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On Monotone Minimal Cuscos

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Abstract

We state the definition of lu^d-minimal set-valued maps and we use it to the characterization of maximal monotonicity for minimal cusco. As a consequence we give a characterization for locally Lipschitz functions which possess a minimal subdifferential.

Key words: Clarke's directional derivative, minimal cusco, convexity, maximal monotone operators.

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1 Introduction

The research of nonsmooth analysis is closely connected with the study of the set-valued maps. Since maximal monotone operators on an open sets in Banach spaces are minimal cuscos, the structure of minimal cuscos has been studied very deeply since 1987 (see e.g. [2], [3], [4], [6], [8], [9]). Special attention has been devoted to the case when the cusco is the generalized gradient of a locally Lipschitz function. We note that a convex function on an open subset of a Banach space is locally Lipschitz, and that its convex subdifferential is maximal monotone operators.

It seems that it is possible to derive some important properties of a minimal cusco only on the base of the properties of some selection of a considered minimal cusco. In [4] was discussed the problem: when can a given (minimal) weak* cusco be represented as the Clarke subdifferential mapping of a real-valued locally Lipschitz function?

Theorem 1.1 *Assume that X is a Banach space and A is a nonempty connected open subset of X. Let $\Phi : A \to 2^{X^*}$ be a locally bounded minimal weak* cusco. Suppose Φ possesses a selection $\sigma : A \to X^*$ such that*

$$\oint_C \sigma(z)dz \leq 0$$

for every closed polygonal path C in A. Then there is some locally Lipschitz function f on A such that $\Phi = \partial f$.

In Section 2, we give the definition of some other notion of minimality of the set-valued map—we call it lu^d-minimality. We discuss its properties and we compare lu^d-minimal maps with minimal cusco maps.

In Section 3, we use lu^d-minimality for the following main results:

Theorem 3.1: Let X be a Banach space, A a nonempty open subset of X, $K > 0$, and $F : X \to 2^{X^*}$ a minimal w^*-cusco. If F possesses a densely defined selection s such that diam(R(s)) $= K$, then diam(R(F)) $= K$.

Theorem 3.2: Let X be a Banach space, A be a nonempty open subset of X and F be a set-valued map from A into subsets of X^* which is a minimal w^*-cusco. If F possesses a densely defined monotone selection, then F is monotone.

Section 3 provides also several consequences of main results, among others we give a characterization of convexity for locally Lipschitz functions which possess a minimal subdifferential.

Throughout this paper we denote the effective domain and the range of the set-valued map $F : X \to 2^Y$, respectively, by $D(F), R(F)$. It means

$$D(F) = \{x \in X, F(x) \neq \emptyset\} \qquad R(F) = \bigcup_{x \in D(F)} F(x).$$

X^* denotes the topological dual of X, and by (X^*, w^*) we mean topological space X^* in its weak* topology. diam A stands for the diametr of the set A. A real-valued function f defined on a nonempty open subset A of a Banach space X is said to be locally Lipschitz on A, if for each $x_0 \in A$ there exist a $K > 0$ and $\delta > 0$ such that

$$|f(x) - f(y)| \leq K\|x - y\| \quad \text{for all } x, y \in B(x_0, \delta) \cap A.$$

The Clarke generalized directional derivative at $x \in A$ in the direction $v \in X$ is given by,

$$f^\circ(x, v) = \limsup_{y \to x,\ t \to 0+} \frac{f(y + tv) - f(y)}{t},$$

and the Clarke generalized gradient of f at x is defined by,

$$\partial f(x) = \{x^* \in X^*, \langle x^*, v\rangle \leq f^\circ(x, v) \text{ for each } v \in X\}.$$

Proposition 1.1 [5, Proposition 2.2.7] *Let X be a Banach space, and let f be a real-valued convex function on an open convex subset A of X, $x \in A$. Then the Clarke subdifferential $\partial f(x)$ agrees with subdifferential in the sense of convex analysis.*

A set-valued map $F : A \to 2^{X^*}$ is said to be monotone on A provided

$$\langle x^* - y^*, x - y\rangle \geq 0$$

whenever $x, y \in A$ and $x^* \in F(x), y^* \in F(y)$. Moreover F is said to be maximal monotone if its graph is not strictly contained in the graph of some other monotone map on A.

