} T^i(A))$ *exists for any* $A \in P$.

Definition. For any given P, T, H and $A \in P$ let us put $h(A, T) = \lim 1/n\, H(\bigvee_{i=0}^{n-1} T^i(A))$, $h(T) = \sup \{h(A, T); A \in P\}$; $h(T)$ is called the *entropy* of the triple (P, T, H).

Examples.

1. Topological entropy. Let X be a topological space, $f: X \to X$ a continuous map, P the family of all finite open coverings of X ($R_1 \leqq R_2$ iff R_2 is a refinement of R_1), $H(A) = \log \operatorname{card} A$, $T(A) = f^{-1}(A)$.

2. Kolmogoroff-Sinaj's entropy. Let (X, S, m) be a probability measure space, $f: X \to X$ a measure preserving transformation, P the family of all finite measurable decompositions A of X such that $A, f^{-1}(A), \ldots, f^{-k}(A)$ are independent for all k, $T(A) = f^{-1}(A)$, $H(A) = -\sum\{m(E) \log m(E); E \in A\}$.

3. Entropy of an automorphism of a Boolean algebra. Let B be a Boolean algebra, f an automorphism of B. Let P be the set of all finite decompositions of the greatest element of B. For $A \in P$ put $H(A) = \log \operatorname{card} A$, $T(A) = f(A)$.

Usually, if "two systems are isomorphic" then their entropies are equal. In general, two triples (P, T, H) and (R, S, G) are equivalent, if there is a bijection $U : P \to R$ with the following properties:

1. $U(A \vee B) = U(A) \vee U(B)$.
2. $T \circ U = U \circ S$.
3. $G(U(A)) = H(A)$.

Theorem 1. *If (P, T, H) and (R, S, G) are equivalent then their entropies are equal.*

We shall illustrate the preceding fact by the following three examples; the first two examples are well-known, the third one leads to a new result.

Let X_n be the set of all sequences $x = \{x_i\}_{i=-\infty}^{\infty}$ of integers $0, 1, \ldots, n-1$. The shift is the map $f : X_n \to X_n$ defined by the formula $f(\{x_n\}_{n=-\infty}^{\infty}) = \{y_n\}_{n=-\infty}^{\infty}$, where $y_n = x_{n+1}$ for every n. There are at least three natural structures on X_n:

1. Topology T_n with the subbase consisting of all cylinders $\{x; x_i = j\}$ and the shift f. It was proved in [1] that the topological entropy $h(f) = \log n$. It follows that there is no homeomorphism $g : X_n \to X_m$ $(n \neq m)$ commuting with the shifts.

2. The (Bernoulli) dynamical system (X_n, S_n, μ, f); here S_n is the σ-algebra generated by the cylinders; $\mu = \displaystyle\mathop{\times}_{i=-\infty}^{\infty} \mu_i$ is the Cartesian product of probability measures μ_i; for all i, $\mu_i = \mu_0$ and μ_0 is defined by means of n-tuple $(p_0, p_1, \ldots, p_{n-1})$, i.e., $\mu_0(i) = p_i$, f is the shift. It is well-known that the Kolmogoroff-Sinaj's entropy $h(f) = -\sum p_i \log p_i$. Hence two Bernoulli systems with different entropies cannot be isomorphic. (Recently D. Ornstein [4] has proved the converse theorem.)

3. σ-algebras S_n generated by the cylinders and the automorphism f induced by the shift. Problem: Is there an isomorphism $g : S_n \to S_m$ commuting with the shifts?

Theorem 2. *If S_n is the σ-algebra generated by the cylinders, f is the automorphism of S_n generated by the shift and $h(f)$ is the entropy introduced in the third example, then $h(f) = \log n$.*

Corollary. *Given $n \neq m$, there is no isomorphism $g : S_n \to S_m$ commuting with the shifts.*

The last corollary was proved also in [5], but in another way.

Of course, also some further theorems can be proved in the general case. So $h(T^k) = k\,h(T)$, $h(T_1 \times T_2) = h(T_1) + h(T_2)$ and if $A \in P$ is an element such that $\{\bigvee_{i=0}^{n-1} T^i(A)\}_{n=0}^{\infty}$ "generates" the set P, then $h(T) = h(T, R)$.

Finally we list further examples satisfying the assumptions of our scheme:

4. Another type of topological entropy. Let P be the family of all open coverings of X having refinements of finite orders, $H(A) = \log \min \{\text{order } B;\ B$ is a refinement of $A\}$. This invariant probably corresponds to the topological dimension. If X is a topological space of finite dimension, then $\dim X \geqq e^{h(T)} - 1$.

5. Group endomorphism entropy (see [1]). Let G be an Abelian group, P the family of all finite subgroups, $A \leqq B$ iff $A \subset B$, T an endomorphism and $H(A) = \log \text{order } A$.

6. Entropy of a measure preserving transformation. Let P be a ring of sets (ordered by the inclusion), H a measure on P, T a measure preserving transformation.

7. Entropy of an operator. P is the system of all integrable functions (ordered as usually), H is the integral, $T(f) = f + g$ where g is a fixed non-positive function.

References

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