Toposym 3

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SOME MAPPING AND FIXED POINT THEOREMS

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1. This remark concerns some mapping and fixed point theorems. Some of these results are related to those of Pochožajev [1], Browder [2], Edelstein [3], Belluce and Kirk [4], Daneš [5] and our paper [6].

Let X, Y be normed linear spaces. In the following we use the symbols "$\to$", "$\rightharpoonup$" to denote strong and weak convergence, respectively. To fix our terminology we introduce here the following notions. A set $M \subset X$ is said to be (a) weakly closed if for each $u_n \in M$, $u_n \rightharpoonup u_0 \Rightarrow u_0 \in M$; (b) weakly compact if for each $u_n \in M$ there is a subsequence u_{n_k} which is weakly convergent in X. A mapping $F : X \to Y$ is said to be

(1) weakly continuous if $u_n \in X$, $u \in X$, $u_n \rightharpoonup u \Rightarrow F(u_n) \rightharpoonup F(u)$;

(2) demicontinuous if u_n, $u \in X$, $u_n \to u \Rightarrow F(u_n) \rightharpoonup F(u)$;

(3) p-positively homogeneous if $F(tu) = t^p F(u)$ for each $u \in X$, $t \geqq 0$, where $p > 0$.

We shall say that a functional φ is quasi-convex on a convex set $M \subset X$ if u, $v \in M$, $\lambda \in [0, 1] \Rightarrow \varphi(\lambda u + (1 - \lambda) v) \leqq \max [\varphi(u), \varphi(v)]$. A functional f is said to be weakly lower-semicontinuous at $u_0 \in X$ if $u_n \in X$, $u_n \rightharpoonup u_0 \Rightarrow f(u_0) \leqq \varliminf_{n \to \infty} f(u_n)$. By $B_\delta(u)$ we denote an open ball of a space X centered at u and with the radius $\delta > 0$.

2. We start with the following

Theorem 1. *Let X be a reflexive Banach space, $F : X \to X$ a mapping such that for some $\lambda > 0$, u, $v \in X$, $u \neq v \Rightarrow \|u - v - \lambda(F(u) - F(v))\| < \|u - v\|$. If $F(X)$ is weakly closed in X, then $F(X) = X$.*

Theorem 2. *Let X, Y be normed linear spaces, Y reflexive, $F : X \to Y$ a mapping such that $F(X)$ is weakly closed in Y. Let $H : X \to Y$ be a p-positively homogeneous map of X onto Y. Suppose that for each $u \in X$ there exist constants α_u, δ_u ($0 \leqq \alpha_u < 1$, $\delta_u > 0$) and a mapping $G_u : X \to Y$ such that $v \in B_{\delta_u}(u) \Rightarrow \|F(v) - F(u) - G_u(v - u)\| \leqq \leqq \alpha_u \|H(v - u)\|$. Assume there is $R > 0$ and $\varepsilon_u \geqq 0$ such that $v \in B_R(0) \Rightarrow \|G_u(v) - - H(v)\| \leqq \varepsilon_u \|H(v)\|$, $u \in X$. If $\varepsilon_u + \alpha_u < 1$ for each $u \in X$, then $F(X) = Y$.*

Corollary 1. *Let X, Y be normed linear spaces, Y reflexive, $K : X \to Y$ a linear (i.e., additive and homogeneous) mapping of X onto Y, $F : X \to Y$ a map such that $(K + F)(X)$ is weakly closed. Assume that for each $u \in X$ there are constants α_u, δ_u $(0 \leqq \alpha_u < 1$, $\delta_u > 0)$ such that $v \in B_{\delta_u}(u) \Rightarrow \|F(v) - F(u)\| \leqq \alpha_u \|K(v - u)\|$. Then $(K + F)(X) = Y$.*

Remark. The conclusions of Theorems 1, 2 remain true if X, Y are normed linear spaces, $F : X \to Y$ is weakly continuous, $F(0) = 0$ and $\{u \in X \mid \|F(u)\| \leqq a\}$ is weakly compact for each $a \geqq 0$. Here we do not assume that $F(X)$ is weakly closed in Y.