Proposition 1.2 [5, Proposition 2.2.9] *Let A be an open convex set, and let f be a real-valued locally Lipschitz function on A. Then f is convex if and only if the map $x \to \partial f(x)$ is monotone.*

A set-valued map F from topological space A into subsets of a linear topological space Y is called an usco (cusco) if it is compact valued (convex and compact) and upper semicontinuous. It is called a minimal usco (minimal cusco) if it is an usco (cusco) whose graph is minimal with respect to set containment among uscos (cuscos). If $Y = X^*$, saying that an usco (cusco) F from A into 2^{X^*} is w^*-usco (cusco) means that we are taking Y to be X^* in its weak* topology. When $\partial f(x)$ is a minimal w*-cusco, we will say f possesses a minimal subdifferential.

Proposition 1.3 [3, Proposition 1.4] *Let G be densely set-valued map from a topological space A into subsets of a separated locally convex topological space Y. If the graph of G is contained in the graph of a cusco map F, then there exists a unique smallest cusco containing G, denoted $CSC(G)$ given by*

$$CSC(G)(x) = \cap\{\overline{co}G(V) : V \textit{ is an open neighbourhood of } x\}.$$

Theorem 1.2 [4, Theorem 3.7] *Let F be a cusco map from a topological space A into subsets of a separated locally convex topological space Y. Then F is a minimal cusco if and only if for every densely defined selection f of F, $CSC(f) = F$.*

Theorem 1.3 [6, Theorem 4.3] *Let F be a cusco map from a topological space A into subsets of a locally convex linear topological space Y. Then the following are equivalent.*

(i) F is a minimal cusco;

(ii) $y^* \circ F$ is a minimal cusco for each $y^* \in Y^*$.

2 lu^d-minimal maps

We can define the lower hull and the upper hull of the functions (see for example [7]).

Definition 2.1 Let A be a topological space and $f : A \to \overline{\mathbf{R}}$ a function. Then a function

$$l(f) = \max\{h \in \overline{\mathbf{R}}^A : (h \leq f)\text{and } h \text{ is lower semicontinuous}\}$$

is called the lower hull of f, and a function

$$u(f) = \min\{h \in \overline{\mathbf{R}}^A : (h \geq f)\text{and } h \text{ is upper semicontinuous}\}$$

is called the upper hull of f.

It is natural to consider the following minimality of the set-valued maps:

Definition 2.2 Suppose that A is a topological space, and suppose that $F : A \to \mathbf{R}$ is a set-valued map. We say that F is lu-minimal if and only if for arbitrary selections f_1 and f_2 of F,
(i) $l(f_1) = l(f_2)$.
(ii) $u(f_1) = u(f_2)$.

Definition 2.3 Let X be a Banach space, $A \subset X$ and let $F : A \to (X^*, w^*)$ be a set-valued map. We say that F is lu-minimal if and only if for each $y \in S_X$ the set-valued map,

$$F_y : A \to \mathbf{R} : F_y(x) = \{\langle x^*, y\rangle : x^* \in F(x)\}$$

is lu-minimal.

It is possible to modify Definition 2.1 for densely defined functions. Then we can modify also Definition 2.2 and Definition 2.3.

Definition 2.4 Let A be a topological space and f is a densely defined function from A into $\overline{\mathbf{R}}$. By the lower hull $l(f)$ of f we mean the lower hull of a function f^∞ for which

$$f^\infty(x) = \begin{cases} f(x) & \text{for } x \in Dom(f) \\ +\infty & \text{otherwice.} \end{cases}$$

By the upper hull $u(f)$ of f we mean the upper hull of a function f_∞ for which

$$f_\infty(x) = \begin{cases} f(x) & \text{for } x \in Dom(f) \\ -\infty & \text{otherwice.} \end{cases}$$

Definition 2.5 Let A be a topological space and $F : A \to \mathbf{R}$ a set-valued map. We say that F is lu^d-minimal if and only if for arbitrary densely defined selections f_1 and f_2 of F,
(i) $l(f_1) = l(f_2)$.
(ii) $u(f_1) = u(f_2)$.

