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PARACOMPACT SUBSETS

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In this paper, we distinguish 3 types of paracompact subsets and 2 types of countably paracompact subsets.

Definition 1. A subset M of a topological space $(X, \mathscr{T})$ is *α-paracompact* (*σ-paracompact*) if every open cover by members of $\mathscr{T}$ has an open locally finite (σ-locally finite) refinement by members of $\mathscr{T}$.

Definition 2. A subset M of a topological space $(X, \mathscr{T})$ is *α-countably paracompact* if every countable open cover by members of $\mathscr{T}$ has an open locally finite refinement by members of $\mathscr{T}$.

In the above definitions the refinements are locally finite or σ-locally finite with respect to all points of X and not just points of M.

Definition 3. A subset M of a topological space is *β-paracompact* (*β-countably paracompact*) if M is a paracompact (countably paracompact) subspace.

We shall need also the following definition.

Definition 4. A subset M of a topological space is *α-collectionwise normal* if for every discrete family $\{D_a\}$, $D_a \subset M$, there is a pairwise disjoint family of open sets $\{G_a\}$ such that $D_a \subset G_a$ for every a.

In the literature the term paracompact subsets generally refers to β-paracompact subsets. Clearly every α-paracompact set is σ-paracompact and every σ-paracompact set in a regular space is β-paracompact. See Michael [7, 834]. In this paper we will prove the following: A β-paracompact subset of a regular normal space is σ-paracompact iff it is α-collectionwise normal and a generalized F_σ; in a regular normal space, closed σ-paracompact subsets are α-paracompact.

In order to prove the latter result, it will first be proved that in a normal space a closed β-countably paracompact subset is α-countably paracompact. It will also be proved that closed subsets of the interior of β-paracompact subsets in normal spaces are α-paracompact (Theorem 1).

The α-paracompact subsets will be shown to behave in many respects as compact subsets (Theorems 4–7 and corollaries 12A and 12B). For instance in a T_2 space, two disjoint α-paracompact subsets are strongly separated. σ-paracompact subsets have certain similarities to Lindelöf subsets (Theorem 13). In the definition of σ-paracompact subsets, σ-locally finite may be replaced by σ-discrete in regular normal spaces.

In general the notation of Kelley will be used. We will use a and b as subscripts with the understanding that a and b are members of arbitrary sets A and B respectively without refering specifically to the index sets. The convention $T_3 = T_1 +$ regular and all similar conventions will be used here. The definitions of locally dense and generalized F_σ set may be found in Corson and Michael [4].

Some Basic Theorems

Theorem 1. *Let F be a closed subset of the interior, G, of a closed β-paracompact (β-countably paracompact) subset M of a topological space $(X, \mathscr{T})$. Then F is α-paracompact (α-countably paracompact).*

Proof. Let $\mathscr{U}$ be an open cover of F. The family consisting of $\{U \cap M : U \in \mathscr{U}\}$ and the set $M \sim F$ is an open cover of M using the relative topology for M and has a locally finite open refinement with respect to M. Let $\mathscr{V}$ consist of members of this refinement contained in a member of $\{U \cap M : U \in \mathscr{U}\}$. Let $\mathscr{W} = \{V \cap G : V \in \mathscr{V}\}$. For $V \in \mathscr{V}$, $V = T \cap M$ where T is open in X. So $W = V \cap G$ is open in X. Since M is closed, $\mathscr{W}$ is locally finite with respect to points of $\sim M$. Hence $\mathscr{W}$ is a locally finite open refinement of $\mathscr{U}$ and F is α-paracompact. The proof involving countably paracompact subsets is similar.

Corollary 1A. *A closed subset of a paracompact (countably paracompact) space is α-paracompact (α-countably paracompact).*

Corollary 1B. *Let F be a closed subset of the interior G of a β-paracompact subset M of a normal topological space $(X, \mathscr{T})$. Then F is α-paracompact.*

Proof. Since $(X, \mathscr{T})$ is normal, there exists an open set V such that $F \subset V \subset \subset \bar{V} \cup G$. $\bar{V}$ is β-paracompact and by Theorem 1, F is α-paracompact.

For countably paracompact subsets there is an analogous result, but we can obtain a stronger result.

