F. Burton Jones

The utility of empty inverse limits

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 223--228.

Persistent URL: http://dml.cz/dmlcz/700723

# THE UTILITY OF EMPTY INVERSE LIMITS

F. B. JONES

Riverside

By an inverse system we shall mean a collection of pairs $\{X_\alpha, f_{\alpha\beta}\}$ where $\alpha$, $\beta$, and $\gamma$ belong to a directed index set $A$, $X_\alpha$ is a nonempty set and $f_{\alpha\beta}$ is a mapping from $X_\alpha$ to $X_\beta$ such that $f_{\beta\gamma}[f_{\alpha\beta}] = f_{\alpha\gamma}$ for $\gamma < \beta < \alpha$. In fact, we shall consider only the special case where $A$ is the set of all ordinals less than a given limit ordinal and the maps $f_{\alpha\beta}$ are "onto". The inverse limit of such a system is the subset $\{x_\alpha\}$ of the cartesian product $\times X_\alpha$ $(\alpha \in A)$ such that $x_\beta = f_{\alpha\beta}(x_\alpha)$ for $\beta < \alpha$. Usually one is interested in an inverse system because the inverse limit is nonempty. However, there are situations when the fact that the inverse limit is empty or trivial is of interest.

In studying Souslin spaces an inverse system whose inverse limit is empty arises naturally. In 1935 G. Kurepa called such systems ramifications and studied them at some length in [8] and [9]. In addition to many positive results (about linearly ordered spaces in particular) he posed some beautiful problems, some of which have apparently turned out to be unsolvable. At about the same time, in connection with the normal Moore space metrization problem [4] (and certain other problems [5]), I also constructed inverse systems of this sort (with $X_\alpha$ countable for each $\alpha < \omega_1$) whose inverse limits were empty. What seemed at that time a promising method of constructing a non-metric normal Moore space has also wound up among the logician's models [11].

About 1950 Tukey raised with Henkin the question as to the existence of an inverse system (with onto bonding maps) whose inverse limit was empty. To understand Henkin's construction [2] it will help to see how a very simple construction would yield such a system if we did not require the bonding maps to be onto.

For each positive integer $i$ let $H_i$ be the set of positive integers greater than $i - 1$. Define $f_{21}(n)$ to be $n$ (in $H_1$) if $n > 1$. And in general, $f_{\alpha\beta}(n) = n$ (in $H_\beta$) for $n \geqq \alpha$. This defines an inverse system $\{H_\alpha, f_{\alpha\beta}\}$ for $\beta < \alpha < \omega_0$ whose inverse limit space is empty. The easiest way to picture this system as a tree is in the first quadrant of the $xy$-plane with $H_1$ being the (positive) integral points of $x$-axis; $H_2$, those on the line $y = 1$, etc., with the branches of the tree being arcs running straight up from $H_1$ to the highest integral point below the main diagonal (the line $y = x$). Clearly the inverse limit space is empty because no arc (or ray) in the union runs all the way to the top of the plane. So when the tree is $\aleph_0$ wide ($|H_n| = \aleph_0$) and $\aleph_0$ tall, no branch need run all the way to the top of the tree. (Actually to complete the tree one must run the branches from $H_1$ down to a common trunk but this just amounts to adding $H_0$ to the system with $H_0$ degenerate, say $H_0 = \{0\}$ and $f_{\alpha 0}(n) = 0$ for all $n \geqq \alpha$ and $0 < \alpha < \omega_0$.) Obviously the maps are not onto. But a similar thing can be done to define a system $\{X_\alpha, f_{\alpha\beta}\}$ for $\beta < \alpha < \omega_1$ where the bonding maps are all "onto".

Begin by letting $X_1$ be the ordinal points in $y = 1$ of $X \times Y$ where both $X$ and $Y$ are "long rays" (from 1 to $\omega_1$ including 1 but not $\omega_1$). For each element $x$ in $X_1$ and countable limit ordinal $\alpha$, run a ray $L_\alpha(x)$ straight up from $x$ which intersects every horizontal ray $y = \beta$ (where $\beta$ is an ordinal less than $\alpha$) but not intersecting $y = \alpha$. Consider all of these rays to be disjoint except that for $\alpha$ and $\beta$ different countable limit ordinals $L_\alpha(x) \cap L_\beta(x) = x$, i.e., no two of the rays emanating from points of $X_1$ intersect except when they emanate from the same point and when they do emanate from the same point, they have that point and no other in common.

