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nX-Complementary Generations of the Harda-Norton Group HN

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ABSTRACT. Let G be a finite group and nX be conjugacy class of elements of order n in G. G is called (l,m,n)-generated, if it is a quotient group of the triangle group $T(l,m,n) = \langle x,y,z | x^l = y^m = z^n = xyz = 1\rangle$, and nX−complementary generated, if, for any arbitrary $x \in G - \{1\}$, there is a $y \in nX$ such that $G = < x, y >$.

In [20] the question of finding all positive integers n such that non-abelian finite simple group G is nX−complementary generated was posed. In this paper we partially answer this question for the sporadic group HN. In fact, we prove that for any element order n of the sporadic group HN, HN is nX−complementary generated if and only if $nX \notin \{2A, 2B, 3A, 5A, 5B\}$.

1. Introduction

A group G is said to be (l,m,n)-generated if it can be generated by two elements x and y such that $o(x) = l, o(y) = m$ and $o(xy) = n$. In this case G is the quotient of the triangle group $T(l,m,n)$ and for any permutation π of S_3, the group G is also $((l)\pi, (m)\pi, (n)\pi)$-generated. Therefore we may assume that $l \leq m \leq n$. By [5], if the non-abelian simple group G is (l,m,n)-generated, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Hence for a non-abelian finite simple group G and divisors l,m,n of the order of G such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, it is natural to ask if G is a (l,m,n)-generated group. The motivation for this question came from the calculation of the genus of finite simple groups [26]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations(for details see [23]).

In a series of papers, [16-21] Moori and Ganief established all possible (p,q,r)−generations and nX-complementary generations, where p,q,r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2, and F_{22}. Also, the author in [2-4] and [8-14], did the same for the sporadic groups Co_1, Th, $O'N$, Ly and He. The

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motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

In what follows, we describe some notations which will be kept throughout. $\Delta(G) = \Delta(lX, mY, nZ)$ denotes the structure constant of G for the conjugacy classes lX, mY, nZ, whose value is the cardinality of the set $\Lambda = \{(x,y)|xy = z\}$, where $x \in lX, y \in mY$ and z is a fixed element of the conjugacy class nZ. Also, $\Delta^\star(G) = \Delta^\star_G(lX, mY, nZ)$ and $\Sigma(H_1 \cup H_2 \cup \cdots \cup H_r)$ denote the number of pairs $(x,y) \in \Lambda$ such that $G = \langle x, y\rangle$ and $\langle x, y\rangle \subseteq H_i$ (for some $1 \leq i \leq r$), respectively. The number of pairs $(x,y) \in \Lambda$ generating a subgroup H of G will be given by $\Sigma^\star(H)$ and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general Conjugacy class of a subgroup H of G with elements of order n will be denoted by nx. Clearly, if $\Delta^\star(G) > 0$, then G is (lX, mY, nZ)-generated and (lX, mY, nZ) is called a generating triple for G. The number of conjugates of a given subgroup H of G containing a fix element z is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H(cf. [24]). In most cases we will calculate this value from the fusion map from $N_G(H)$ into G stored in GAP, [22].

Let G be a group and nX a conjugacy class of elements of order n in G. Following Woldar [25], the group G is said to be nX-complementary generated if, for any arbitrary non-identity element $x \in G$, there exists a $y \in nX$ such that $G =< x, y >$. The element $y = y(x)$ for which $G =< x, y >$ is called complementary.

Now we discuss techniques that are useful in resolving generation type questions for finite groups. We begin with a result of [6] that, in certain situations, is very effective at establishing non-generations.

Theorem 1.1. ([6]) *Let G be a finite centerless group and suppose lX, mY and nZ are G-conjugacy classes for which $\Delta^\star(G) = \Delta^\star_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then $\Delta^\star(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

A further useful result that we shall often use is a result from Conder, Wilson and Woldar [6], as follows:

Lemma 1.2. *If G is nX-complementary generated and $(sY)^k = nX$, for some integer k, then G i /complemen s sY-complementary generated.*

Further useful results that we shall use are:

Lemma 1.3.([18]). *Let G be a $(2x, sY, tZ)-$generated simple group then G is $(sY, sY, (tZ)^2)-$generated.*

Lemma 1.4. *Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, x_2, \cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

