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ENTROPY COMPACTIFICATION OF THE TRANSONIC FLOW

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1. Introduction

Let us consider a compressible, irrotational, steady, adiabatic, isentropic and inviscid fluid in a bounded, simply connected domain $\Omega \subset \mathbb{R}^n$, n = 2,3, with Lipschitz boundary. The relation between the presure p and the density ρ is

$$(1.1) \qquad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\varkappa}, \qquad 1 < \varkappa < 2,$$

where quantities with the zero index correspond to the speed $\vec{v} = 0$. If $\vec{v}$ is the velocity vector, then the condition of the irrotational flow is

$$(1.2) \qquad \text{rot } \vec{v} = 0 .$$

The flow satisfies the continuity equation

$$(1.3) \qquad \text{div } (\rho\vec{v}) = 0$$

and the Euler equation of motion:

$$(1.4) \qquad \vec{v} \text{ grad } \vec{v} = - \frac{1}{\rho} \text{ grad } p .$$

This implies for the potential of the velocity is satisfied the equation

$$(1.5) \qquad \text{div } (\rho\nabla u) = 0 ,$$

where

$$(1.6) \qquad \rho = \rho(|\nabla u|^2) = \rho_0 \left(1 - \frac{\varkappa-1}{2a_0^2} |\nabla u|^2\right)^{\frac{1}{\varkappa-1}}$$

and a is the speed of the sound. If the Mach number defined as

$$M \overset{\text{def}}{=} \frac{|\nabla u|}{a} \text{ is } < 1 \Leftrightarrow |\nabla u|^2 < \frac{2a_0^2}{1+\varkappa}$$

the flow is subsonic and the equation (5) is elliptic. In the opposite case the flow is supersonic and the equation (5) is hyperbolic. A flow with subsonic and supersonic regions is called transonic.

It is important to underline that the equation (5) does not contain an information about the behaviour of the entropy on the shock surfaces. The entropy condition across the shock: $|\nabla u|$ is decreasing.

This can be formulated, for example, in the form

(1.7) $\Delta u \le K < \infty$;

we shall consider in the next only physical speed, i.e. such that

(1.8) $$|\nabla u| \le \frac{\sqrt{2}a_0}{\sqrt{\varkappa - 1}} .$$

We suppose the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $u = 0$ on Γ_1 and $\rho(|\nabla u|^2)\frac{\partial u}{\partial \nu} = g$ on Γ_2.

The transonic flow problem was considered numerically by many authors. We mention here a book by R. Glowinski [1]; there are excelent numerical results by many authors: M.O. Bristeau, R.Glowinski, J. Periaux, P. Perrieer, O. Pironneau, G.Poirer, M.Feistauer, A.Jameson, K.Kozel, J.Polášek, M.Vavřincová. They used entropy conditions of the type (7), upwinding iterations and viscosity approximations. We shall do the same in the next. The entropy condition (7) is compactifying, which follows from some slight generalisation of the result by F.Murat [2]. More complete discussion of the result is in M.Feistauer, J.Nečas [3], M.Feistauer, J.Madel, J.Nečas [4], J.Nečas [5]. A justification of the finite element approximation is discussed in Ph.G.Ciarlet, J.Mandel, J.Nečas [6].

2. Formulation of the problem, compactness by entropy

We look for a week solution to the equation (1.5), i.e. for $u \in W^{1,\infty}(\Omega)$, such that for $v \in V = \{v \in W^{1,2}(\Omega);\ v = 0 \text{ on } \Gamma_1\}$

(2.1) $$\int_\Omega \rho(|\nabla u|^2)\nabla u \nabla v dx = \int_{\partial\Omega} g v dS, \quad g \in L^\infty(\partial\Omega) .$$

If $\Gamma_1 = 0$, we suppose $\int_{\partial\Omega} g dS = 0$.

We can give also Dirichlet data on a part $\Gamma_t \subset \Gamma_2$, where $\Gamma_t \subset \{x \in \partial\Omega;\ (\vec{v},\vec{\nu}) < 0\}$. For an illustration, let us consider a paralel flow: $\vec{v} = (u,0,0)$, $u = u(x_1)$. For $w = \frac{u}{a_0}$ let us consider $w \in W^{1,\infty}((0,1))$, $w(1) = 0$, $|w'|^2 \le \frac{2}{\varkappa-1}$, satisfying with $\mu(s) = (1 - \frac{\varkappa-1}{2}s)^{1/(\varkappa-1)}$

(2.2) $(\mu(w'^2)w')' = 0$ in $(0,1)$,

(2.3) $w'(0)\mu(w'(0)^2) = A$

(2.4) $w'' \le K$, $K > 0$.

So first $A \in [0,0.57]$ which is clear from teh Fig. 1; the general solution of (2.2), (2.3), (2.4) is sketched on the Fig. 2 and is unique, the Cauchy data in the origine being prescribed.

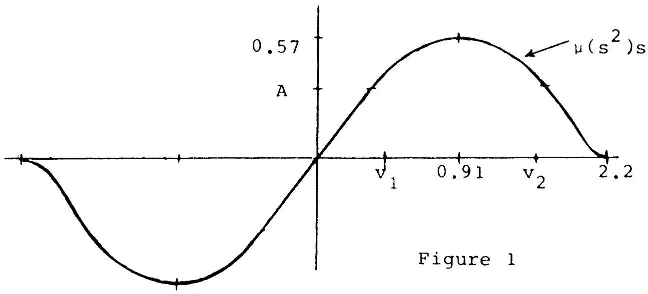

Figure 1

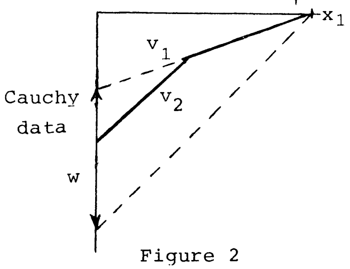

Figure 2

Let us mention that a uniqueness of the entropx solution is probably not true in more dimensions. The existence of the solution will follow from some "à posteriori" conditions given by an ideal computer.

2.5. Definition.

Let $h \in C^1([0,s_0])$, $\frac{2a_0^2}{\varkappa+1} < s_0 < \frac{2a_0^2}{\varkappa-1}$, $h(s) > 0$ in $(0,s_0]$ and let it satisfy here the monotony condition: $h(s) + 2sh'(s) > 0$. A transonic flow is called h-entropic if $\forall\varphi \in D_+(\Omega)$: $(\varphi \geq 0)$

(2.6) $$-\int_\Omega h(|\nabla u|^2)\nabla u\nabla\varphi dx \leq K\int_\Omega \varphi dx, \quad K \in \mathbf{R}^1 .$$

(2.6') Examples:

(1) $h(s) \equiv 1$, $s_0 < \frac{2a_0^2}{\varkappa-1}$,

$$-\int_\Omega \nabla u\nabla\varphi dx \leq K\int_\Omega \varphi dx \Leftrightarrow \Delta u \leq K, \ M = \infty ,$$

(2) $h(s) = s\rho(s)$, $s_0 < \frac{6a_0^2}{3\varkappa-1}$,
$M = \sqrt{3}$: entropy by viscosity ,

(3) $h(s) = -\rho(s)\ln\left(1 - \frac{\varkappa-1}{2a_0^2}s\right)$,
$s_0 < \frac{2a_0^2}{\varkappa-1}\tau_0$, $\tau_0 = [\frac{1}{2}\ln(1-\tau_0)](1-6\tau_0)$;

$M = 1.91$: Hugoniot's entropy,

(4) $h(s) = -\rho'(s)s$, $s_0 < \frac{6a_0^2}{\varkappa-1}$,
natural entropy, $M = 2.23$. □

In a formal way: the monotony condition for h and $s > \frac{2a_0^2}{\varkappa+1}$ is sufficient and necessary, for the solution satisfies the entropy condition on the shock surface.

2.7. Theorem (F.Murat). Let $\{G_n\}$ be a sequence of functionals defined on $W^{1,2}(\Omega)$, $G_n \rightharpoonup G$. Let for $h \in D_+(\Omega)$, $\langle G_n, h\rangle \geq 0$. Then $G_n \to G$ in $[W^{1,p}(\Omega)]'$, $\forall p > 2$.

Idea of the proof: $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega$, $\psi \in D_+(\Omega)$, $\psi(x) = 1$ in Ω_1. Let $h \in D(\Omega)$, supp $h \subset \Omega_1$. There is

(2.8) $$-\|h\|_{C(\bar{\Omega})}\,\psi \leq h \leq \|h\|_{C(\bar{\Omega})}\,\psi ,$$

and it follows

(2.9) $$|\langle G_n, h\rangle| \leq \langle G_n, \psi\rangle \|h\|_{C(\bar{\Omega})},$$

hence G_n is a sequence of Radon measures. Let u_n be defined by

(2.10) $$\int_\Omega (\nabla u_n \nabla h + u_n h)dx = \langle G_n, h\rangle , \ \forall h \in W^{1,2}(\Omega) ,$$

$$\Omega_2 \subset \bar{\Omega}_2 \subset \Omega_1, \quad q > n, \ h \in W_0^{1,q}(\Omega_1) .$$

Then

(2.11) $$„-\Delta u_n + u_n = G_n" \in [W_0^{1,q}(\Omega_1)]'$$

and

(2.12) $$\|u_n\|_{W^{1,q'}(\Omega_2)} \leq C(\Omega_2) .$$

Because $W^{1,q}(\Omega_1) \hookrightarrow\hookrightarrow C(\bar{\Omega}_1) \Rightarrow \{u_n\}$ is convergent in $W^{1,q'}(\Omega_2)$ to u. An usual interpolation technic as well as the estimate

(2.13) $$\left|\int_{\Omega\setminus\Omega_2} \nabla(u_n - u)\nabla h dx\right| \leq \left(\int_{\Omega\setminus\Omega_2} |\nabla(u_n - u)|^2 dx\right)^{\frac{1}{2}} \cdot$$
$$\left(\int_{\Omega\setminus\Omega_2} |\nabla h|^2 dx\right)^{\frac{1}{2}} \leq \left(\int_{\Omega\setminus\Omega_2} |\nabla(u_n - u)|^2 dx\right)^{\frac{1}{2}} \cdot$$
$$\cdot \left(\int_{\Omega\setminus\Omega_2} |\nabla h|^p dx\right)^{\frac{1}{p}} \cdot |\Omega\setminus\Omega_2|^{\frac{1}{2} - \frac{1}{p}}$$

gives the result.

2.14 Theorem.

Let $$E_h = \{u; \|u\|_{W^{1,2}(\Omega)} \leq C, |\nabla u|^2 \leq s_0 ,$$

$$\forall \varphi \in D_+(\Omega) : -\int_\Omega h(|\nabla u|^2)\nabla u \nabla\varphi dx \leq K \int_\Omega \varphi dx\} .$$

Then E_h is compact in $W^{1,2}(\Omega)$.

Idea of the proof: Put $\langle G_n, h\rangle = K \int_\Omega h dx + \int_\Omega h(|\nabla u_n|^2) \cdot$ $\cdot \nabla u_n \nabla h dx$, $\langle G, h\rangle = K \int_\Omega h dx + \int_\Omega h(|\nabla u|^2)\nabla u \nabla h dx$.

It follows from the theorem 2.7: $\langle G_n - G, u_n - u\rangle \to 0$, provided $u_n \to u$. For the pairing $\langle G_n - G, u_n - u\rangle$ we use the Leray-Lions trick from the theory of monotone operators. □

3. Solution of the transonic problem by use of the alternating functional

The equation (1.5) is the Euler's equation to the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega \left(\int_0^{|\nabla u|^2} \rho(t)dt\right)dx - \int_{\partial\Omega} guds. \tag{3.1}$$

Let $|\nabla u|^2 \le \frac{2a_0^2}{\varkappa - 1}$, and define $w \in W^{1,2}(\Omega)$ by

$$\int_\Omega \rho(|\nabla u|^2)\nabla w \nabla h dx = \int_{\partial\Omega} ghds \tag{3.2}$$

for $h \in W^{1,2}(\Omega)$, $h = 0$, on Γ_1. (In the case $\Gamma_1 = 0$, $\Gamma_t = 0$, we suppose $\int_\Omega wdx = 0$.)

Define the alternating functional by

$$\psi(u) = \Phi(u) - \Phi(w(u)) . \tag{3.3}$$

Because $\rho'(s) \le 0$, we have with $c > 0$ (for details see [3], [4], [5])

$$c\|u - w(u)\|^2_{W^{1,2}(\Omega)} \le \psi(u). \tag{3.4}$$

So u is a solution of the transonic problem iff $\psi(u) = 0$. But if V is the space od solutions in $W^{1,2}(\Omega)$ then we have

3.4. Theorem

The alternating functional attains on $E_h \cap V \cap \{u = u_0 \text{ on } \Gamma_t\}$ its minimum in some point u. If $\psi(u) = 0$, then u is a h-entropic solution of the transonic problem.

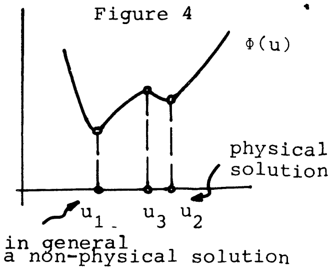

Figure 4

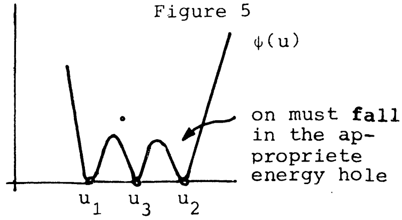

Figure 5

4. Viscosity method

Let us consider a complet system of gas: p - pressure, ρ-density, T -temperature, $\vec{v}$ - velocity vector, provided:

(4.1) $0 < T_1 \le T \le T_2 < \infty, \quad T \in W^{1,2}(\Omega)$,

(4.2) $0 < \rho_1 \le \rho, \quad \rho \ln \rho \in L^2(\Omega)$,

(4.3) $p = R\rho T$, R is the gas-constant,

(4.4) $\vec{v} \in [W^{1,2}(\Omega)]^3, \quad |v|^2 \le \frac{2a_0^2}{\varkappa - 1}$,

(4.5) $\mu = \mu(T),\ \lambda = \lambda(T),\ \mu, \lambda \in C(\mathbf{R}_+),\ \lambda = -\theta\frac{2}{3}\mu,\ \theta < 1,\ \Gamma_1 = 0,$ $\Gamma_t = 0$,

(4.6) $\mathrm{div}(\rho\vec{v}) = 0$ in Ω, $\rho v.\nu = g$ on $\partial\Omega$,

(4.7) $\rho v.\nabla v + \nabla p = \nabla(\lambda \,\mathrm{div}\, \vec{v}) + 2\,\mathrm{div}(\mu \underset{\sim}{e})$, in Ω,

$$\underset{\sim}{e} = \{e_{ij}(v)\},\ 2e_{ij}(v) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i},$$

(4.8) $\vec{v} = \vec{v}^0$ on $\partial\Omega$, $\|\vec{v}^0\|_{[W^{1,2}(\Omega)]^3} \le c,\ |\vec{v}^0| \le \frac{2a_0^2}{\varkappa - 1}$,

(4.9) $c_v \mathrm{div}(\rho . \ln \frac{T}{\rho^{\varkappa - 1}} . \vec{v}) = k \frac{\Delta T}{T} + \frac{E(\vec{v})}{T}$ in Ω,

$$E(v) = \lambda(\mathrm{div}\, v)^2 + 2\mu \mathrm{tr}\, \underset{\sim}{ee},\ c_v,\ k \in \mathbf{R}_+^1,$$

(4.10) $c_v g \ln \frac{T}{\rho^{\varkappa - 1}} - k \frac{1}{T}\frac{\partial T}{\partial \nu} = h$ on $\partial\Omega$, $\|h\|_{L^1(\partial\Omega)} \le c$.

Let $\beta_n > 0$, and take $\mu_n = \mu.\beta_n$, $\lambda_n = \lambda\beta_n$, $k_n = k\beta_n$. We let go $\beta_n \to 0$ and look for an optimal control problem: let the cost functional, where $\vec{v} = \vec{\varepsilon} + \nabla u$, be

(4.11) $I(\vec{v}) = \int_\Omega |\vec{\varepsilon}|^2 dx + \int_\Omega [\rho - \rho_0 (1 - \frac{\varkappa - 1}{2a_0^2}|\vec{v}|^2)^{\frac{1}{\varkappa - 1}}]^2 dx$

and let us look for $I(v) \to 0$, "à posteriori" entropy condition: $p \in W^{1,1}(\Omega)$,

(4.12) $\nabla p . \vec{v} \ge -K$.

4.13 Remark

$$\int_{\partial\Omega} h dS \ge k \int_\Omega \frac{|\nabla T|^2}{T^2} dx + \int_\Omega \frac{E(\vec{v})}{T} dx .$$

Open problem: how to estimate the pressure p? If $I(\vec{v}) < \infty$, then

L^2 estimate of ρ follows.

4.14 Definiton

A sequence $\{u_n\}$, $\int_\Omega u_n dx = 0$, $|\nabla u_n|^2 \le s_0$ is h - entropic, if $\forall \varphi \in D_+(\Omega)$:

$$(4.15) \qquad - \int_\Omega h(|\nabla u_n|^2)\nabla u_n \nabla\varphi dx \le K \int_\Omega \varphi dx + \langle R_n, \varphi\rangle ,$$

where

$$R_n \to 0 \quad \text{in } [W^{1,2}(\Omega)]' .$$

4.16 Theorem

Let T_n, p_n, ρ_n, $\vec{v}^n$ be a sequence to solutions of (4.1) - (4.10) with $I(\vec{v}^n) \to 0$ and satisfying $\nabla\rho_n \vec{v}^n \ge -K$. Suppose, without the loss of generality, that $u_n \to u$. Then $\{u_n\}$ is $h(s) = \rho(s)s$ - entropic, $u_n \to u$, and u is a h-entropic solution to the transonic problem, Provided $|\nabla u_n|^2 \le s_0 < \frac{6a_0^2}{3\varkappa - 1}$.

Idea of the proof: Let $\varphi \in D_+(\Omega)$, multiply (4.7) by $\vec{v}^n\varphi$ and integrate by parts. We get

$$(4.17) \qquad \int_\Omega \rho_n v_j^n \frac{\partial v_i^n}{\partial x_j} v_i^n \varphi dx = - \frac{1}{2} \int_\Omega \rho_n |\vec{v}^n|^2 v_j^n \frac{\partial\varphi}{\partial x_j} dx =$$

$$= - \int_\Omega \frac{\partial p_n}{\partial x_i} v_i^n \varphi dx - \int_\Omega \lambda_n (\text{div } \vec{v}^n)^2 \varphi dx - \int_\Omega \lambda_n \text{div } \vec{v}^n . v_i^n \frac{\partial\varphi}{\partial x_i} dx -$$

$$- 2 \int_\Omega \mu_n e_{ij}(\vec{v}^n) e_{ij}(\vec{v}^n)\varphi dx - 2 \int_\Omega \mu_n e_{ij}(\vec{v}^n) v_i^n \frac{\partial\varphi}{\partial x_j} dx \le$$

$$\le K \int_\Omega \varphi dx - \int_\Omega \lambda_n \text{div } \vec{v}^n v_i^n \frac{\partial\varphi}{\partial x_i} dx - 2 \int_\Omega \mu_n e_{ij}(\vec{v}^n) v_i^n \frac{\partial\varphi}{\partial x_j} dx =$$

$$\stackrel{\text{def}}{=} K \int_\Omega \varphi dx + \langle S_n, \varphi\rangle .$$

But we have, because of 4.13

$$(4.18) \qquad \|S_n\|_{[W^{1,2}(\Omega)]'} \le c\beta_n^{\frac{1}{2}}$$

and replacing in $\int_\Omega \rho_n |\vec{v}^n|^2 v_j^n \frac{\partial\varphi}{\partial x_i}$, ρ_n by $\rho_0 (1 - \frac{\varkappa - 1}{2a_0^2}|\vec{v}^n|^2)^{\frac{1}{\varkappa - 1}}$ and then $\vec{v}^n$ by ∇u_n, the result follows. □

Solving the system (4.1)-(4.10) with the cost functional (4.11), the inequality (4.12) can be expected, but must be supposed. For to cancel this condition, let us first formulate (4.6) in the weak sense:

$$(4.19) \qquad \int_\Omega \rho v_i \frac{\partial \varphi}{\partial x_i}\, dx = \int_{\partial\Omega} g\varphi\, dS .$$

Any approximate solution to (4.19) satisfies in fact

$$(4.20) \qquad \int_\Omega \rho v_i \frac{\partial \varphi}{\partial x_i}\, dx = \int_{\partial\Omega} g\varphi\, dS + \langle R,\varphi\rangle ,$$

where the term R represents small material sources in Ω and a small flux of the material through the $\partial\Omega$. If we choose R in an appropriate way, we get automatically (4.15) provided the cost functional

$$(4.21) \qquad J(\vec{v}) = \int_\Omega |\vec{\varepsilon}|^2 dx + \int_\Omega [\rho - \rho_0 (1 - \frac{\varkappa-1}{2a_0^2}|\vec{v}|^2)^{\frac{1}{\varkappa-1}}]^2 dx + \int_\Omega [\ln \frac{T}{\rho^{\varkappa-1}} - \ln \frac{T_0}{\rho_0^{\varkappa-1}}]^2 dx$$

tends to zero; here $T_0 = \frac{1}{R}\frac{p_0}{\rho_0}$.

For to precise the conditions, let $\beta_n \to 0$, $\beta_n > 0$, λ_n, μ_n, k_n as before and put $\alpha_n \to 0$ in the way that $\alpha_n \beta_n^{-\frac{1}{2}} \to \infty$. Let us suppose $T_0 > \rho_0^{\varkappa-1}$. Put in (4.20)

$$(4.22) \qquad \langle R_n,\varphi\rangle = \int_\Omega \rho_n v_i^n \frac{\partial\varphi}{\partial x_i} dx - \int_\Omega \rho_n^{1-\alpha_n} v_i^n \frac{\partial\varphi}{\partial x_i} dx + \int_\Omega \rho_n \Big(1 - \frac{\ln \frac{T_n}{\rho_n^{\varkappa-1}}}{\ln \frac{T_0}{\rho_0^{\varkappa-1}}}\Big) v_i^n \frac{\partial\varphi}{\partial x_i} dx + \int_\Omega f_n \varphi\, dx + \langle G_n,\varphi\rangle + \langle H_n,\varphi\rangle ,$$

where

$$\|f_n\|_{L^\infty(\Omega)} \le K\alpha_n ,\quad \frac{1}{\alpha_n}\|G_n\|_{[W^{1,2}(\Omega)]'} \to 0 ,$$

$$\|H_n\|_{[W^{1,2}(\Omega)]'} \to 0 \quad \text{and} \quad \langle H_n,\varphi\rangle \ge 0 \ \forall\varphi \in D_+(\Omega).$$

4.23 Theorem.

Let $h(s) = -\rho(s)\ln (1 - \frac{\varkappa-1}{2a_0^2} s)$ (see examples (2.6′)) and T_n, p_n, ρ_n, $\vec{v}^n$ a sequence of solutions to (4.1)-(4.10), with $J(\vec{v}^n) \to 0$ and with (4.20), (4.22). Let us suppose without the loss of generality $u_n \to u$. Suppose $|\nabla u_n|^2 < s_0$ (see (2.6'), ([illegible])). Then $\{u_n\}$ is h-entropic and u is a h-entropic solution to the transonic problem.

4.24 Remark

The term $\int_\Omega \rho^{1-\alpha} v_i \frac{\partial \rho}{\partial x_i} dx$ gives "vanishing" in the continuity equation.

Idea of the proof to the theorem 4.23: We have from (4.9) as before

$$(4.25) \qquad \beta_n \int_\Omega \frac{1}{T_n^2} |\nabla T_n|^2 dx + \beta_n \int_\Omega |\nabla \vec{v}^n|^2 dx \le c < \infty .$$

Take $\varphi \in D_+(\Omega)$. It follows from (4.9)

$$(4.26) \qquad -c_v \int_\Omega \rho_n \ln \frac{T_n}{\rho_n^{\varkappa-1}} v_i^n \frac{\partial \varphi}{\partial x_i} dx = + k_n \int_\Omega \frac{|\nabla T_n|^2}{T_n^2} \varphi dx -$$

$$- k_n \int_\Omega \frac{\nabla T_n}{T_n} \nabla \varphi dx + \int_\Omega \frac{E_n(\vec{v}^n)}{T_n} \varphi dx .$$

Multiply (4.26) by $\frac{1}{c_v} \frac{1}{\ln \frac{T_0}{\rho_0^{\varkappa-1}}}$ and add to (4.20).

We get

$$(4.27) \qquad \frac{1}{\alpha_n} \int_\Omega (\rho_n - \rho_n^{1-\alpha_n}) v_i^n \frac{\partial \varphi}{\partial x_i} dx = - \frac{1}{\alpha_n} \int_\Omega f_n \varphi dx - \frac{1}{\alpha_n} \langle G_n, \varphi \rangle -$$

$$- \frac{1}{\alpha_n} \langle H_n, \varphi \rangle - \frac{k_n}{\alpha_n} \int_\Omega \frac{|\nabla T_n|^2}{T_n^2} \varphi dx - \frac{1}{\alpha_n} \int_\Omega \frac{E_n(\vec{v}^n)}{T_n} \varphi dx +$$

$$+ \frac{k_n}{\alpha_n} \int_\Omega \frac{\nabla T_n}{T_n} \nabla \varphi dx$$

and the result follows as in the theorem 4.16.

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