Theorem 2. *Let F be a closed β-countably paracompact set in a normal space. Then F is α-countably paracompact.*

Proof. Let $\{U_n\}$ be a countable open cover of F. It follows from a theorem of Dowker [5,220] that there is a family of relative open sets with respect to F, $\{V_n\}$, such that $\{V_n\}$ covers F and $\bar{V}_n \subset U_n$, where $\bar{V}_n$ is the relative closure with respect to F of V_n. Since F is closed $\bar{V}_n$ is closed in X. By the normality property, there exists an open cover $\{W_n\}$ (open in X) of F such that $\bar{V}_n \subset W_n \subset \bar{W}_n \subset U_n$. Set $W = \bigcup W_n$. Let G be open and such that $F \subset G \subset \bar{G} \subset W$. Set $T_1 = U_1 \cap G$; $T_n = (U_n \cap G) \sim$

$\sim \bigcup_{k=1}^{n-1} \overline{W}_k$. Clearly $\{T_n\}$ is open in X and is a refinement of $\{U_n\}$. If $x \in F$, there exists m such that $x \in U_m$ and $x \notin U_k$ for $k < m$; so $x \in T_m$ and $\{T_n\}$ is a cover of F. It remains to show that $\{T_n\}$ is locally finite in regard to all points of X. If $x \notin W$, $\sim \overline{G}$ is a neighborhood of x not intersecting T_n for any n. If $x \in W$, there exists m such that $x \in W_m$ and $x \notin W_k$ for $k < m$. W_m intersects at most a finite number of T_n.

Later we will show that in regular collectionwise normal spaces, an analogous result is satisfied for paracompact subsets. We now use Theorem 2 to relate σ-paracompact subsets to α-paracompact subsets.

Theorem 3. *Let M be a σ-paracompact, α-countably paracompact subset in a topological space $(X, \mathscr{T})$. Then M is α-paracompact.*

Proof. Let $\mathscr{U}$ be an open cover of M. There is an open σ-locally finite refinement of $\mathscr{U}$, $\mathscr{V} = \bigcup \mathscr{V}_n$ where each $\mathscr{V}_n$ is locally finite. Let $W_n = \bigcup\{V : V \in \mathscr{V}_n\}$. Then $\{W_n\}$ is a countable open cover of M with countable open locally finite refinement $\{T_n\}$. Let $\mathscr{S}_n = \{V \cap T_n : V \in \mathscr{V}_n\}$, $\mathscr{S} = \bigcup \mathscr{S}_n$. For $x \in X$, there exists a neighborhood N_x intersecting a finite number of $\{T_n\}$, $T_{x1}, T_{x2}, \ldots, T_{xm}$. There exist neighborhoods $N_{x1}, N_{x2}, \ldots, N_{xm}$ intersecting a finite number of members of $\mathscr{S}_{x1}, \mathscr{S}_{x2}, \ldots, \mathscr{S}_{xm}$, respectively. Set $N_{x0} = N_x$. The intersection $\bigcap_{i=0}^{m} N_{xi}$ is a neighborhood of x intersecting a finite number of members of $\mathscr{S}$.

Corollary 3. *A σ-paracompact closed subset of a normal regular space is α-paracompact.*

Proof. Theorems 2 and 3 and the fact that in regular spaces σ-paracompact subsets are β-paracompact and hence β-countably paracompact.

Properties of α-Paracompact Subsets

We now turn to some theorems about α-paracompact subsets, in which these subsets behave similar to compact subsets. We will use the terminology A is strongly separated from B to indicate that there are disjoint open sets U and V containing A and B respectively.

Theorem 4. *Let $(X, \mathscr{T})$ be T_2 and let M be α-paracompact and let $x \notin M$. Then x and M are strongly separated.*

Proof. Since X is T_2, for $y \in M$, there exists open U_y such that $x \notin \overline{U}_y$. The cover $\{U_y : y \in M\}$ has an open locally finite refinement $\{V_a\}$. $x \notin \overline{V}_a$ so $\bigcup V_a$ and $\sim \bigcup \overline{V}_a$ are disjoint open sets containing M and $[x]$ respectively.

Corollary 4. *In a T_2 space, every α-paracompact subset is closed.*

The proofs of the next two theorems are similar to that of Theorem 4.

Theorem 5. *Let $(X, \mathcal{T})$ be T_2 and let M and N be disjoint α-paracompact subsets. Then M and N are strongly separated.*

A. H. Stone [8,363] proved that a necessary and sufficient condition for a space S to be metrizable where $S = S_1 \cup S_2$ are open and metrizable is that $\mathrm{Fr}(S_1)$ and $\mathrm{Fr}(S_2)$ are strongly separated. From the above theorem this will happen iff these boundaries are α-paracompact.

Theorem 6. *Let $(X, \mathcal{T})$ be regular and let M be α-paracompact and F closed, $M \cap F = \emptyset$. Then M is strongly separated from F.*

It is clear that the known theorems of Dieudonné that T_2 paracompact spaces are T_4 and regular paracompact spaces are normal follow from Theorems 5 and 6 respectively.

Theorem 7. *In a regular space the closure of an α-paracompact subset is α-paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of $\overline{M}$ where M is α-paracompact. Let $\{V_a\}$ be an open locally finite refinement of $\mathcal{U}$ that covers M. For each x and each V_a such that $x \in V_a$, there is an open set W_{xa} such that $x \in W_{xa} \subset \overline{W}_{xa} \subset V_a$. The family $\{W_{xa}\}$ is an open cover of M and has an open locally finite refinement $\{T_b\}$. $\overline{M} \subset \bigcup \overline{T}_b \subset \subset \bigcup V_a$, so $\{V_a\}$ is a cover of $\overline{M}$.

σ-Paracompact subsets

In this section the properties of σ-paracompact subsets and the relations of these subsets with β-paracompact subsets are discussed.

The next theorem is a modification of a theorem of Bing [3,177].

Theorem 8. *A σ-paracompact subset in a regular space is α-collectionwise normal.*

Proof. Let M be σ-paracompact and let $\{D_a\}$ be a discrete family of subsets of M. Let $\mathcal{U}$ be an open cover of M such that the closure of each member intersects at most one member of $\{D_a\}$. $\mathcal{U}$ has an open σ-locally finite refinement $\mathcal{V} = \bigcup \mathcal{V}_n$ where each $\mathcal{V}_n$ is a locally finite family. For each a, let W_{an} be the union of members of $\mathcal{V}_n$ intersecting D_a. Set $T_{an} = W_{an} \sim \bigcup_{k=1}^{n} \bigcup_b \{\overline{W}_{bk} : b \neq a\}$ and set $T_a = \bigcup_{n=1}^{\infty} T_{an}$. $\{T_a\}$ is a family of pairwise disjoint open sets such that $D_a \subset T_a$.

Theorem 9. *A locally dense β-paracompact subset of a regular space $(X, \mathcal{T})$ is α-collectionwise normal.*

Proof. Let M be β-paracompact and locally dense. M is then dense in an open set G using the relative topology for G. Let $\{D_a\}$, $D_a \subset M$, be discrete in X and hence discrete in M. Since M is β-paracompact, there is a pairwise disjoint family of sets,

$\{U_a\}$, open in M such that $D_a \subset U_a$. $U_a = V_a \cap M$ where V_a is open in G and hence open in X. Assume $V_a \cap V_b \neq \emptyset$ for some $a \neq b$. Then there is a non-null subset of $G \sim M$ contrary to M being dense in G. So the members of $\{V_a\}$ are pairwise disjoint.

Theorem 10. *Let $(X, \mathcal{T})$ be a normal regular topological space. Let a β-paracompact, α-collectionwise normal, subset M be a generalized F_σ-subset. Then every open cover of M (using the topology for X) has an open σ-discrete refinement and M is σ-paracompact.*

Because of the similarity of the proof to the proof of Theorem 5.2 of Corson and Michael [4,356], we omit the proof.

Theorem 11. *A σ-paracompact subset of a regular space $(X, \mathcal{T})$ is a generalized F_σ.*

Proof. Let M be σ-paracompact. Let G be an open set such that $M \subset G$. For $x \in M$, there is a closed neighborhood C_x such that $C_x \subset G$. Let $\mathcal{V} = \bigcup \mathcal{V}_n$ be an open σ-locally finite refinement of $\{\text{int } C_x : x \in M\}$ where each $\mathcal{V}_n$ is locally finite. For each n let $F_n = \bigcup(\overline{V} : V \in \mathcal{V}_n\}$. The set $H = \bigcup F_n$ is such that $M \subset H \subset G$ so that M is a generalized F_σ.

Corollary 11A. *A β-paracompact subset of a normal regular space is σ-paracompact iff it is α-collectionwise normal and a generalized F_σ.*

Proof. Theorems 8, 10, and 11.

Corollary 11B. *A closed β-paracompact subset of a normal regular space is α-paracompact iff it is α-collectionwise normal.*

Proof. Theorem 8 and Corollaries 3 and 11A.

Corollary 11C. *Let $(X, \mathcal{T})$ be normal and regular. A β-paracompact, locally dense subset is σ-paracompact iff it is a generalized F_σ.*

Proof. Theorem 9 and Corollary 11A.

Corollary 11D. *A subset M of a normal regular space $(X, \mathcal{T})$ is σ-paracompact iff every open cover by members of $\mathcal{T}$ has an open σ-discrete refinement.*

Proof. Theorems 8, 10, and 11.

Corollary 11E. *In collectionwise normal, perfectly normal spaces, every β-paracompact subset is σ-paracompact.*

Theorem 12. *Let a σ-paracompact subset M be the complement of an α-paracompact subset in a T_2 space. Then M is an F_σ.*

Proof. By Theorem 4, there exists a closed neighborhood C_x of x such that $C_x \subset M$ for $x \in M$. The proof is now similar to Theorem 11 noting that M is open.

Corollary 12A. *An α-paracompact subset M of a T_2 hereditary Lindelöf space is a G_δ.*

Corollary 12B. *An α-paracompact subset in a T_2 space with a σ-locally finite base is a G_δ.*

For compact subsets, one may substitute point-countable base for σ-locally finite base in the above corollary. See Aull [2].

Theorem 6 shows that two disjoint subsets one α-paracompact and the other closed are strongly separated in a regular space. For σ-paracompact subsets we have the following theorem.

Theorem 13. *Let F and H be two closed σ-paracompact subsets in a regular space. Then F and H are strongly separated.*

Proof. For $x \in F$, there exists a closed neighborhood C_x such that $C_x \cap H = \emptyset_n$ The family $\{C_x : x \in F\}$ has an open σ-locally finite refinement $\mathscr{U} = \bigcup_n \mathscr{U}_n$ and each $\mathscr{U}.$ is an open locally finite family. If $U \in \mathscr{U}$, $\overline{U} \cap H = \emptyset$. Let $V_n = \bigcup\{U : U \in \mathscr{U}_n\}$. V_n is open and $\overline{V}_n \cap H = \emptyset$. Similarly, one can construct a countable open cover of H, $\{W_n\}_n$, such that $\overline{W}_n \cap F = \emptyset$. Let $S_n = V_n \sim \bigcup_{k=1}^{n} \overline{W}_k$ and $S = \bigcup S_n$ and let $T_n = W_n \sim \sim \bigcup_{k=1}^{n} \overline{V}_k$ and $T = \bigcup T_n$. S and T are disjoint open sets containing F and H respectively.

Corollary 13. *In a regular space, two disjoint closed Lindelöf subsets are strongly separated.*

The example of Niemytski of a T_{3a} space that is not T_4 contains a closed σ-paracompact subset that is not α-paracompact and a closed β-paracompact subset that is not σ-paracompact.

Example. Let X be the upper half plane including the x-axis. If $y_0 > 0$, let Hausdorff neighborhoods of (x_0, y_0) be the usual neighborhoods of the plane relativized with respect to X. If $y_0 = 0$, let Hausdorff neighborhoods of $(x_0, 0)$ consist of open circles with center (x_0, y) and radius y with the point $(x_0, 0)$ for each $y > 0$.

It is known that the set R of rationals on the x-axis and the set I of irrationals on the x-axis are disjoint closed sets which are not strongly separated. See Vaidyanathaswamy [9, 153]. R is σ-paracompact, but not α-paracompact by Theorem 6 and by Theorem 3 not even α-countably paracompact. By Theorem 13, I is not σ-paracompact though it is β-paracompact.

Some Further Remarks

In general countable unions of β-paracompact subsets are not β-paracompact as pointed out by Corson and Michael [4,356]. On the other hand countable unions of σ-paracompact subsets are σ-paracompact. From this fact we can show that there exists a β-paracompact closed set in a perfectly normal T_1 space that is not σ-paracompact.

The closed set F_p of [3, example H] of Bing has this property. F_p and its complement $F \sim F_p$ are both metrizable, the latter being an F_σ and in fact σ-paracompact. Corson and Michael [4,359] noted that this space F is not paracompact. Thus F_p cannot be σ-paracompact. It is interesting to note that F_p is α-countably paracompact.

However with β-countably paracompact subsets we have the following theorem.

Theorem 14. *Let $\{M_n\}$ be a countable family of F_σ, β-countably paracompact subsets in a normal space $(X, \mathscr{T})$. Then $M = \bigcup M_n$ is β-countably paracompact.*

Proof. Let $\{U_k\}$ be a countable open cover of M. Mansfield [6,445] has shown that a normal space is countably paracompact iff every countable open cover has a closed σ-discrete refinement. Let Q be a member of $\{M_n\}$. So there is a σ-discrete relative closed refinement of $\{U_k\}$, $\mathscr{B} = \bigcup \mathscr{B}_n$, where each $\mathscr{B}_n$ is discrete and closed with respect to Q. There exists a countable family of closed sets $\{F_i\}$ such that $Q = \bigcup F_i$. For fixed i and n, the family $\{F_i \cap B : B \in \mathscr{B}_n\}$ is closed and discrete with respect to X and the theorem follows, again using the result of Mansfield.

For some additional properties of α-countably paracompact subsets, see Aull [1].

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