We calculated h for suitable triples in Table II. Throughout this paper our notation is standard and taken mainly from [1], [16] and [17]. We will prove the following theorem:

Theorem. *The Harada-Norton group HN is nX-complementary generated if and only if $nX \notin \{2A, 2B, 3A, 5A, 5B\}$.*

2. $nX-$Complementary Generations for HN

In this section we obtain all of the nX-complementary generations of the Harada-Norton group HN. We will use the maximal subgroups of HN listed in the ATLAS extensively, especially those with order divisible by 19. We listed in Table I, all the maximal subgroups of HN and in Table II, the fusion maps of the maximal subgroup $U_3(8).3_1$ into HN (obtained from GAP) that will enable us to evaluate $\Delta^{\star}_{HN}(pX, qY, rZ)$, for prime classes pX, qY and rZ. In this table h denotes the number of conjugates of the maximal subgroup H containing a fixed element z (see Lemma 1.4). For basic properties of the group HN and information on its maximal subgroups the reader is referred to [7]. It is a well known fact that HN has exactly 14 conjugacy classes of maximal subgroups, as listed in Table I.

In [25], Woldar proved that every sporadic simple group is $pX-$ complementary generated, for the greatest prime divisor p of the order of the group. As a consequence of a result in the mentioned paper, a group G is nX-complementary generated if and only if G is (pY, nX, t_pZ)-generated, for all conjugacy classes pY with representatives of prime order and some conjugacy class t_pZ (depending on pY). By the mentioned result of Woldar HN is $19X$-complementary generated, for $X \in \{A, B\}$.

Lemma 2.1. *The sporadic group HN is not $2X-$, $3A-$ and $5X-$ complementary generated, in which $X \in \{A, B\}$.*

Proof. For any positive integer n, $T(2,2,n) \cong D_{2n}$, the dihedral group of order $2n$. Since HN is simple and for all $n \geq 3$ the dihedral group D_{2n} is not simple, HN is not $(2X, 2X, nY)$-generated, for all conjugacy classes of involutions and any HN-class nY. Thus, HN is not $2X-$complementary generated. We now show that HN is not $3A-$, $5A-$ and $5B-$complementary generated. To do this, we assume that $nX \in \{3A, 5A, 5B\}$ and consider the conjugacy class $2A$. Using a simple program in GAP language [22], we can see that for any conjugacy class t_2Z, we have:

$$\Delta_{HN}(2A, nX, t_2Z) < |C_{HN}(t_2Z)|.$$

Therefore, by Theorem 1.1, $\Delta^{\star}_{HN}(2A, nX, t_2Z) = 0$ and HN is not $3A-$, $5A-$ and $5B-$complementary generated. □

In the following lemma, we prove that for every conjugacy class nX with elements of prime order, other than $2X, 3A$ and $5X, X \in \{A, B\}$, HN is $nX-$complementary generated.

Lemma 2.2. *The Harada-Norton group HN is pX-complementary generated, in which p is an odd prime divisor of $|HN|$ and $pX \neq 3A, 5A, 5B$.*

Proof. By the Woldar's result, mentioned above, the group HN is $19X$-complementary generated for $X \in \{A, B\}$. So, it is enough to investigate the prime divisors of $|HN|$ distinct from 2 and 19. Set $A = \{5C, 5D, 5E, 11A\}$. Suppose $nX \in A$ and consider the conjugacy class $19A$. Then for every prime class pY, there is no maximal subgroup of HN that contains $(pY, nX, 19A)-$generated proper subgroups. On the other hand, we can see that $\Delta_{HN}(pY, nX, 19A) > 0$ and so

$$\Delta^{\star}_{HN}(pY, nX, 19A) = \Delta_{HN}(pY, nX, 19A) > 0.$$

Therefore, the Harada-Norton group HN is $nX-$complementary generated. We now prove that HN is $3B-$complementary generated. To do this, we assume that $B = \{2A, 5A, 5B, 5C, 5D, 5E, 11A\}$ and consider the conjugacy class $19A$.

