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In: Ravi P. Agarwal and František Neuman and Jaromír Vosmanský (eds.): Proceedings of Equadiff 9, Conference on Differential Equations and Their Applications, Brno, August 25-29, 1997, [Part 1] Survey papers. Masaryk University, Brno, 1998. CD-ROM. pp. 159--168.

Persistent URL: http://dml.cz/dmlcz/700264

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pp. 159–168

Generalized Linking Theorem and Nonlinear Equations in Unbounded Domains

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Abstract. We consider the following three problems: existence of nontrivial solutions for a semilinear Schrödinger equation in $\mathbb{R}^N$, existence of homoclinics for a first order Hamiltonian system and existence of time–periodic motions in an infinite chain of particles. The common feature of these problems is that the associated Euler–Lagrange functional has the so–called linking geometry and the Palais–Smale condition is not satisfied.

AMS Subject Classification. 34C15, 34C37, 35J65, 58E05

Keywords. Generalized linking theorem, Schrödinger equation, Hamiltonian system, chain of particles

1 Introduction

The purpose of this paper is to survey some recent results in the theory of nonlinear differential equations in unbounded domains. Solutions of these equations will be found as critical points of an associated Euler-Lagrange functional Φ in a suitable Hilbert space. We consider three different problems whose common feature is that Φ has the so-called linking geometry and the Palais-Smale condition is not satisfied. To be more specific, in Section 2 we will be concerned with the problem of existence of a nontrivial solution of the Schrödinger equation $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$ in a situation when 0 is in a gap of the spectrum of the operator $-\Delta + V$ and the nonlinearity f is superlinear at $u = 0$ and $|u| = \infty$. In Section 3 the existence of homoclinic solutions of a Hamiltonian system in $\mathbb{R}^{2N}$ is considered, and in Section 4 we turn our attention to the problem of existence of time-periodic solutions for an infinite chain of particles with nearest neighbour interaction (the so-called Fermi-Pasta-Ulam model). The arguments presented here are very sketchy. Complete proofs may be found in the original work to which the reader is referred.

Our starting point is the following generalized linking theorem:

Theorem 1. *Let E be a separable real Hilbert space and suppose that $\Phi \in C^1(E, \mathbb{R})$ satisfies the following hypotheses:*
(i) $\Phi(u) = \frac{1}{2}\langle Lu, u\rangle - \psi(u)$, where L is a bounded selfadjoint linear operator, ψ is bounded below, weakly sequentially lower semicontinuous and $\nabla\psi$ is weakly sequentially continuous.
(ii) $E = Y \oplus Z$, where Y, Z are L-invariant and the quadratic form $\langle Lu, u\rangle$ is negative definite on Y and positive definite on Z.
(iii) There are constants $b, \rho > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq b$, where $B_\rho := \{u \in E : \|u\| < \rho\}$.
(iv) There is $z_0 \in Z$, $\|z_0\| = 1$, and $R > \rho$ such that $\Phi|_{\partial M} \leq 0$, where $M := \{u = y + \lambda z_0 : y \in Y,\ \|u\| \leq R,\ \lambda \geq 0\}$.
Then there exists a sequence (u_n) such that $\nabla\Phi(u_n) \to 0$ and $\Phi(u_n) \to c$ for some $c \in [b, \sup_M \Phi]$.

The above result extends linking theorems of Rabinowitz [21,22] and Benci-Rabinowitz [9,22] (in the first of them Y is assumed to be finite-dimensional, in the second $\nabla\psi$ is compact). Theorem 1 may be found in [19], see also [34]. We would like to emphasize that in the problems considered in the next sections both Y and Z are infinite-dimensional and $\nabla\psi$ is not compact.