Theorem 3. *Let X, Y be normed linear spaces, X reflexive, $F : \overline{B_R(0)} \to Y$ a given mapping, $G : X \to Y$ a suitable p-positively homogeneous mapping of X onto Y so that $u, v \in B_R(0) \Rightarrow \|F(u) - F(v) - G(u - v)\| \leqq \alpha \|G(u - v)\|$, for some $\alpha \in [0, 1)$. Suppose there is a point $u_0 \in B_R(0)$ such that $f(u_0) < \min_{\|u\| = R} f(u)$, where $f(u) = \|F(u)\|$, $u \in \overline{B_R(0)}$. If either* a) *F is weakly continuous on $\overline{B_R(0)}$, or* b) *F is demicontinuous on $\overline{B_R(0)}$ and $f(u)$ is quasi-convex on $\overline{B_R(0)}$, then there exists $u^* \in B_R(0)$ such that $F(u^*) = 0$.*

Theorem 4. *Let X, Y be normed linear spaces, $M \subset X$ an open subset, $F : M \to Y$, $G : X \to Y$ mappings such that $f(u) = \|F(u) + G(u)\|$ is weakly lower-semicontinuous on M and that G is a linear mapping from X onto Y. Suppose that $\{u \in M \mid f(u) \leqq c\}$ is weakly compact and non-void for some $c \geqq 0$. If for each point $u \in M$ there exist constants α_u, δ_u $(0 \leqq \alpha_u < 1$, $\delta_u > 0)$ so that $B_{\delta_u}(u) \subset M$ and $v \in B_{\delta_u}(u) \Rightarrow \|F(v) - F(u)\| \leqq \alpha_u \|G(v - u)\|$, then there exists a point $u^* \in M$ such that $F(u^*) + G(u^*) = 0$.*

As a simple consequence of Theorem 4 one can obtain a new fixed-point theorem for a class of nonlinear mappings which are called local contractions (compare [3]).

A mapping F defined on an open subset M of a normed linear space X with values in X is said to be a feeble local contraction on M if for each $u \in M$ there are constants α_u, δ_u $(\alpha_u \in [0, 1)$, $\delta_u > 0)$ such that $v \in B_{\delta_u}(u) \subset M \Rightarrow \|F(v) - F(u)\| \leqq \alpha_u \|v - u\|$.

Theorem 5. *Let X be a normed linear space, $M \subset X$ an open subset, $F : M \to X$ a feeble local contraction on M such that $\{u \in M \mid \|u - F(u)\| \leqq c\}$ is weakly compact and nonvoid for some $c \geqq 0$. If either* a) *F is weakly continuous on M, or* b) *M is convex, F is demicontinuous and $\psi(u) = \|u - F(u)\|$ is quasi-convex on M, then there is $u^* \in M$ such that $u^* = F(u^*)$.*

Theorem 6. *Let X be a normed linear space, M a non-void subset of X, $F : M \to M$ such that $u, v \in M$, $u \neq v \Rightarrow \|F(u) - F(v)\| < \|u - v\|$. If either* a) *$X$ is* *reflexive and* (id $-$ F)(M) *is weakly closed, or* b) (id $-$ F)(M) *is weakly closed and weakly compact, then there is a unique point* $u^* \in M$ *such that* $F(u^*) = u^*$.

Theorem 7. *Let* X *be a reflexive Banach space,* M *an open subset of* X, $M \neq \emptyset$, $F : M \to X$ *a feeble local contraction on* M. *If* (id $-$ F)(M) *is weakly closed, then there is a point* $u^* \in M$ *such that* $u^* = F(u^*)$.

Remark. In comparison with Banach's contraction principle we need not assume in Theorem 5 that X is complete, M is closed, and that F is a map of M into M.

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