Now let $X_2$ be the set of all points of $y = 2$ belonging to rays $L_\alpha(x)$ for $x$ in $X_1$. So for each $x$ in $X_1$ and each countable limit ordinal $\alpha$ there is a unique point of $L_\alpha(x)$ in $X_2$ and $f_{21}$ maps this point to $x$. Again for $x$ in $X_2$ and limit ordinal $\alpha < \omega_1$ run a ray straight up from $x$ up to but not intersecting the horizontal line $y = \alpha$ and consider all of these disjoint except their common emanation point in case they emanate from the same point. The set $X_3$ is the set of all points at the $y = 3$ level which belong to vertical rays thus defined emanating from points of $X_1 \cup X_2$. The definition of the bonding maps is obvious. And in fact, the entire construction is now obvious, $X_{\omega_0}$ (for example) being those points of $y = \omega_0$ belonging to those rays running straight up from points of $X_\beta$ for $\beta < \omega_0$ as already defined. Hence if the inverse limit of the system were non-empty some branch would have to run straight up from some line $y = \alpha$ ($\alpha < \omega_1$) to (but not intersecting) the line $y = \omega_1$, for if a path changed branches infinitely many times, say at $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ then such a path would not reach $y = \lim \alpha_i$. But no branch runs straight up to $y = \omega_1$. So the inverse limit is empty.

At this point one can see that even when the bonding maps are "onto" it is not surprising for the inverse limit to be empty when $|X_\alpha| \geqq \aleph$ ($\alpha \in A$) and $|A| \leqq \aleph$ unless some countable subset of $A$ is cofinal with $A$. On the other hand, if $A$ is a limit ordinal (i.e., the set of all ordinals less than $A$) and $\aleph$ is a cardinal such that $|X_\alpha| < < \aleph < |A|$ for $\alpha < A$, the inverse limit is non-empty. The spirit of this condition would be violated if some subsequence of $A$ cofinal with $A$ had cardinality less than $|A|$. Roughly speaking, if the tree is substantially taller than it is wide, then from some level on, the branches run straight on up to the top. See [5] for a more accurate statement of this rather simple fact. So the more interesting cases occur when the height is larger than the width but just barely so. This is the case in an example due to Higman and Stone, which I shall describe next.

If the spaces in the inverse system $\{X_\alpha, f_{\alpha\beta}\}$ are groups or rings, etc., and the mappings are "onto" homomorphisms, the inverse limit is a group or ring, etc., in such a natural way that its natural projection onto $X_\alpha$ will be an "onto" homomorphism. Zelinsky raised the question as to the necessity of the projection being "onto" and Higman and Stone constructed an example where the cardinality of the inverse limit is too small for the projection be "onto" (in fact, contained only one element). Basic to their construction (for the case of groups, etc.) is the following inverse system whose inverse limit is empty [3].

For each initial closed interval $[1, \alpha]$ of the countable ordinals $(\alpha < \omega_1)$ let $G_\alpha$ denote the collection of all increasing bounded functions from $[1, \alpha]$ to the real numbers. Now for each $\alpha < \omega_1$, $X_\alpha$ is defined by induction to be a countable subset of $G_\alpha$ subject to the following conditions:

(1) if $\beta < \alpha < \omega_1$ and $x_\alpha$ belongs to $X_\alpha$, there is one and only one element $x_\beta$ of $X_\beta$ such that $x_\beta \subset x_\alpha$, and

(2) if $\beta < \alpha < \omega_1$, $n$ is a natural number, and $x_\beta$ belongs to $X_\beta$, then there is an element $x_\alpha$ of $X_\alpha$ such that $x_\beta \subset x_\alpha$ and $x_\beta(\beta) - x_\alpha(\alpha) < 1/2^n$.

