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SOME SOLVED AND UNSOLVED CANONICAL PROBLEMS OF DIFFRACTION THEORY

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1. Introduction

Mathematical diffraction theory is concerned with the following boundary value problem in case of an incoming or primary time-harmonic wave-field $\mathrm{Re}[\Phi_{pr}(\underline{x})e^{-i\omega t}]$:
Given an obstacle $\Omega \subset R^n$; $n = 2$ or 3; with boundary $\Gamma = \partial\Omega$. Find the scattered field $\Phi_{sc}(\underline{x})$ in $\Omega_a := R^n - \overline{\Omega}$, s.th.

(1.1) $(\Delta + k^2)\Phi_{sc}(\underline{x}) = 0 \quad \text{for} \quad x \in \Omega_a$

with a wave-number $k = k_1 + ik_2 \in C_{++} - \{0\}$ fulfilling a boundary condition

(1.2a) $B_1[\Phi_{sc}(\underline{x})]|_\Gamma := \Phi_{sc}(\underline{x})|_\Gamma = f(\underline{x})$ of Dirichlet-type
or

(1.2b) $B_2[\Phi_{sc}(\underline{x})]|_\Gamma := (\frac{\partial}{\partial n} + i\, p(\underline{x}))\Phi_{sc}(\underline{x})|_\Gamma = g(\underline{x})$

of $\left\{\begin{matrix}\text{Neumann } (p \equiv 0) \\ \text{Impedance } (p \not\equiv 0)\end{matrix}\right\}$ - type.

In the case of edges E and/or vertices $V \subset \Gamma$ existing the "edge condition"

(1.3) $\Phi_{sc}(\underline{x}) = O(1)$ and $\nabla\Phi_{sc}(\underline{x}) \in L^2_{loc}(\Omega_a)$

should hold. Besides this the scattered field should be "outgoing", i.e. "Sommerfeld's radiation conditions" should hold

(1.4) $\Phi_{sc}(\underline{x}) = \vartheta(e^{-k_2 r})$, $(\frac{\partial}{\partial r} - i.k)\Phi_{sc}(\underline{x}) = \vartheta(e^{-k_2 r}/r^{\frac{n-1}{2}})$

as $r = |\underline{x}| \to \infty$

For smooth compact boundaries Γ this problem has completely been solved, e.g. by the boundary integral equation method (BEM) (c.f. e.g. COLTON-KRESS (1983) [2]) or by means of Sobolev space methods (c.f. e.g. LEIS (1985) [11]). Generalizations to piecewise smoothly bounded domains were carried out by GRISVARD (1980) [6] and COSTABEL (1984) [4], e.g.

2. The Sommerfeld Half-Plane Problem

There are a number of "canonical diffraction problems" with domains whose boundaries extend to infinity and having corners and cusps. The most famous one is the "Sommerfeld half-plane problem", the first diffraction problem having been treated in a mathematically rigorous way (1896) [15].

Applying the well-known representation formula for outgoing solutions of the Helmholtz equation (1.1) the Sommerfeld half-plane problems leads to the following integral or integro-differential equations (of the first kind) of the Wiener-Hopf type:

$$(2.1)\qquad \int_0^\infty H_0^{(1)}(k|x-x'|)\, I(x')dx' = -4i.\Phi_{pr}(x,0) \quad \text{for} \quad x \geq 0$$

in the case of the Dirichlet problem and

$$(2.2)\qquad \left(\frac{d^2}{dx^2} + k^2\right) \int_0^\infty H_0^{(1)}(k|x-x'|)\, Q(x')dx' = 4i\, \frac{\partial\Phi_{pr}}{\partial y}(x,0) \text{ for } x > 0$$

in the case of the Neumann problem with the unknown jumps

$$(2,3)\qquad I(x') := \frac{\partial\Phi_{sc}}{\partial y}(x',+0) - \frac{\partial\Phi_{sc}}{\partial y}(x',-0) \qquad \text{for } x' > 0$$

and

$$(2.4)\qquad Q(x') := \Phi_{sc}(x',+0) - \Phi_{sc}(x',-0) \qquad \text{for } x' \geq 0 ,$$

respectively.

The theory of such equations, but of the second kind, in $L^p(\mathbf{R}_+)$ or $W^{m,p}(\mathbf{R}_+)$-spaces for $m \in N_0$, $1 \leq p \leq \infty$ has been developed by M.G. KREIN (1958/62) [9], E.Gerlach (1969) [5] and, combined with other integral operators than 1-convolutions, by G.THELEN (1985) [17].

To solve the equations (2.1) or (2.2) on the half-line, or more directly the original boundary value problem, one applies a one-dimensional Fourier transform to the scattered wave function

$$(2.5)\qquad \hat{\Phi}_{sc}(\lambda,y) := \int_{-\infty}^{\infty} e^{i\lambda x}\Phi_{sc}(x,y)dx, \quad \lambda \in \mathbf{R}, \quad y \gtrless 0 .$$

The usual, or S'-distributional Fourier transform technique leads to the following "function-theoretic Wiener-Hopf equations", in the case of a damping medium, i.e. Im $k = k_2 > 0$, and an incoming plane wave:

$$(2.6)\qquad \hat{E}_-(\lambda) + \tfrac{1}{2}\,\hat{I}_+(\lambda)/\sqrt{\lambda^2 - k^2} = [\,i(\lambda + k\cos\theta)]^{-1}$$

and

$$(2.7)\qquad \hat{V}_-(\lambda) + \tfrac{1}{2}\,\hat{Q}_+(\lambda).\sqrt{\lambda^2 - k^2} = -k \sin\theta\, [\lambda + k\cos\theta]^{-1} ,$$

respectively, for the Dirichlet and Neumann case with the unknown F-transforms $\hat{E}_-$, $\hat{V}_-$ being holomorphic for Im $\lambda < k_2$ and I_+, Q_+ being holomorphic for Im $\lambda > -k_2 \cos\theta$. The equations (2.6) and (2.7) are equivalent to "non-normal Riemann boundary value problems on a line" parallel to the real λ-axis.

The well-known steps of factorization of $\gamma(\lambda) := \sqrt{\lambda^2 - k^2}$ into

$\gamma_+(\lambda).\gamma_-(\lambda)$, the multiplication of (2.6) and (2.7) by γ_- and by γ_-^{-1}, respectively, then additive decomposition of $\gamma_-.[\lambda + k\cos\theta]^{-1}$ and $\gamma_-^{-1}.[\lambda + k\cos\theta]^{-1}$ in the λ-strip gives after rearrangement and application of Liouville's theorem the explicite solutions to eqs. (2.6) and (2.7) as

(2.8) $\hat{I}_+(\lambda) = 2\sqrt{2k}\cos\theta/2.\gamma_+(\lambda)[\lambda + k\cos\theta]^{-1}$

and

(2.9) $\hat{Q}_+(\lambda) = -2i\sqrt{2k}\sin\theta/2.\gamma_+^{-1}(\lambda)[\lambda + k\cos\theta]^{-1}$

for $\operatorname{Im}\lambda > -k_2\cos\theta$.

These functions being known allow to calculate $\Phi_{sc}(x,y)$ in both cases after applying an inverse F-transform and shifting the line of integration in the complex λ-plane to get all informations relevant, i.e. the edge behaviour an the far field in the geometrically different regions.

This functiontheoretic method has been applied successfully to a big number of canonical problems in microwave theory and to other diffraction problems, e.g. for systems of parallel semi-infinite plates (A.E.Heins (1948) [7]), or cascades of such (J.F. Carlson, A.E. Heins (1946/50) [1]), or cylindrical semi-infinite pipes (e.g. L.A. Vajnshtejn (1948) [18]).

The "canonical mixed Sommerfeld half-plane problems", where there are given different boundary conditions on the faces $\delta_\pm$ of the semi-infinite screen $\delta := \{(x,y) \in \mathbf{R}^2 : y = 0,\ x \geq 0\}$, may be transformed by the same Fourier technique into a 2×2-functiontheoretic system of Wiener-Hopf equations

(2.10) $\underline{\hat{\Phi}}_-(\lambda) = \underline{\underline{K}}(\lambda)\underline{\hat{\Phi}}_+(\lambda) + \underline{\hat{r}}(\lambda)$ for $-k_2\cos\theta < \operatorname{Im}\lambda < k_2$

with the known 2×2-function matrix

(2.11) $$\underline{\underline{K}}(\lambda) := \begin{pmatrix} \sqrt{(\lambda-k)/(\lambda+k)} & 1 \\ -1 & \sqrt{(\lambda+k)/(\lambda-k)} \end{pmatrix}$$

and the unknown 2×1-function-vectors

(2.12) $$\underline{\hat{\Phi}}_-(\lambda) := \begin{pmatrix} \sqrt{\lambda-k}\ .\ \hat{E}_-(\lambda) \\ \hat{V}_-(\lambda)/\sqrt{\lambda-k} \end{pmatrix},$$

$$\underline{\hat{\Phi}}_+(\lambda) := -\frac{1}{2}\begin{pmatrix} \sqrt{\lambda+k}\ .\ \hat{\Phi}_+(\lambda,-0) \\ \hat{\Phi}'_+(\lambda,+0)/\sqrt{\lambda+k} \end{pmatrix}.$$

The matrix $\underline{\underline{K}}(\lambda)$ - or a closely related one - has been factorized into $[\underline{\underline{K}}_-(\lambda)]^{-1}\underline{\underline{K}}_+(\lambda)$ only (1982/83) by A.E.Heins [8], (1981) by A.D.Rawlins [14] and (1981/85) by the present author [12], independently by different methods. Now the solution of the mixed Sommerfeld problem may be written down explicitely and gives full information on the behaviour of Φ_{sc}, $\nabla\Phi_{sc}$ as $r \to 0$ and $r \to \infty$, respectively, which is now different at the edge compared to the one-boundary-condition-problems. The corresponding mixed boundary value problems for systems of parallel semi-infinite plates or a tube are unsolved up to now due to the lack of a known explicit factorization of the 2×2-function matrices involved (c.f. e.g. the authors paper (1984/85) [12]!).

The Sommerfeld half-plane problems have been generalized to the so called "Quarter-plane Problems of Diffraction Theory" where the half-plane, i.e. the screen $\delta \subset \mathbf{R}^3$, is replaced by a screen $\Sigma \subset \mathbf{R}^3$ which is the quarter-plane $\mathbf{R}^2_{++} := \{(x,y,z) \in \mathbf{R}^3 : z = 0,\ x \geq 0,\ z \geq 0\}$ with two semi-infinite lines as edges meeting in the corner E at the origin. Like for an arbitrary plane screen $\Sigma \subset \mathbf{R}^2_{xy}$ the 2-dimensional F-transform applied to the unknown scattered field $\Phi_{sc}(\underline{x})$, $\underline{x} \in \mathbf{R}^3$, leads to the following "Two-dimensional Wiener-Hopf functional equations"

$$(2.13)\quad \gamma^{-1}(\lambda_1,\lambda_2)\hat{I}_\Sigma(\lambda_1,\lambda_2) - \hat{\Phi}_{\mathbf{R}^2\setminus\Sigma}(\lambda_1,\lambda_2,0) = -\Phi_{pr,\mathbf{R}^2\setminus\Sigma}(\lambda_1,\lambda_2,0)$$

and

$$(2.14)\quad \gamma(\lambda_1,\lambda_2)\hat{Q}_\Sigma(\lambda_1,\lambda_2) - (\tfrac{\partial}{\partial z}\Phi)\hat{}_{\mathbf{R}^2\setminus\Sigma}(\lambda_1,\lambda_2,0) = -(\tfrac{\partial}{\partial z}\Phi_{pr})_{\mathbf{R}^2\setminus\Sigma}(\lambda_1,\lambda_2,0)$$

where $\gamma(\lambda_1,\lambda_2) := \sqrt{\lambda_1^2 + \lambda_2^2 - k^2}$ and the indices Σ and $\mathbf{R}^2\setminus\Sigma$ refer to the 2D-F-transforms of the restrictions to Σ and $\mathbf{R}^2\setminus\Sigma$, respectively.

Up to now there exists no explicit factorization of the multiplication operator γ with respect to the complementary projectors $\hat{P}_\Sigma$, $\hat{Q}_\Sigma := I - \hat{P}_\Sigma$ in spaces $FL^p(\mathbf{R}^2)$ or $FW^{s,p}(\mathbf{R}^2)$, $s > 0$, $1 < p \leq 2$ (∞). But there exists now a very general theory for "general Wiener-Hopf or Toeplitz operators" of the form

$$(2.15)\quad P_2A|_{P_1X}\, u = v \in P_2Y$$

for bijective continuous operators $A : X \to Y$ acting between two Banach-spaces X,Y with bounded projectors $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$. This theory by F.-O.Speck (1983/85) [16] gives necessary and sufficient conditions for the general invertibility and Fredholm property of operators of type (2.15) in dependance on factorization properties of A w.r.t. (P_1,P_2).

3. Canonical Transmission Problems

Another big class of canonical diffraction problems exists given by the following specification:

Given a primary time-harmonic wave-field $\mathrm{Re}[\Phi_{pr}(\underline{x})e^{-i\omega t}]$ and a region $\Omega_1 \subset \mathbf{R}^n$, $n = 2$ or 3, and finitely many disjoint regions $\Omega_2,\dots,\Omega_N \subset \mathbf{R}^n$, s. th. $\bigcup_{j=1}^{N} \overline{\Omega}_j = \mathbf{R}^n$.. Then one looks for a scattered field $\Phi_{sc}(\underline{x})$, $x \in \mathbf{R}^n$, s. th. $\Phi_{sc}(\underline{x})|_{\Omega_j} \in C^2(\Omega_j) \cap C^1(\overline{\Omega}_j\setminus\{0\})$ and

(3.1) $(\Delta + k_j^2)\Phi_{sc}(\underline{x}) = 0$ in Ω_j, $j = 1,\ldots,N$,

fulfilling the "transmission conditions"

(3.2a) $\Phi_{sc,j}(\underline{x}) - \Phi_{sc,t}(\underline{x}) = F_{jl}(\underline{x})$

and on $\partial\Omega_j \cap \partial\Omega_l \neq \phi$

(3.2b) $\rho_j \cdot \frac{\partial\Phi_{sc,j}}{\partial n_j}(\underline{x}) + \rho_l \cdot \frac{\partial\Phi_{sc,l}}{\partial n_l}(\underline{x}) = G_{jl}(\underline{x})$

with prescribed data F_{jl}, G_{jl} from the primary field on the common boundary parts $\partial\Omega_j \cap \partial\Omega_l$.
Additionally the edge conditions $\Phi_j(\underline{x}) = \Phi(x)|_{\Omega_j} = O(1)$ and $\nabla\Phi_j \in L^2_{loc}(\Omega_j)$ and the radiation condition for $\Phi_1(x)$ as $|x| = r \to \infty$ have to hold.

Again in the case of smoothly bounded domains with compact boundaries $\partial\Omega_j$ this "transmission or interface problem" has been solved by the boundary integral method and in the case of two-dimensional polygonal domains by M.Costabel and E.Stephan (1985) [3].

In the special case of two different media (i.e. N = 2) and a plane interface (i.e. $\partial\Omega_1 = \partial\Omega_2 = \mathbf{R}xy$ or $= \mathbf{R}^1_x$) the problem is elementary and gives, for a plane wave as the primary wave-function, the well-known relations from Snellius' law and the reflection and transmission coefficients explicitely. The corresponding "two-dimensional Sommerfeld half-plane problems with two media" are unsolved up to now - as far as an explicit representation is concerned - due to the unknown matrix factors of the 2X2-Wiener-Hopf function matrices involved here having two different square roots $\sqrt{\lambda^2 - k_1^2}$ and $\sqrt{\lambda^2 - k_2^2}$ to be taken into account [12].

A very important canonical transmission problem is the so-called "Dielectric Wedge Problem", i.e. the case of $\Omega_1 = \mathbf{R}^2_{++}$ and $\Omega_2 = \mathbf{R}^2 \backslash \mathbf{R}^2_{++}$ in $\mathbf{R}^2$ or the corresponding "Dielectric Octant Problem" in $\mathbf{R}^3$-space: This has been generalized to the "Four-Quadrant-Transmission-Problem" in $\mathbf{R}^2$ with the four quadrants filled with different media. Applying 2D-Fouriertransformation the restrictions of the unknown scattered field may be represented by the 1D - F - transformed Cauchy-data on the semi-infinite lines, the boundaries of the quadrants. For $\hat{\Phi}_1(\lambda_1,\lambda_2)$ one gets e.g.

(3.3) $\hat{\Phi}_1(\Lambda_1,\lambda_2) = [i\lambda_2 . \hat{f}_1^{(1)}(\lambda_1) + g_1^{(1)}(\lambda_1) + i\lambda_1 . \hat{f}_2^{(1)}(\lambda_2) +$
$+ \hat{g}_2^{(1)}(\lambda_2)] \cdot (\lambda_1^2 + \lambda_2^2 - k_1^2)^{-1}$ for $\mathrm{Im}\,\lambda_1$, $\mathrm{Im}\,\lambda_2 > -\beta_1, -\beta_2$
with $\beta_1^2 + \beta_2^2 < (\mathrm{Jm}k_1)^2$.

Due to the transmission conditions (3.2) the total sum of all numerators of the $\hat{\Phi}_j(\lambda_1,\lambda_2)$ is a known function $Z(\lambda_1,\lambda_2)$. Dividing by the known $N(\lambda_1,\lambda_2,k^2)$ with an appropriate $k \in \mathbb{C}_+$ one arrives at the "Four-part Wiener-Hopf functional equation"

$$(3.4)\qquad \sum_{j=1}^{4} \left(1 + \frac{k^2-k_j^2}{\lambda_1^2+\lambda_2^2-k^2}\right)\hat{P}_j\hat{\Phi}(\lambda_1,\lambda_2) = \frac{Z(\lambda_1,\lambda_2)}{N(\lambda_1,\lambda_2,k^2)}$$

holding for a pair of strips of $\mathbb{C}^2$. Here we have

$$(3.5)\qquad \hat{\Phi}_j(\lambda_1,\lambda_2) := \hat{P}_j\hat{\Phi}(\lambda_1,\lambda_2) := (F_{2\chi Q_j}:F_2^{-1}\Phi)(\lambda_1,\lambda_2)$$

It has been shown (e.g. by N.Latz (1968) [10]) that in the case of $\operatorname{Im} k_j > 0$ the auxiliary k may be chosen in such a way that eq. (3.4) is uniquely solvable in $FL^p(\mathbb{R}^2)$, $1 < p \le 2$, for any $\Phi_{pr}(\underline{x}) \in L^p(\mathbb{R}^2)$. The present author has derived quite recently (1984) [13] a 4X4-system of integral equations for the Fourier-cosine transforms of the normal derivatives on the bounding semi-axis's of the four quadrants Q_j. This system is uniquely solvable in the case of $\operatorname{Im} k_j > 0$ and $|k_j - k_\nu|$ and $|\rho_j - \rho_\nu|$ small by Banach's fixed point theorem in the spaces $(L^q(\mathbb{R}_+))^4$ for $2 \le q < \infty$, but the general case of four different wave numbers k_j is still unsolved.

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