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SIMPLE CATEGORIES OF TOPOLOGICAL SPACES

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This contribution deals with productive and closed-hereditary classes of uniformizable Hausdorff spaces (= spaces), i.e., [16], with the epireflective subcategories of the category of all spaces. Such classes can be described as the classes $\mathcal{K}(\mathcal{E})$ of all $\mathcal{E}$-compact spaces for suitable classes $\mathcal{E}$ of spaces; here $\mathcal{E}$-compact means that the space can be embedded as a closed subspace into a product of spaces from $\mathcal{E}$ (see [8]). Clearly, if a class $\mathcal{C}$ is productive and closed-hereditary then $\mathcal{C} = \mathcal{K}(\mathcal{C})$ but the problem is to find a nice and small class $\mathcal{E}$ such that $\mathcal{C} = \mathcal{K}(\mathcal{E})$. If $\mathcal{E}$ may be found to be a set or, which is the same, to consist of a single space E, then $\mathcal{C}$ is called simple – [8]. The simple class $\mathcal{K}(E)$ generated by a space E and called E-compact spaces was introduced and studied in [17], [2].

Up today there is no known characterization of simple classes among all the productive and closed-hereditary classes. Here we want to present two comments concerning the problem: the first one describes a nice class of non-simple categories, and the second one shows a connection of the problem with set-theory.

We need some more concepts: $\mathcal{E}$-regular spaces are the homeomorphs of subspaces of products of spaces from $\mathcal{E}$ (hence if $\mathcal{E}$ is productive then $\mathcal{E}$-regular spaces are just homeomorphs of subspaces of members from $\mathcal{E}$). Relatively measurable cardinals are infinite cardinals α admitting non-trivial two valued measures which are β-additive for all $\beta < \alpha$; we denote by $\{\alpha_\xi\}$ the class of all relatively measurable cardinals arranged into the increasing transfinite sequence together with a symbol α_η following all the cardinals if all the relatively measurable cardinals form a set with the index-set $\{\xi \mid \xi < \eta\}$ (hence $\alpha_0 = \omega_0$ and an uncountable relatively measurable cardinal α_ξ is the first cardinal admitting a non-trivial two-valued measure which is α_ζ-additive for all $\zeta < \xi$). The term "nonmeasurable cardinal" is used in the sense of [6], i.e. any cardinal smaller than α_1. The symbol α^+ means the successor of cardinal α. For more details on $\mathcal{E}$-compact and $\mathcal{E}$-regular spaces or epireflective subcategories or relatively measurable cardinals we refer the reader to [8] or [9] or [15], respectively, where one can find further references.

The proofs of the following three assertions can be found in [13].

Theorem 1. *Let $\mathcal{C}$ be a productive class of spaces containing all the discrete spaces of cardinalities α_ζ, $\zeta < \xi$ for a given ordinal $\xi > 0$. If $\mathcal{C}$ is preserved by* *perfect maps onto $\mathscr{C}$-regular spaces, then $\mathscr{C}$ is not a part of $\mathscr{K}(E)$ for any space E of cardinality smaller than α_ξ.*

Corollary. *Let $\mathscr{C}$ be a productive and closed-hereditary class of spaces containing a space which is not countably compact. If $\mathscr{C}$ is preserved by perfect maps onto $\mathscr{C}$-regular spaces, then $\mathscr{C}$ is a part of no $\mathscr{K}(E)$ for any space E of nonmeasurable cardinality.*

Corollary. *If E is a space of nonmeasurable cardinality which is not countably compact, then the class $\mathscr{K}(E)$ is not preserved by perfect maps onto E-regular spaces.*

In the assertions stated above, the perfect maps (i.e. closed continuous maps with compact preimages of points) may be replaced by at most 2 to 1 closed continuous maps (at most 2 to 1 means that preimages of points contain at most two points).

The proof of Theorem 1 is based on the proof of a theorem in [5]; in fact, for nonmeasurable cardinals, Theorem 1 is a reformulation of a theorem from [5] without requiring regularity of certain cardinals.

It is clear that all the productive and closed-hereditary classes $\mathscr{C}$ of compact spaces are preserved by perfect maps onto $\mathscr{C}$-regular spaces without any regard to the simplicity of $\mathscr{C}$. So there remains a problem what is the situation if $\mathscr{C}$ is a productive and closed-hereditary class of countably compact spaces containing a space which is not compact. We believe the solution of this problem can be then generalized to other relatively measurable cardinals. Another problem is to find a general way how to find generators for simple classes; in most cases (e.g. [11], [12]) they were found by chance.