Choosing $X_1$ to be some arbitrary countably infinite subset of $G_1$ there would be no difficulty in constructing the system if one realizes that for each $n$ and each $\alpha < \omega_1$ there is an increasing function from $[1, \alpha]$ of the ordinals into $[0, 1/2^n]$ of the real numbers. Furthermore, the bonding maps $f_{\alpha\beta}$ are defined so that for $x_\alpha$ in $X_\alpha$, $f_{\alpha\beta}(x_\alpha) = x_\beta$ in $X_\beta$ if and only if $x_\beta \subset x_\alpha$. From this property it follows immediately that the inverse limit is empty because there exists no increasing function from $[1, \omega_1)$ to the real numbers (uncountable subsets of the real numbers contain points which are condensation points from both sides).

This construction is quite like an old one of mine (about 1946) in which the branches of the tree were arcs whose lengths were diadic rational numbers. The functions $x_\alpha$ above for $[0 < \beta \leqq \alpha]$ could be chosen so that the arc length from $H_0$ to its end point in $H_\beta$ is $x_\alpha(\beta)$ [6].

The use of inverse systems with empty inverse limits in examples is perhaps to be expected. But such systems are quite useful in the proof of certain kinds of theorems. For instance, Roy in proving Arhangelskii's theorem on the cardinality of first countable, Lindelöf spaces $S$ uses an inverse system $\{X_\alpha, f_{\alpha\beta}\}$ where for each $\alpha < \omega_1$, $X_\alpha$ is a closed covering of $S$ with $|X_\alpha| \leqq \mathbf{c}$. While the bonding maps preserve inclusion (i.e., $f_{\alpha\beta}^{-1}(x_\beta)$ is a decomposition of $x_\beta$ for $\alpha = \beta + 1$ when $f_{\alpha\beta}^{-1}(x_\beta)$ is nondegenerate) and the maps are onto, the tree does not branch at $x_\beta \in X_\beta$ if $|x_\beta| \leqq \mathbf{c}$, i.e., if $f_{\alpha\beta}(x_\alpha) = x_\beta$ and $|x_\beta| \leqq \mathbf{c}$ then $x_\alpha = x_\beta$. So the inverse system itself does not have an empty inverse limit. However the subsystem of those sets $x_\alpha$ such that $|x_\alpha| > \mathbf{c}$ does have an empty inverse limit and it is this fact that is central in the proof.

Perhaps it may be instructive to outline Roy's proof [10], leaving unproved certain set theoretic lemmas which are either known or in any case easy to establish.

**Arhangelskii's Theorem** [1]. *Let $S$ denote a Hausdorff space in which the First Countability Axiom holds true $(\chi(S) = \aleph_0)$. Then if $S$ is Lindelöf $|S| \leqq \mathbf{c}$.*

Let $X_0 = \{S\}$. Let $X_1$ denote a collection of closed subsets of $S$ covering $S$ such that $|X_1| \leqq \mathbf{c}$. Define $f_{10}$ from $X_1$ onto $X_0$ in the obvious way: $f_{10}(x_1) = S$

for all $x_1$ in $X_1$. Construct $X_1$ so that one of its elements $x_1^c$ has the following three properties:

(1) $|x_1^c| \leqq \mathbf{c}$,

(2) if $x_1 \in X_1 - \{x_1^c\}$ then $x_1 \cap x_1^c = \emptyset$ (if $|x_1^c| \leqq \mathbf{c}$, then $x_1^c$ is the intersection of at most $\mathbf{c}$ open subsets of $S$), and

(3) if $|S| \leqq \mathbf{c}$, $X_1 - \{x_1^c\} = \emptyset$.

In general, given $X_\beta$, $X_\alpha$ is defined from $X_\beta$ in exactly the same way when $\alpha = \beta + 1$ so that:

If $x_\beta \in X_\beta$, $f_{\alpha\beta}^{-1}(x_\beta)$ is a collection of no more than $\mathbf{c}$ closed subsets of $x_\beta$ covering $x_\beta$ which contains an element $x_\alpha^c$ such that

(1) $|x_\alpha^c| \leqq \mathbf{c}$,

(2) if $x_\alpha \in f_{\alpha\beta}^{-1}(x_\beta) - \{x_\alpha^c\}$ then $x_\alpha \cap x_\alpha^c = \emptyset$, and

(3) if $|x_\beta| \leqq \mathbf{c}$ then $x_\alpha^c = x_\beta$ and $f_{\alpha\beta}^{-1}(x_\beta) - \{x_\alpha^c\} = \emptyset$.