If $pY \in B$ then by Table I and II, there is no maximal subgroup of HN that contains $(pY, 3B,\ 19A)-$generated proper subgroups. Therefore, $\Delta^{\star}_{HN}(pY,\ 3B, 19A) = \Delta_{HN}(pY, 3B, 19A) > 0$. Thus, HN is $(pY, 3B, 19A)-$generated. On the other hand, by Lemma 1.3, since HN is $(2A, 3B, 19A)-$generated, it is $(3B, 3B, (19A)^2 = 19B)-$ generated. Using the character table of HN [7], we can see that $19A^{-1} = 19B$. Thus, the sporadic group HN is $(3B, 3B, 19A)-$generated. Suppose $pY = 2B$. Then amongst the maximal subgroups of HN with order divisible by 19, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $U_3(8).3_1$. Our calculations give,

$$\Delta^{\star}(HN) \geq \Delta(HN) - 1(57) = 2565 - 57 > 0,$$

proving the generation of HN by this triple. Using a similar argument as in above, we can prove that $(3A, 3B, 19A)$ and $(7A, 3B, 19A)$ are generating triples for HN. We now consider the conjugacy class $pY = 19A$. Since HN is $(2B, 3B, 19A)-$generated, for any permutation $\pi \in S_3$, the group HN is also $((2B)\pi, (3B)\pi, (19A)\pi)$-generated. Thus the Harada-Norton group HN is $(2B, 19A, 3B)-$generated and by Lemma 1.3, it is $(19A, 19A, (3B)^2 = 3B)$-generated. This shows that the sporadic group HN is $(19A, 3B, 19A)$-generated and using a similar argument, we can see that it is $(19B, 3B, 19A)$-generated. Therefore, HN is $3B-$complementary generated.

We can apply a similar method to show that HN is $7A-$complementary generated. This completes the proof. □

Lemma 2.3. *The Harada-Norton group HN is $nX-$complementary generated, for $n = 4, 6$ and $X \in \{A, B, C\}$.*

Proof. The proof for the cases $n = 4$ and $n = 6$ is similar. Hence, we investigate only the case $n = 6$. Using the character table of HN [22], we can see that $6C^2 = 3B$. Since HN is $3B-$complementary generated, by Lemma 1.2, it is $6C-$complementary generated. Next we show that HN is $6A-$complementary generated. Consider the conjugacy class $19A$. Then for every prime class pY, there is no maximal subgroup of HN that contains $(pY, 6A, 19A)-$generated proper subgroups. On the other hand, we can see that $\Delta_{HN}(pY, 6A, 19A) > 0$ and so

$\Delta^{\star}_{HN}(pY, nX, 19A) = \Delta_{HN}(pY, nX, 19A) > 0$. Therefore, the Harada-Norton group HN is $6A-$ complementary generated.

We now prove that HN is $6B-$complementary generated. To do this, we assume that $B = \{2A, 5A, 5B, 5C, 5D, 5E, 11A\}$ and consider the conjugacy class $19A$. If $pY \in B$ then by Table I and II, there is no maximal subgroup of HN that contains $(pY, 6B, 19A)-$generated proper subgroups. Therefore, $\Delta^{\star}_{HN}(pY, 6B, 19A) = \Delta_{HN}(pY, 6B, 19A) > 0$. Thus, HN is $(pY, 6B, 19A)-$generated. Suppose $pY = 2B$. Then amongst the maximal subgroups of HN with order divisible by 19, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $U_3(8).3_1$. Our calculations give,

$$\Delta^{\star}(HN) \geq \Delta(HN) - 1(342) = 125210 - 342 > 0,$$

proving the generation of HN by this triple. Using a similar argument as in above, we can prove that $(3A, 6B, 19A)$, $(3B, 6B, 19A)$, $(7A, 6B, 19A)$, $(19A, 6B, 19A)$ and $(19B, 6B, 19A)$ are generating triples for HN. Therefore, HN is $6B-$complementary generated. This completes the proof. □

Lemma 2.4. *The Harada-Norton group HN is $10X-$, $15Y-$ and $25Z-$ complementary generated, in which $X \in \{A, B, C, D, E, F, G, H\}$, and $Y \in \{A, B, C\}$ and $Z \in \{A, B\}$.*

Proof. Set

$$A = \{10A, 10B, 10C, 10D, 10E, 10F, 10G, 10H, 15A, 15B, 15C, 25A, 25B\}.$$

We consider the conjugacy class $19A$. If $nX \in A$ and pY is an arbitrary prime class of HN then $\Delta_{HN}(pY, nX, 19A) > 0$ and there is no maximal subgroup of HN that contains $(pY, nX, 19A)-$generated proper subgroups. Therefore, $\Delta^{\star}_{HN}(pY, nX, 19A) = \Delta_{HN}(pY, nX, 19A) > 0$. Thus, HN is $(pY, nX, 19A)-$generated. This shows that for any $nX \in A$, HN is $nX-$complementary generated. □

Set $T = \{8A, 8B, 9A, 12A, 12B$, $12C$, $14A$, $20A$, $20B$, $20C$, $20D$, $20E$, $21A$, $22A$, $30A$, $30B$, $30C$, $35A$, $35B$, $40A$, $40B\}$. In Lemmas 2.1-2.4, we proved that if $nX \notin T \cup \{2A, 2B, 3A, 5A, 5B\}$ then the Harada-Norton group HN is $nX-$complementary generated. In the following lemma, we used these results and Lemma 1.2 to prove the $nX-$complementary generations of the conjugacy classes of T.

Lemma 2.5. *For every $nX \in T$, the Harada-Norton group HN is $nX-$ complementary generated.*

Proof. Using the character table of HN [7], we can see that:

$$(8A)^2 = (12A)^3 = 4B, (8B)^2 = (12A)^3 = 4A, (9A)^3 = 3B, (22A)^2 = 11A, (21A)^3 = 7A$$

$$(12C)^3 = 4C, (20A)^5 = (20B)^5 = 4A, (20C)^5 = 4B, (20D)^5 = (20E)^5 = 4C$$

$$(30A)^5 = 6A, (30B)^5 = (30C)^5 = 6C, (35A)^5 = (35B)^5 = 7A$$

By Lemma 1.3 and Lemmas 2.1-2.4, HN is $nX-$complementary generated, for $nX \in T-\{40A, 40B\}$. On the other hand, since $(40A)^2 = (40B)^2 = 20C$ and HN is $20C-$complementary generated, it is $40X-$complementary generated, $X \in \{A, B\}$, proving the lemma. □

We are now ready to state the main result of this paper:

Theorem. *The Harada-Norton group HN is nX-complementary generated if and only if $nX \notin \{2A, 2B, 3A, 5A, 5B\}$.*

Proof. The proof follows from the Lemmas 2.1-2.5. □

Table I

The Maximal Subgroups of HN

Group	*Order*	*Group*	*Order*
A_{12}	$2^9.3^5.5^2.7.11$	$2.HS.2$	$2^{11}.3^2.5^3.7.11$
$U_3(8).3_1$	$2^9.3^5.7.19$	$2^{1+8}.(A_5 \times A_5).2$	$2^{14}.3^2.5^2$
$(D_{10} \times U_3(5)):2$	$2^6.3^2.5^4.7$	$5^{1+4}:2^{1+4}.5.4$	$2^7.5^6$
$2^6.U_4(2)$	$2^{12}.3^4.5$	$(A_6 \times A_6).D_8$	$2^9.3^4.5^2$
$2^3.2^2.2^6.(3 \times L_3(2))$	$2^{14}.3^2.7$	$5^2.5.5^2.4A_5$	$2^4.3.5^6$
$M_{12}.2$	$2^7.3^3.5.11$	$HNM12$	$2^7.3^3.5.11$
$3^4:2(A_4 \times A_4).4$	$2^7.3^6$	$3^{1+4}:4A_5$	$2^4.3^6.5$

Table II

The Partial Fusion Maps of $U_3(8).3_1$ into HN

$U_3(8).3_1$-class	2a	3a	3b	3c	3d	3e	3f	3g	3h	3i
$\to HN$	2B	3A	3A	3B	3A	3A	3B	3B	3B	3B
$U_3(8).3_1$-class	4a	4b	4c	6a	6b	6c	6d	6e	6f	6g
$\to HN$	4A	4C	4C	6B	6B	6B	6B	6C	6C	6C
$U_3(8).3_1$-class	6h	7a	9a	9b	9c	12a	12b	12c	12d	12e
$\to HN$	6C	7A	9A	9A	9A	12B	12B	12C	12C	12C
h		20								
$U_3(8).3_1$-class	12f	19a	19a	21a	21b					
$\to HN$	12C	19A	19B							
h		1	1							

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