The class $\mathscr{S}$ of all spaces having a complete uniformity is productive and closed-hereditary. Is this class simple? Realize that $\mathscr{S}$ contains the class of all compact spaces which is somehow connected with the first relatively measurable cardinal $\alpha_0 = \omega_0$, and $\mathscr{S}$ also contains the bigger class of all realcompact spaces which is connected with α_1. Defining other bigger and bigger classes which are connected with other relatively measurable cardinals we obtain a monotone transfinite sequence of classes the union of which is the $\mathscr{S}$. Now it is sufficient to prove that each such class is simple and we receive the following

Theorem 2. *The class $\mathscr{S}$ of all spaces having complete uniformities is simple if and only if the class of all relatively measurable cardinals is a set.*

For the sake of completeness we shall describe here generators of the classes mentioned above. All the details and proofs may be found in [12]. Denote by $S(\alpha)$ the metrizable hedgehog with α prickles, i.e. α copies of the closed unit interval $[0, 1]$ sewed together in the point 0; denote $S_0 = S(1) = [0, 1]$ and for an ordinal

$\xi > 0$, $S_\xi = \Pi\{S(\alpha_\zeta) \mid \zeta < \xi\}$, where α_ξ are relatively measurable cardinals (see the paragraph before Theorem 1) – we need not take here all the ζ's smaller than ξ but only a cofinal part so that we may also define $S_{\xi+1} = S(\alpha_\xi)$. The covering character of a uniformity (see [14]) is the first cardinal α such that the uniform space contains no uniformly discrete subspaces of cardinality α; a space is said to be pseudo-α-compact if the covering character of its fine uniformity is at most α – [3], [14].

The following theorem may be regarded as a generalization of a Shirota's theorem from [18].

Theorem 3. *Let P be a space and $\alpha_\zeta \in \{\alpha_\xi\}$. Then the following conditions* (1)–(4) *are equivalent to each other; if $\zeta > 0$ then* (5) *is equivalent to* (1)–(4).

(1) *P is S_ζ-compact.*

(2) *P has a complete uniformity and no closed discrete (or closed discrete C^*-embedded or closed discrete C-embedded, respectively) subspace of cardinality α_ζ.*

(3) *P has a complete uniformity and no uniformly discrete subspace of cardinality α_ζ (i.e., P has a complete uniformity and is pseudo-α_ζ-compact).*

(4) *P has a complete uniformity with covering character at most α_ζ.*

(5) *P has a complete uniformity with covering character at most* $\sup\{\alpha_\eta^+ \mid \eta < \zeta\}$.

We restate Theorem 3 once more for the special case $\zeta = 1$ with additional characterizations by means of Herrlich's α-compact spaces [7] and of van der Slot's α-ultracompact spaces [19] (for the definitions see parts (g), (h), (i) of the next theorem). For $\zeta = 0$ we would obtain well-known characterizations of compact spaces.

Theorem 4. *For a space P, the following conditions are equivalent:*

(a) *P is realcompact.*

(b) *P can be embedded as a closed subspace into a power of the metrizable hedgehog $S(\omega_0)$.*

(c) *P has a complete uniformity and each of its closed discrete (or closed discrete C^*-embedded or closed discrete C-embedded, respectively) subspaces is of nonmeasurable cardinality.*

(d) *P has a complete uniformity and is pseudo-α_1-compact.*

(e) *P has a complete uniformity with covering character at most α_1.*

(f) *P has a complete uniformity with covering character at most ω_1.*

(g) *P has a complete uniformity and is α_1-compact (i.e., each maximal filter of zero-sets in P with nonmeasurable intersection property is fixed).*

(h) *P has a complete uniformity and is ω_1-ultracompact (i.e., each ultrafilter with countable intersection property for closed sets converges).*

(i) *P has a complete uniformity and is α_1-ultracompact (i.e., each ultrafilter with nonmeasurable intersection property for closed sets converges).*

Using the foregoing assertions for the classes $\mathscr{C}_\alpha$ or $\mathscr{U}_\alpha$ of all α-compact or α-ultracompact spaces, respectively, we obtain the following interesting results:

(α) If α is not a relatively measurable cardinal, then $\mathscr{C}_\alpha$ is not preserved by perfect maps.

(β) If α is not a relatively measurable cardinal, then $\mathscr{C}_\alpha$ is a proper subclass of $\mathscr{U}_\alpha$ (a partial solution of Problem 8 from [10]).

(γ) For any uncountable cardinal α and any nonmeasurable cardinal β there is an α-ultracompact zerodimensional space which is not β-compact.

(δ) α-compactness and β-ultracompactness coincide on spaces with complete uniformities for α, β having the same first greater relatively measurable cardinal.

There remains an open problem what is the connection between α-compactness and α-ultracompactness for uncountable relatively measurable α; under the condition S posed on relatively measurable cardinals in [1] these concepts coincide so that the problem seems to be in a close connection with properties of relatively measurable cardinals. Any way, for every α, α-ultracompact spaces are just uniformizable perfect images of α-compact spaces (see [4]).

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