2 Schrödinger equation

Consider the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N \\ u(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{2.1}$$

where V and f are continuous and 1-periodic with respect to x_j, $1 \leq j \leq N$. It is known [23, Theorem XIII.100] that under such conditions the operator $-\Delta + V$ in $L^2(\mathbb{R}^N)$ has purely continuous spectrum which is bounded below (but not above) and consists of closed disjoint intervals. Intervals (a, b) such that $\sigma(-\Delta + V) \cap [a, b] = \{a, b\}$ will be called *spectral gaps* of $-\Delta + V$. Let $F(x, u) := \int_0^u f(x, \xi)\, d\xi$ and suppose that f and V satisfy the following hypotheses:

- **(A1)** V is 1-periodic in x_j, $1 \leq j \leq N$, continuous, and 0 lies in a gap of the spectrum of $-\Delta + V$.
- **(A2)** f is 1-periodic in x_j, $1 \leq j \leq N$, and continuous.
- **(A3)** $f(x, u)/u \to 0$ uniformly in x as $u \to 0$.
- **(A4)** There are $c > 0$ and $p \in (2, 2^*)$ such that $|f(x,u)| \leq c(1 + |u|^{p-1})$, where $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or 2.
- **(A5)** There is $\gamma > 2$ such that $0 < \gamma F(x, u) \leq uf(x, u)$ whenever $u \neq 0$.

Since $f(x, 0) = 0$ according to (A3), it is clear that (2.1) has the trivial solution $u = 0$.

Theorem 2. *If the hypotheses* (A1)–(A5) *are satisfied, then* (2.1) *has at least one nontrivial solution.*

For the proof, we consider the functional

$$\begin{aligned}\Phi(u) &:= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx \\ &= \frac{1}{2}\langle Lu, u\rangle - \psi(u)\end{aligned}$$

on the (real) Sobolev space $E := H^1(\mathbb{R}^N)$. Since

$$|f(x,u)| \le c_0(|u| + |u|^{p-1}) \tag{2.2}$$

according to (A3)–(A4), $\Phi \in C^1(E,\mathbb{R})$ [34, Lemma 3.10] and it is easy to see that $\nabla\Phi(u) = 0$ if and only if u is a (weak) solution of the equation in (2.1). Moreover, it can be shown that $u \in E$ and $\nabla\Phi(u) = 0$ imply $u(x) \to 0$ as $|x| \to \infty$.

To verify (i) of Theorem 1 we observe that $\psi \ge 0$ and ψ is weakly sequentially lower semicontinuous according to Fatou's lemma. Moreover,

$$\langle \nabla\psi(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v\,dx,$$

and weak sequential continuity of $\nabla\psi$ follows from (2.2) since if $u_n \rightharpoonup u$, then $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$ and $L^p_{loc}(\mathbb{R}^N)$.

Since 0 is in a gap of the spectrum of $-\Delta + V$, E decomposes as a direct sum of two infinite-dimensional L-invariant subspaces Y, Z such that $\langle Lu, u\rangle$ is negative definite on Y and positive definite on Z (cf. [28, Section 9]). Hence (ii) of Theorem 1. The quadratic form $\langle Lu, u\rangle$ is positive definite on Z and, according to (A3), $\psi(u) = o(\|u\|^2)$ as $u \to 0$; therefore $\Phi(u) \ge b > 0$ for $u \in \partial B_\rho \cap Z$ provided ρ is small enough. This gives (iii). Since $\langle Lu, u\rangle$ is negative definite on Y and $\psi \ge 0$, $\Phi|_Y \le 0$. Using the fact that $p > 2$ one can show that $\Phi \le 0$ on the set $\{u \in M : \|u\| = R\}$ whenever R is large enough. Hence also (iv) is satisfied.

Now it follows from Theorem 1 that there exists a sequence (u_n) such that $\Phi(u_n) \to c > 0$ and $\nabla\Phi(u_n) \to 0$. Furthermore, it can be shown that (u_n) is bounded, so $u_n \rightharpoonup \bar{u}$ after passing to a subsequence. Since $\nabla\Phi$ is weakly sequentially continuous, $\nabla\Phi(\bar{u}) = 0$. If $\bar{u} \ne 0$, the proof is complete. So assume $\bar{u} = 0$. According to a lemma due to P. L. Lions (see [12, Lemma 2.18], [20, Lemma I.1] or [34, Lemma 1.21]), if (u_n) is bounded and there exists $r > 0$ such that