More specifically, this is how $X_\alpha$ is constructed if $\beta$ is not a limit ordinal.

When $\beta$ is a limit ordinal define $X_\beta$ as follows: If $\{x_\gamma\}$ $(\gamma < \beta)$ is an element of the inverse limit of $\{X_\gamma, f_{\gamma\delta}\}$ then $\bigcap x_\gamma$ $(\gamma < \beta)$ is an element of $X_\beta$ and conversely. Obviously $X_\beta$ is a closed cover of $S$ and the definition of $f_{\beta\gamma}$ for $\gamma < \beta$ is the natural one. Furthermore, if $\beta < \omega_1$, $|X_\beta| \leqq \mathbf{c}$.

Returning to the definition of $|X_\alpha|$ when $\alpha = \beta + 1$ and $\beta$ is a limit ordinal with $x_\beta = f_{\alpha\beta}(x_\alpha^c)$ and $x_\beta = \bigcap x_\gamma$ as above, we require that $x_\alpha^c \cup \{x_\gamma^c \mid f_{\gamma(\gamma-1)}(x_\gamma^c) = f_{\gamma(\gamma-1)}(x_\gamma)\}^*$ (for all $\gamma < \beta$) be closed. (* means union.)

Now suppose that the subsystem of all those elements of $X_\alpha$ for all $\alpha < \omega_1$ whose cardinality exceeds $\mathbf{c}$ has a non-empty inverse limit. Let $\{x_\alpha\}$ be an element of this limit. Then $|x_\alpha| > \mathbf{c}$ for each $\alpha < \omega_1$ and $\{x_\alpha^c\}^*$ for which $f_{\alpha(\alpha-1)}(x_\alpha^c) = f_{\alpha(\alpha-1)}(x_\alpha)$ is closed. But $\{x_\alpha^c\}^*$ is covered by $\{S - x_\alpha\}$ $(\alpha < \omega_1)$ but by no countable subcollection. Since $S$ is Lindelöf this is a contradiction. Hence the subsystem has an empty inverse limit. It follows that the collection of all elements of $\bigcup X_i$ of cardinality $\mathbf{c}$ or less covers $S$. But $|\bigcup X_i| \leqq \mathbf{c}$; so $|S| \leqq \mathbf{c}$.

When one grasps the salient points of this construction, one sees that it generalizes to cardinals larger than $\aleph_0$. (Juhász [7] has done this for Arhangelskii's argument.)

**Theorem** (Juhász). *Suppose that $S$ is a Hausdorff space and $\aleph$ is a transfinite cardinal such that $\chi(p) \leqq \aleph$ for each $p$ in $S$. Then if each open cover of $S$ contains an open subcover of cardinality $\aleph$ or less,* $|S| \leqq \exp \aleph$.

First let us check some elementary lemmas.

**Lemma 1.** *If $p$ is a limit point of a point set $M$ then $p$ is the limit of a net $T$ of distinct points of $M$ such that* $|T| \leqq \aleph$.

**Lemma 2.** *If $M$ is a closed subset of $S$, such that $|M| \leqq \exp \aleph$, then $M$ is the intersection of no more than* $\exp \aleph$ *open sets.*

Proof. For each point $p$ of $M$ let $\mathscr{U}(p)$ denote a topological basis at $p$ such that $|\mathscr{U}(p)| \leqq \aleph$. If $U$ is any open set containing $M$ some subcollection $\mathscr{V}$ of $\bigcup_{p \in M} \mathscr{U}(p)$ covers $M$ such that $U \supset \mathscr{V}^*$. Without loss of generality we may assume that $|\mathscr{V}| \leqq \aleph$. Hence the total number of such collections $\mathscr{V}$ required so that we have at least one for each $U \supset M$ is $(\exp \aleph)^{\aleph}$ and $(\exp \aleph)^{\aleph} = \exp \aleph$.