Definition 2.6 Suppose that X is a Banach space, $A \subset X$, and suppose that $F : A \to (X^*, w^*)$ is a set-valued map. We say that F is lu^d-minimal if and only if for each $y \in S_X$ the set-valued map

$$F_y : A \to \mathbf{R} : F_y(x) = \{\langle x^*, y\rangle : x^* \in F(x)\}$$

is lu^d-minimal.

It follows immediately from definitions that lu^d-minimal set-valued map from a subset A of a Banach space X into subsets of X^* is lu-minimal. The converse is not true since for example the Dirichlet function is lu-minimal, but it is not lu^d-minimal.

In the case when the set-valued map is upper semicontinuous, the two notions coincide:

Proposition 2.1 *Let A be a topological space and suppose that $F : A \to \mathbf{R}$ is an upper semicontinuous set-valued map. Then F is lu-minimal if and only if F is lu^d-minimal.*

Proof Let us assume that F is not lu^d-minimal. There exist two densely defined selections f_1 and f_2 of F and $x_0 \in A$ such that either $l(f_1)(x_0) \neq l(f_2)(x_0)$ or $u(f_1)(x_0) \neq u(f_2)(x_0)$.

Let us assume for example that $l(f_2)(x_0) > l(f_1)(x_0)$. We can take $c \in \mathbf{R}$ and $\varepsilon > 0$ such that

$$l(f_1)(x_0) < c - \varepsilon < c + \varepsilon < l(f_2)(x_0).$$

Then there exists a neighbourhood U of x_0 such that

$$\forall x \in (U \cap Dom(f_2)) : f_2(x) > c + \varepsilon. \tag{1}$$

There is also a net $\{x_i\}_{i\in I}$, $x_i \to x_0$, such that

$$\forall i \in I : f_1(x_i) < c - \varepsilon. \tag{2}$$

Since F is upper semicontinuous and (1) is true, for every $y \in U$ there exists $z_y \in F(y)$ such that $z_y \geq c + \varepsilon$. Let us consider the selections f^1, f^2 of F; $f^2(y) = z_y$ whenever $y \in U$, $f^1(x) = f_1(x_i)$ whenever $x = x_i$ for some $i \in I$, and $f^1(x) = f^2(x)$ in otherwise. It follows immediately from (2) that $l(f^2)(x_0) > l(f^1)(x_0)$, so F is not lu-minimal.

If $u(f_1)(x_0) \neq u(f_2)(x_0)$, then we can proceed in an analogous way. □

Theorem 2.1 *Suppose that X is a Banach space, $A \subset X$ and suppose that $F : A \to (X^*, w^*)$ be an upper semicontinuous set-valued map. Then F is lu-minimal if and only if F is lu^d-minimal.*

Proof The composition of two upper semicontinuous set-valued maps is again upper semicontinuous (see for example [1]), then the proof follows immediately from Proposition 2.1. □

It could be useful to compare the two concepts of minimality of the set-valued maps.

Lemma 2.1 *Let A be a topological space and let $F : A \to \mathbf{R}$ be a cusco. Suppose that $x_0 \in A$ and f is any densely defined selection of F. Then*

$$CSC(f)(x_0) = [l(f)(x_0), u(f)(x_0)]. \tag{3}$$

Proof For any $\varepsilon > 0$ there exists a neighbourhood U of x_0 such that

$$\forall x \in (U \cap D(f)) : l(f)(x_0) - \varepsilon \leq l(f)(x) \leq f(x) \leq u(f)(x) \leq u(f)(x_0) + \varepsilon.$$

Hence, from Proposition 1.3,

$$CSC(f)(x_0) \subset [l(f)(x_0), u(f)(x_0)].$$

Let us assume that for example $l(f)(x_0) \notin CSC(f)(x_0)$. We can take $c \in \mathbf{R}$,

$$l(f)(x_0) < c < \min CSC(f)(x_0). \tag{4}$$

From the construction of $CSC(f)$ it follows that there exists a neighbourhood V of x_0 such that

$$\forall x \in (V \cap D(f)) : f(x) > c,$$

but it is a contradiction with the first inequalities in (4). Analogously we can proceed when $u(f)(x_0) \notin CSC(f)(x_0)$. Therefore (3) is true. □