$$\lim_{n\to\infty}\sup_{a\in\mathbb{R}^N}\int_{|x-a|<r} u_n^2\,dx = 0, \tag{2.3}$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (2, 2^*)$. Hence either $u_n \to 0$ in $L^p(\mathbb{R}^N)$ or there exists a sequence $(a_n) \subset \mathbb{Z}^N$ and $r, \delta > 0$ such that

$$\int_{|x-a_n|<r} u_n^2\,dx \ge \delta$$

for almost all n. In the first case one shows that $u_n \to 0$ in E which is impossible since $\Phi(u_n) \to c > 0$. In the second one $v_n(x) := u_n(x + a_n) \rightharpoonup \bar{v} \neq 0$ after taking a subsequence. Since Φ is invariant with respect to the action of $\mathbb{Z}^N$ given by

$$(a * u)(x) = u(x + a), \qquad u \in E, \quad a \in \mathbb{Z}^N, \tag{2.4}$$

we have $\Phi(v_n) = \Phi(u_n)$ and $\nabla\Phi(v_n) \to 0$, so $\bar{v}$ is the nontrivial solution we were looking for.

Theorem 2 and its proof are taken from [19], see also [34]. Earlier versions of this result, under the assumption that the function F is strictly convex, have been obtained by Alama and Li [1], and Buffoni, Jeanjean and Stuart [10]. Although the techniques in [1] and in [10] are very different, in both papers the problem is eventually reduced to that of finding a critical point of a functional having the mountain pass geometry. The hypothesis that F is convex has been removed, first by Troestler and Willem [33], and then by Kryszewski and Szulkin [19]. An extension of Theorem 2 has been recently found by Bartsch and Ding [8]. They considered the situation where 0 is a left endpoint of a gap in the spectrum of $-\Delta + V$, i.e. $[0, \beta] \cap \sigma(-\Delta + V) = \{0\}$ for some $\beta > 0$.

We would also like to mention the work of Heinz, Küpper and Stuart, see [16,28] and the references there, and that of Troestler [32], on bifurcation into spectral gaps for (2.1) with $V(x)$ replaced by $V(x) - \lambda$.

It follows immediately from the periodicity assumptions on V and f that if u is a solution of (2.1), then so is $a * u$ (cf. (2.4)) for any $a \in \mathbb{Z}^N$. Two solutions u_1 and u_2 are said to be *geometrically distinct* if $a * u_1 \neq u_2$ for any $a \in \mathbb{Z}^N$. The problem of finding the number of geometrically distinct solutions of (2.1) has been studied by several authors. If $\sigma(-\Delta + V) \subset (0, \infty)$ (i.e. if the quadratic form $\langle Lu, u\rangle$ is positive definite), it has been shown by Coti Zelati and Rabinowitz [12] that there are infinitely many such solutions. The same result remains true if 0 is in a spectral gap of $-\Delta + V$ and $f(x, u) = W(x)|u|^{p-2}u$, where $W > 0$ and $2 < p < 2^*$ [2]. For nonconvex F it has been shown in [8,19] that (2.1) has infinitely many geometrically distinct solutions under the additional assumption that f is odd in u. It seems to be an open problem to decide whether oddness of f is really needed here.

3 Hamiltonian systems

Let

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

be the standard symplectic $2N \times 2N$- matrix. In this section we will be concerned with the question of existence of homoclinic solutions for the Hamiltonian system

$$\dot{z} = JH_z(z, t), \qquad z \in \mathbb{R}^{2N}. \tag{3.1}$$

Recall that a solution z of (3.1) is said to be *homoclinic* (to 0) if $z \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$. Suppose that $H(z,t) = \frac{1}{2}Az \cdot z + F(z,t)$ satisfies the following assumptions:

(B1) A is a constant symmetric $2N \times 2N$-matrix and $\sigma(JA) \cap i\mathbf{R} = \emptyset$.

(B2) F and F_z are 1-periodic in t and continuous.

(B3) $F_z(z,t)/|z| \to 0$ uniformly in t as $z \to 0$.

(B4) There exists $\gamma > 2$ such that $0 < \gamma F(z,t) \le z \cdot F_z(z,t)$ for all $z \neq 0$.

(B5) There exist $c, r > 0$ such that $|F_z(z,t)|^2 \le cz \cdot F_z(z,t)$ for all $|z| \le r$.

(B6) There exist $c, R > 0$ and $q \in (1,2)$ such that $|F_z(z,t)|^q \le cz \cdot F_z(z,t)$ for all $|z| \ge R$.

It follows from (B6) that

$$|F_z(z,t)| \le \tilde{c}(1 + |z|^{p-1}) \tag{3.2}$$

for some $\tilde{c} > 0$ and $p = q/(q-1)$.

Theorem 3. *If the hypotheses* (B1)–(B6) *are satisfied, then* (3.1) *has at least one homoclinic solution.*

Let $E := H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$ be the Sobolev space of functions $z \in L^2(\mathbb{R}, \mathbb{R}^{2N})$ such that their Fourier transform $\hat{z}$ satisfies

$$\int_{\mathbb{R}} (1 + |\xi|^2)^{1/2} |\hat{z}(\xi)|^2 \, d\xi < \infty.$$

Then E is a Hilbert space and

$$\langle z, v \rangle := \int_{\mathbb{R}} (1 + |\xi|^2)^{1/2} \hat{z}(\xi) \cdot \overline{\hat{v}(\xi)} \, d\xi$$

is an inner product in E. Consider the functional

$$\begin{aligned} \Phi(z) &:= \frac{1}{2} \int_{\mathbb{R}} (-J\dot{z} - Az) \cdot z \, dt - \int_{\mathbb{R}} F(z,t) \, dt \\ &= \frac{1}{2} \langle Lz, z \rangle - \psi(z). \end{aligned}$$

According to (3.2) and (B3), $|F_z(z,t)| \le c_0(|z| + |z|^{p-1})$. Hence using the argument of [34, Lemma 3.10] and the fact that E is continuously embedded in $L^s(\mathbb{R}, \mathbb{R}^{2N})$ for each $s \ge 2$ (see e.g. [28, Lemma 10.4]) it is easy to show that $\Phi \in C^1(E, \mathbb{R})$ and $\nabla\Phi(z) = 0$ if and only if z is a solution of (3.1). Moreover, $F_z(z(.),.) \in L^2(\mathbb{R}, \mathbb{R}^{2N})$ for such z. It follows therefore from (3.1) that $z \in H^1(\mathbb{R}, \mathbb{R}^{2N})$, so $z(t) \to 0$ as $|t| \to \infty$. Hence critical points $z \neq 0$ of Φ are homoclinic solutions of (3.1).

According to (B1), $-i\xi J - A$ is an invertible matrix and $(-i\xi J - A)^{-1}$ is uniformly bounded with respect to $\xi \in \mathbb{R}$. Hence it follows from Plancherel's formula that L is bounded, selfadjoint and has a bounded inverse. So E decomposes as in (ii) of Theorem 1. See [28, Section 10] for more details. In particular, it is shown in [28] that the spectrum of $-J\frac{d}{dt} - A$ is unbounded, from above and below, in $H^1(\mathbb{R}, \mathbb{R}^{2N})$. Therefore $\langle Lz, z\rangle$ is positive and negative definite on subspaces of infinite dimension.

Other hypotheses of Theorem 1 are verified in the same way as in the preceding section. Hence we obtain a sequence (z_n) such that $\Phi(z_n) \to c > 0$ and $\nabla\Phi(z_n) \to 0$. Moreover, (z_n) can be shown to be bounded. The argument is the same as for the Schrödinger equation and may be found in [7]. The proof of boundedness makes essential use of (B4)–(B6). Finally, since Φ is invariant with respect to the action of $\mathbb{Z}$ given by $(a * z)(t) = z(t + a)$ ($z \in E$, $a \in \mathbb{Z}$), cf. (2.4), an application of P. L. Lions' lemma gives a solution $\bar{z} \neq 0$. Here a remark is in order: in [12, Lemma 2.18] and [34, Lemma 1.21] the space is $H^1(\mathbb{R}^N)$; however, a simple adaptation of the argument in [12,34] shows that if (z_n) is bounded in $H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$ and (2.3) is satisfied, then $z_n \to 0$ in $L^s(\mathbb{R}, \mathbb{R}^{2N})$ for all $s \in (2, +\infty)$.

Theorem 3 for Hamiltonian systems with strictly convex F is due to Coti Zelati, Ekeland and Séré [11]. They reformulated the problem in terms of a dual functional which has the mountain pass geometry. The convexity assumption has been removed by Hofer and Wysocki [17] and Tanaka [29]. The proof in [29] is obtained by constructing a sequence of subharmonic solutions of (3.1) and passing to the limit. A truncation argument is also used there in order to weaken some of the hypotheses (in particular, in [29] it is assumed that $q = 1$ in (B6), so F need not satisfy any growth restriction like (3.2)). An extension of Theorem 3 in a similar spirit as in [8] has been obtained by Ding and Willem [14]. They allowed A to be t-dependent, 1-periodic and such that $[0, \beta] \cap \sigma(-J\frac{d}{dt} - A) = \{0\}$ for some $\beta > 0$.

It has been shown by Séré [25,26] that if F is strictly convex, then (3.1) has infinitely many geometrically distinct homoclinic solutions. Recently Ding and Girardi [13] have obtained a result on the existence of infinitely many homoclinics for F which is even in z but not necessarily convex. In [7] it will be shown that the same result remains valid for F invariant with respect to an action of a more general symmetry group. Also for Hamiltonian systems it seems to be unknown whether such invariance condition can be removed if F is nonconvex.

4 Infinite chain of particles

Consider a chain of particles arranged linearly in a doubly infinite sequence. Assume that each particle has unit mass and that it interacts only with its nearest neighbours. Denote the displacement of the i-th particle from its original position by q_i and let ϕ denote the potential of interaction. Then the equations of motion for this chain are

$$\ddot{q}_i = \phi'(q_{i-1} - q_i) - \phi'(q_i - q_{i+1}), \qquad i \in \mathbb{Z}. \tag{4.1}$$

If $\phi(x) = \frac{1}{2}\beta x^2$, $\beta > 0$, the system is linear and it is possible to explicitly find normal mode solutions of (4.1), see [31].

Nonlinear systems of this kind (for a finite number of particles) were considered for the first time by Fermi, Pasta and Ulam in [15]. They wanted to verify numerically the conjecture that while there is no exchange of energy between different modes when the system is linear, already a perturbation by a small nonlinear term causes the energy to be gradually shared by the modes. Contrary to what they expected, they found that only little energy was shared and the system returned periodically to the initial state. In a subsequent research Toda has found that if the force of interaction is exponential, then the system (4.1) is integrable and there exist both periodic solutions of finite energy and soliton solutions. See [31] and the references there for more information.

Suppose now that $\phi(x) = \frac{1}{2}\beta x^2 + V(x)$ and β, V satisfy the following conditions:

(C1) $\beta > 0$.
(C2) V is continuously differentiable.
(C3) $V'(x)/x \to 0$ as $x \to 0$.
(C4) There is $\gamma > 2$ such that $0 < \gamma V(x) \le V'(x)x$ whenever $x \ne 0$.

Note that since $\phi'(x)$ has the same sign as x, the potential ϕ is purely attractive.

Theorem 4. *If the hypotheses* (C1)–(C4) *are satisfied, then* (4.1) *has a nontrivial T-periodic solution of finite energy for each $T > 0$.*

Let $q := \{q_i\}_{i\in\mathbb{Z}}$, $S^1 := [0,T]/\{0,T\}$,

$$\langle q, p\rangle := \sum_i \int_0^T \left(\dot{q}_i(t)\dot{p}_i(t) + (q_i(t) - q_{i+1}(t))(p_i(t) - p_{i+1}(t))\right) dt,$$

$\|q\|^2 = \langle q, q\rangle$ and

$$E := \left\{ q \in H^1(S^1, \mathbb{R})^{\mathbb{Z}} : \int_0^T q_0(t)\, dt = 0,\ \|q\| < \infty \right\}.$$