**Lemma 3.** *If $\mathscr{M}$ is a collection of disjoint closed point sets $M$ such that $|\mathscr{M}| \leqq \aleph$ and $|M| \leqq \exp \aleph$, then $|\mathrm{Cl}\, \mathscr{M}^* - \mathscr{M}^*| \leqq \exp \aleph$.*

Indication of proof. If $p \in \mathrm{Cl}\, \mathscr{M}^* - \mathscr{M}^*$, then $p$ is the limit of a net $T$ obtained by selecting no more than one point from each element of $\mathscr{M}$. The total number of such nets $T$ is at most $(\exp \aleph)^{\aleph} = \exp \aleph$.

Now for the space in the theorem we construct our inverse system $\{X_\alpha, f_{\alpha\beta}\}$ for all $\alpha, \beta < \omega^+$ where $\omega^+$ is the smallest ordinal such that $|\omega^+| = \aleph^+$ (the smallest cardinal greater than $\aleph$). Suppose that $\alpha = \beta + 1$ and $x_\beta$ is an element of $X_\beta$. Then if $|x_\beta| \leqq \exp \aleph$, $f_{\alpha\beta}^{-1}(x_\beta)$ contains just one element $x_\alpha$ of $X_\alpha$; otherwise, $f_{\alpha\beta}^{-1}(x_\beta)$ is a collection of closed subsets of $x_\beta$ covering $x_\beta$ one of whose elements $x_\alpha^c$ is of cardinality no more than $\exp \aleph$ and is disjoint from all the others. If $\beta$ is a limit ordinal, $x_\beta = \bigcap f_{\beta\gamma}(x_\beta)$ for all $\gamma < \beta$ and $x_\alpha^c$ contains all of the limit points of $\{x_\gamma^c \mid f_{\alpha(\gamma-1)}(x_\alpha^c) = f_{\beta(\gamma-1)}(x_\beta)\}^*$ which belong to $x_\beta$.

Again, as in Roy's proof of Arhangelskii's theorem, the subsystem composed of those elements of $\bigcup X_\alpha$ $(\alpha < \omega^+)$ whose cardinality exceed $\exp \aleph$ has an empty inverse limit. Consequently the subcollection of $\bigcup X_\alpha$ of those of cardinality $\exp \aleph$ or less covers $S$. Since the cardinality of this collections is $\exp \aleph$ or less, the cardinality of its union and hence of $S$ is also $\exp \aleph$ or less.

## References


[1] *A. V. Arhangelskii:* On the cardinality of first countable compacta. Dokl. Akad. Nauk SSSR *187* (1969), 967—978.

[2] *L. Henkin:* A problem on inverse mapping systems. Proc. Amer. Math. Soc. *1* (1950), 224—225.

[3] *G. Higman and A. H. Stone:* On inverse systems with trivial limits. J. London Math. Soc. *29* (1954), 233—236.

[4] *F. B. Jones:* On certain well-ordered monotone collections of sets. J. Elisha Mitchell Scientific Society *69* (1953), 30—34.

[5] *F. B. Jones:* On a property related to separability in metric spaces. J. Elisha Mitchell Scientific Society *70* (1954), 30—33.

[6] *F. B. Jones:* Remarks on the normal Moore space metrization problem. Annals Math. Stud. *60* (1966), 115—119.

[7] *I. Juhász:* Arhangelskii's solution of Alexandroff's problem. Mathematisch Centrum Publication ZW 1969—013, Amsterdam, 1969.

[8] *G. Kurepa:* Ensembles ordonnés et ramifiés. Publications Mathématiques de l'Université de Belgrade *4* (1935), 1—138.

[9] *G. Kurepa:* Ensembles linéaires et une classe de tableaux ramifiés (Tableaux ramifiés de M. Aronszajn). Publications Mathématiques de l'Université de Belgrade *6—7* (1938), 129—160.

[10] *P. Roy:* The cardinality of first countable spaces. Bull. Amer. Math. Soc. (to appear).

[11] *F. D. Tall:* Set-theoretic consistency results and topological theorems concerning the normal Moore space conjecture and related problems. Dissertation, University of Wisconsin, 1969.

UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA