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Thermal Effects in an Elastic Plate-beam Structure

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Abstract. We consider a linear model for a $2-D$ hybrid elastic structure consisting of a thermo-elastic plate which has a beam attached to its free end. We show that the interplay of parabolic dynamics and hyperbolic dynamics in the model yields analyticity for the entire system. This result provides an easy route to uniform stability.

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1 Introduction and Statement of the Problem

We consider well-posedness of the following model, Pr (P), for the transversal vibrations of a hybrid structure consisting of a thin rectangular thermo-elastic plate which is clamped along three edges, while to its free edge a thin beam with ends clamped to the adjoining clamped edges of the plate, is attached:

$$
\begin{aligned}
w_{tt} + \Delta^2 w + \alpha\Delta\theta &= 0 \text{ in } \Omega_T \\
w = 0 &= \frac{\partial w}{\partial n} \text{ on } \partial\Omega_T - \Gamma_T \\
\beta\theta_t - \eta\Delta\theta - \alpha\Delta w_t &= 0 \text{ in } \Omega_T \\
\theta &= 0 \text{ on } \partial\Omega_T - \Gamma_T
\end{aligned}
$$

This is an overview article.

$$
\begin{aligned}
w_{tt} - [w_{xxx} + (2-\nu)w_{xyy}] + w_{yyyy} - \alpha\frac{\partial\theta}{\partial n} + b\theta_{yy} &= 0 \text{ on } \Gamma_T \\
\frac{\partial w}{\partial n} &= 0 \text{ on } \Gamma_T \\
w = 0 &= w_y \text{ at } \partial\Gamma_T \\
\beta\theta_t + \eta\frac{\partial\theta}{\partial n} - \kappa\theta_{yy} - bw_{yyt} &= 0 \text{ on } \Gamma_T \\
\theta &= 0 \text{ at } \partial\Gamma_T, \\
w(x,y,0) = w_0(x,y),\ w_t(x,y,0) = w_1(x,y),\ \theta(x,y,0) &= \theta_0(x,y) \text{ in } \Omega \\
w(a,y,0) = \mu_0(y),\ w_t(a,y,0) = \mu_1(y),\ \theta(a,y,0) &= \theta_1(y) \text{ on } \Gamma.
\end{aligned}
$$

Here Ω denotes the interior of the plate with corner points $(0,0),(a,0),(a,\ell)$ and $(0,\ell)$, while Γ is the line joining $(a,0)$ and (a,ℓ) and $\partial\Gamma$ its end-points.

The constitutive equations in $Pr\ (P)$ are "contact" equations in the sense that the deflections as well as the temperatures of the plate and the beam match at the interface for $t>0$, but not necessarily initially. Thus the $1-D$ biharmonic equation and heat equation along Γ form a system of dynamic boundary conditions for the thermo-elastic plate equations. By allowing for interaction between the plate and the beam, the partial differential equations along Γ contain additional terms: the third order space derivatives of the displacement variable w in the beam equation represent the combined shear force and twisting moment exerted by the plate on the beam, while the conormal derivative of the thermal variable θ in the heat equation along Γ reflects the flux of heat from the plate to the beam across the interface Γ.

2 Implicit Evolution Equation for $Pr\ (P)$

We formulate $Pr\ (P)$ as an implicit evolution problem, $Pr\ (AEP)$, of the form
Find U such that

$$
\begin{aligned}
\frac{d}{dt}(BU(t)) + AU(t) = 0,\ U \in \mathcal{D} \subset X,\ t > 0 \\
\lim_{t\to 0^+} BU(t) = y \in Y
\end{aligned}
$$

with A and B operators from a Banach space X to a second Banach space Y. The construction of a unique solution of $Pr\ (AEP)$ with representation $U(t) = S(t)y$ entails the construction of a double family of evolution operators [4], viz. $\langle\{\mathcal{S}(t),\mathcal{E}(t)\}\rangle = \langle\{S(t): Y \to X | t > 0\}, \{E(t): Y \to Y | t > 0\}\rangle$, with $E(t) =: BS(t)$ a semigroup in Y. The evolution from an initial state in Y to a solution in the space X, is generated by the jointly closed operator pair $\langle -A, B\rangle : \mathcal{D} \to Y \times Y$ in which $\mathcal{R}(B)$ is dense in Y.

3 Mathematical Setting for $Pr\ (P)$

We define the following spaces and operators:

$X_0 =: L^2(\Omega)$ with inner product $(,)_0$ and norm $\|.\|_0$.

$H^m(\Omega) = H^{m,2}(\Omega)$ denotes the usual Sobolev spaces with inner products $(,)_m$ and norms $\|.\|_m$ when $m > 0$ and the Hilbert space $L^2(\Omega)$ when $m = 0$. $(,)_{m,\Gamma}$ and $\|.\|_{m,\Gamma}$ denote the inner products and norms in $H^m(\Gamma)$.

For $u \in H^m(\Omega)$ we denote the trace of u on Γ by γu.

We define the following subspaces of X_0:

$$X_1 =: \{w \in H^1(\Omega)\Big| w = 0 \text{ on } \partial\Omega - \Gamma, \gamma w \in H_0^1(\Gamma)\}.$$

$$X_2 =: \{w \in H^2(\Omega)\Big| w = 0 = \tfrac{\partial w}{\partial n} \text{ on } \partial\Omega - \Gamma, \tfrac{\partial w}{\partial n} = 0 \text{ on } \Gamma, \gamma w \in H_0^2(\Gamma)\}.$$

The spaces $X_i, i = 0, 1, 2$ are endowed with the inner products $(,)_i$ and the norms $\|.\|_i$. For X_2 we also use the equivalent inner product $((,))_2$ given by

$$a(w, z) = (w_{xx}, z_{xx})_0 + 2(1-\nu)(w_{xy}, z_{xy})_0 + (w_{yy}, z_{yy})_0 + \nu(w_{xx}, z_{yy})_0 + \nu(w_{yy}, z_{xx})_0.$$

The associated norm will be denoted by $|||.|||_2$.

$Y_0 =: X_0 \times L^2(\Gamma)$. The (usual) inner product and norm are denoted by $(,)_{Y_0}$ and $\|.\|_{Y_0}$.

The domains D_1 and D_2 are defined by

$$D_1 =: \{w \in H^4(\Omega)\Big| w = 0 = \tfrac{\partial w}{\partial n} \text{ on } \partial\Omega - \Gamma, \tfrac{\partial w}{\partial n} = 0 \text{ on } \Gamma, \gamma w \in H^4(\Gamma) \cap H_0^2(\Gamma)\}.$$

$$D_2 =: \{\theta \in H^2(\Omega)\Big| \theta = 0 \text{ on } \partial\Omega - \Gamma, \gamma\theta \in H^2(\Gamma) \cap H_0^1(\Gamma)\}.$$

The operators A, B and $C_j, j = 1, 2, 3$ from X_0 into Y_0 are defined by

$$Aw =: \left\langle \Delta^2 w, -[\gamma(w_{xxx} + (2-\nu)w_{xyy})] + (\gamma w)_{yyyy} \right\rangle,$$

$$Bw =: \langle w, \gamma w \rangle,\ w \in D_1 = \mathcal{D}(A).$$

$$C_1\theta =: \left\langle \alpha\Delta\theta, -\alpha\gamma\tfrac{\partial\theta}{\partial n} + b(\gamma\theta)_{yy} \right\rangle,\ \theta \in D_2,$$

$$C_2\dot{w} =: \tfrac{1}{\beta}\left\langle -\alpha\Delta\dot{w}, -b(\gamma\dot{w})_{yy} \right\rangle,\ \dot{w} \in X_2,$$

$$C_3\theta =: \tfrac{1}{\beta}\left\langle -\eta\Delta\theta, \eta\gamma\tfrac{\partial\theta}{\partial n} - \kappa(\gamma\theta)_{yy} \right\rangle,\ \theta \in D_2.$$

Observing that $\mathcal{R}(B) = \{\langle w, \gamma w \rangle, w \in D_1\}$ is a proper subset of Y_0, we define the following subsets of $X_1 \times H_0^1(\Gamma)$ and $X_2 \times H_0^2(\Gamma)$:

$Y_1 = \mathcal{C}\ell(B[X_1]), Y_2 = \mathcal{C}\ell(B[X_2])$, with closures taken in Y_0.

Y_1 will be endowed with the norm $\|Bw\|_{Y_1} = (\|\nabla w\|_0^2 + \|(\gamma w)_y\|_{0,\Gamma}^2)^{\frac{1}{2}}$ and Y_2 with the norm $|||Bw|||_{Y_2} \equiv ((Bw, Bw))_{Y_2} = (a(w, w) + \|(\gamma w)_{yy}\|_{0,\Gamma}^2)^{\frac{1}{2}}$.

To cast $Pr\ (P)$ in the abstract form $Pr\ (AEP)$, we define product spaces equipped with product space inner products and norms, viz. the "finite energy" space $\mathcal{X}_\mathcal{E}$ and its accompanying space $\mathcal{Y}_\mathcal{E}$ and a weaker space $\mathcal{X}$, with accompanying space $\mathcal{Y}$:

$\mathcal{X}_{\mathcal{E}} =: X_2 \times (X_0)^2, \mathcal{Y}_{\mathcal{E}} =: Y_2 \times (Y_0)^2.$

In $\mathcal{X}_{\mathcal{E}}$ we define the domain
$\mathcal{D}_{\mathcal{E}} =: \{U_{\mathcal{E}} = (w, \dot{w}, \theta), w \in D_1, \dot{w} \in X_2, \theta \in D_2\}$

The linear operators $\mathcal{A}$ and $\mathcal{B}$ on the common domain $\mathcal{D}_{\mathcal{E}}$ are now defined by
$$\mathcal{A}U_{\mathcal{E}} =: \begin{bmatrix} 0 & -B & 0 \\ A & 0 & C_1 \\ 0 & C_2 & C_3 \end{bmatrix} U_{\mathcal{E}},\ \mathcal{B}U_{\mathcal{E}} =: \begin{bmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix} U_{\mathcal{E}}, U \in \mathcal{D}_{\mathcal{E}}.$$

To define the weaker spaces $\mathcal{X}, \mathcal{Y}$ we first define
$H =: \{w \in X_0 \Big| A^{\frac{1}{2}} w = 0\},\ W_{X_0} = H^{\perp}.$ [1][2]

$Z =: \{y = \langle y_1, y_2\rangle \in Y_0 \Big| A^{\frac{1}{2}} y_1 = 0\},\ W_{Y_0} = Z^{\perp}.$

$\mathcal{X} =: (W_{X_0} \cap X_0) \times (X_0)^2,\ \mathcal{Y} =: (W_{Y_0} \cap Y_0) \times (Y_0)^2.$

To define a domain $\mathcal{D}$ in $\mathcal{X}$, we introduce the variable $U =: (u, \dot{w}, \theta), Bu =: -A^{\frac{1}{2}} w.$

$\mathcal{D} =: \{U = (u, \dot{w}, \theta), u \in (W_{X_0} \cap H^2(\Omega)), \dot{w} \in X_2, \theta \in D_2.\}$

The linear operators $\mathcal{L}$ and $\mathcal{M}$ from $\mathcal{X}$ to $\mathcal{Y}$ are now defined by

$$\mathcal{L}U =: \begin{bmatrix} 0 & A^{\frac{1}{2}} & 0 \\ -A^{\frac{1}{2}} & 0 & C_1 \\ 0 & C_2 & C_3 \end{bmatrix} U,\ \ \mathcal{M}U =: \begin{bmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix} U,\ \ U \in \mathcal{D}.$$

$Pr\ (P)$ may now be cast in the form of implicit evolution problems, viz. $Pr\ (AEP)_I$:

$$\frac{d}{dt}(\mathcal{B}U_{\mathcal{E}}(t)) + \mathcal{A}U_{\mathcal{E}}(t) = 0,\ \mathcal{U}_{\mathcal{E}} \in \mathcal{D}_{\mathcal{E}},\ t > 0,$$
$$\lim_{t \to 0^+} \mathcal{B}U_{\mathcal{E}}(t) = \mathcal{G} \in \mathcal{Y}_{\mathcal{E}}$$

or $Pr\ (AEP)_{II}$:

$$\frac{d}{dt}(\mathcal{M}U) + \mathcal{L}U = 0,\ U \in \mathcal{D},\ t > 0,$$
$$\lim_{t \to 0^+} \mathcal{M}U(t) = \mathcal{F} \in \mathcal{Y}.$$

4 Main Results

Th e reader is referred to [3] for the detailed proofs.

Lemma 1.

$$\begin{aligned} & Re\,\{(\mathcal{A}U_{\mathcal{E}}, \mathcal{B}U_{\mathcal{E}})_{\mathcal{Y}_{\mathcal{E}}}\} \\ &= Re\{-((B\dot{w}, Bw))_{Y_2} + (Aw, B\dot{w})_{Y_0} + (C_1\theta, B\dot{w})_{Y_0} + (C_2\dot{w} + C_3\theta, B\theta)_{Y_0}\} \\ &= \eta\|\nabla\theta\|_0^2 + \kappa\|(\gamma\theta)_y\|_{0,\Gamma}^2, \end{aligned}$$

$Re\{(\mathcal{L}U, \mathcal{M}U)_{\mathcal{Y}}\} = \eta\|\nabla\theta\|_0^2 + \kappa\|(\gamma\theta)_y\|_{0,\Gamma}^2.$

We prove

Theorem 2. *The operator pair* $\langle -\mathcal{A}, \mathcal{B}\rangle$ *generates a unique uniformly bounded double family* $\langle \mathcal{S}, \mathcal{E}\rangle = \langle \{S(t) : \mathcal{Y}_{\mathcal{E}} \to \mathcal{X}_{\mathcal{E}} | t > 0\}, \{E(t) : \mathcal{Y} \to \mathcal{Y} | t > 0\}\rangle$ *of evolution operators. Thus Pr* $(AEP)_I$ *has unique solution* $U_{\mathcal{E}} \in C((0,\infty); \mathcal{D}_{\mathcal{E}})$ *with representation* $U_{\mathcal{E}}(t) = S(t)\mathcal{G}$ *for any* $\mathcal{G} \in \mathcal{R}(\mathcal{B})$ *and each* $t \in (0,\infty)$.

Corollary 3. *Pr* (P) *in* (w, w_t, θ) *can be associated with a uniformly bounded evolution operator* $S(t)\colon \mathcal{Y}_{\mathcal{E}} \to \mathcal{D}_{\mathcal{E}} \subset \mathcal{X}_{\mathcal{E}}$ *in the sense that* $S(t)\mathcal{G} = U_{\mathcal{E}}(t) = (w, \dot{w}, \theta)$ *solves Pr* $(AEP)_I$ *for any* $\mathcal{G} = (G_1, G_2, G_3) = (\langle g_1, \gamma g_1\rangle, \langle g_2, \gamma g_2\rangle, \langle g_3, \gamma g_3\rangle)$ *such that*

$$\begin{cases} g_1 \in D_1,\ \gamma g_1 \in H^4(\Gamma) \cap H_0^2(\Gamma) \\ g_2 \in X_2,\ \gamma g_2 \in H_0^2(\Gamma) \\ g_3 \in D_2,\ \gamma g_3 \in H^2(\Gamma) \cap H_0^1(\Gamma). \end{cases}$$

The restriction that each $G_i, i = 1,2,3$, of $\mathcal{G}$ is of the form $\langle g_i, \gamma g_i\rangle$, may be interpreted as meaning that the initial displacement, velocity and temperature in the plate and the beam should match along Γ.

Theorem 4. *The operator pair* $\langle -\mathcal{L}, \mathcal{M}\rangle$ *generates a unique analytic uniformly bounded double family* $\langle \mathcal{S}, \mathcal{E}\rangle = \langle \{S(t) : \mathcal{Y} \to \mathcal{X} | t > 0\}, \{E(t) : \mathcal{Y} \to \mathcal{Y} | t > 0\}\rangle$ *of evolution operators. Thus Pr* $(AEP)_{II}$ *has unique solution* $U \in C((0,\infty); \mathcal{D})$ *with representation* $U(t) = S(t)\mathcal{F}$ *for any* $\mathcal{F} \in \mathcal{Y}$ *and each* $t \in (0,\infty)$.

Corollary 5. *Pr* (P) *in* (w, w_t, θ) *can be associated with an analytic evolution operator* $S(t) : \mathcal{Y} \to \mathcal{D} \subset \mathcal{X}$ *in the sense that* $S(t)\mathcal{F} = U(t) = (u, \dot{w}, \theta)$ *solves Pr* $(AEP)_{II}$ *for any* $\mathcal{F} = (\langle f_1, f_2\rangle, \langle g_1, g_2\rangle, \langle h_1, h_2\rangle)$ *such that*

$$\begin{cases} f_1 \in X_0, 0 \neq \mathcal{E}(\langle f_1, \gamma f_1\rangle),\ f_2 \in L^2(\Gamma) \\ g_1 \in X_0,\ g_2 \in L^2(\Gamma) \\ h_1 \in X_0,\ h_2 \in L^2(\Gamma) \end{cases}$$

with $2\mathcal{E}(\langle f_1, \gamma f_1\rangle) = a(f_1, f_1) + \|(\gamma f_1)_{yy}\|_{0,\Gamma}^2$ *the elastic potential energy.*

With the aid of Lemma 1 we obtain uniform stability for $Pr\ (AEP)_{II}$:

Theorem 6. *There exist constants* $M, \sigma > 0$ *such that for* $t > 0$, *the unique solution* $U \in C((0,\infty); \mathcal{D})$ *of Pr* $(AEP)_{II}$, *represented as* $U(t) = S(t)\mathcal{F}$ *for any* $\mathcal{F} \in \mathcal{Y}$, *satisfies*

$$\|S(t)\mathcal{F}\|_{\mathcal{X}} \leq M \exp(-\sigma t)\|\mathcal{F}\|_{\mathcal{Y}}.$$

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