Then E is a Hilbert space and

$$\begin{aligned}\Phi(q) &:= \sum_i \frac{1}{2} \int_0^T \left(\dot{q}_i^2 - \beta(q_i - q_{i+1})^2\right) dt - \sum_i \int_0^T V(q_i - q_{i+1})\, dt \\ &= \frac{1}{2}\langle Lq, q\rangle - \psi(q)\end{aligned}$$

is defined on E. Moreover, $\Phi \in C^1(E, \mathbb{R})$ and critical points of Φ are T-periodic solutions of (4.1) [6]. Note that if $q = \{q_i\}$ is a solution of (4.1), so is $\tilde{q} = \{q_i + \sigma\}$

for any constant $\sigma \in \mathbb{R}$. Therefore the condition that $\int_0^T q_0(t)\,dt = 0$ which appears in the definition of E is a way of normalizing (4.1) by dividing out the constants.

Let $T \in (0, \pi/\sqrt{\beta})$ be fixed. Then it can be shown that Φ satisfies all hypotheses of Theorem 1. The proof is similar to that in Section 2 but more technical. Here we only show how the decomposition $E = Y \oplus Z$ is obtained and refer to [6] for the other parts.

We first make a side remark that $\langle Lq, q\rangle$ is negative definite if $\beta < 0$, $Lq = 0$ for all constant sequences q if $\beta = 0$ and it has been shown in [6] that L has no bounded inverse if $\beta \geq \pi^2/T^2$.

Let $Y := \{q \in E : q_i = \mathit{const.}$ for all $i\}$ and $Z := Y^\perp$. Suppose $T \in (0, \pi/\sqrt{\beta})$; then $0 < \beta < \pi^2/T^2$ and a simple computation using Wirtinger's inequality shows that $\langle Lq, q\rangle$ is positive definite on Z. Clearly, $\langle Lq, q\rangle$ is negative definite on Y. Hence by Theorem 1 there exists a sequence $(q^{(n)})$ such that $\Phi(q^{(n)}) \to c > 0$ and $\nabla\Phi(q^{(n)}) \to 0$. Moreover, it can be shown $(q^{(n)})$ is bounded, so we may assume it is weakly convergent. If $q^{(n)} \rightharpoonup \bar{q} \neq 0$, we are done. Otherwise one shows there is a sequence (i_n) of integers such that if $\tilde{q}_i^{(n)} := q_{i+i_n}^{(n)} + \sigma^{(n)}$, then $\tilde{q}^{(n)} \rightharpoonup \tilde{q} \neq 0$ ($\sigma^{(n)}$ is chosen in order to have $\int_0^T \tilde{q}_0^{(n)}(t)\,dt = 0$). Since Φ is invariant with respect to the action of $\mathbb{Z}$ given by $(k * q_i)(t) = q_{i+k}(t) + \sigma_k$, a familiar argument shows that $\tilde{q}$ is a T-periodic solution of (4.1). Moreover, the energy $\frac{1}{2}\langle L\tilde{q}, \tilde{q}\rangle + \psi(\tilde{q})$ is finite.

Suppose $T \geq \pi/\sqrt{\beta}$; then we can find an integer k such that $T/k < \pi/\sqrt{\beta}$. So (4.1) has a T/k-periodic solution which of course is T-periodic as well.

The special role played by the number $T_0 := \pi/\sqrt{\beta}$ raises the question of the behaviour of solutions as $T \nearrow T_0$. A partial answer may be found in [6] where it has been shown that if there exist $c > 0$ and $p \in (2, 4)$ such that $V(x) \geq c|x|^p$, then nontrivial solutions of (4.1) bifurcate at T_0. More precisely, there exist nontrivial solutions of arbitrarily small energy and L^∞-norm, with a period arbitrarily close to T_0.

The study of chains of particles by variational methods has been initiated by Ruf and Srikanth in [24]. They considered finite chains with different kinds of (nonlinear) potential. In a series of papers Arioli, Gazzola and Terracini considered the infinite chain (4.1) with $\beta < 0$ [3,5] (potential repulsive for small and attractive for large displacements) and $\beta = 0$ [4]. Theorem 4 here is a special case of a more general result contained in [6]. Finally, let us also mention two papers, by Smets and Willem [27], and by Tarallo and Terracini [30], on solitary waves for systems of equations like (4.1).

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