Lemma 2.2 *Suppose that A is a topological space, $F : A \to \mathbf{R}$ a cusco. Then F is the minimal cusco if and only if F is lu^d-minimal.*

Proof As a consequence of Theorem 1.2, F is a minimal cusco if and only if for any two densely defined selections f_1 and f_2 of F it holds $CSC(f_1) = CSC(f_2)$. The rest of the proof now follows directly from the previous lemma. □

Proposition 2.2 *Let X be a Banach space, A an open subset of X, and let $f : A \to \mathbf{R}$ be a locally Lipschitz function. Then the set-valued map $x \to \partial f(x)$ is the minimal w^*-cusco if and only if it is lu^d-minimal.*

Proof The proof follows immediately from Theorem 1.3 and Lemma 2.2. □

Corollary 2.1 *Let X be a Banach space, A an open subset of X, and let $f : A \to \mathbf{R}$ be a continuous convex function. Then the set-valued map $x \to \partial f(x)$ is lu^d-minimal.*

3 Applications of lu^d-minimality

In this section we give some applications of lu^d-minimality. We derive some properties of a minimal w*-cusco only on the base of such properties of an arbitrary selection of F.

Theorem 3.1 *Let X be a Banach space, A a nonempty open subset of X, $K > 0$, and $F : X \to 2^{X^*}$ a minimal w^*-cusco. If F possesses a densely defined selection s such that* $\operatorname{diam}(R(s)) = K$, *then* $\operatorname{diam}(R(F)) = K$.

Proof Clearly

$$\operatorname{diam}(R(F)) \geq K.$$

Now let us assume that $\operatorname{diam}(R(F)) > K$, i.e. there are $x, y \in A, x^* \in F(x), y^* \in F(y), \varepsilon > 0$, and $h \in S_X$ such that

$$\langle x^* - y^*, h\rangle > K + \varepsilon.$$

Hence

$$\langle -y^*, h\rangle > K + \varepsilon - \langle x^*, h\rangle \tag{5}$$

Let us consider two sequences $\{x_n\} \subset A \cap D(s)$ and $\{y_n\} \subset A \cap D(s)$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \langle s(x_n), h\rangle = \limsup_{z\to x} \langle s(z), h\rangle,$$

$$\lim_{n\to\infty} y_n = y \text{ and } \lim_{n\to\infty} \langle s(y_n), h\rangle = \liminf_{z\to y} \langle s(z), h\rangle.$$

From assumptions,

$$\langle s(x_n) - s(y_n), h\rangle \leq K.$$

Hence

$$\langle s(y_n), h\rangle \geq -K + \langle s(x_n), h\rangle. \tag{6}$$

Adding inequalities (5), (6) and letting $n \to \infty$ we derive

$$\begin{aligned}\liminf_{z\to y} \langle s(z), h\rangle - \langle y^*, h\rangle &\geq \limsup_{z\to x} \langle s(z), h\rangle - \langle x^*, h\rangle + \varepsilon \\ &> \limsup_{z\to x} \langle s(z), h\rangle - \langle x^*, h\rangle.\end{aligned}$$

Now consider the case when $\limsup_{z\to x} \langle s(z), h\rangle - \langle x^*, h\rangle < 0$ and define a selection t of F as follows

$$t(z) = \begin{cases} s(z) & \text{if } z \in D(s) - \{x\}, \\ x^* & \text{if } z = x. \end{cases}$$

Then

$$u(\langle s(\cdot), h\rangle)(x) < \langle x^*, h\rangle = u(\langle t(\cdot), h\rangle)(x),$$

which is a contradiction with Proposition 2.2. Finally assume that $\limsup_{z\to x} \langle s(z), h\rangle - \langle x^*, h\rangle \geq 0$, then $\liminf_{z\to y} \langle s(z), h\rangle - \langle y^*, h\rangle > 0$. Consider a selection t' of F given by

$$t'(z) = \begin{cases} s(z) & \text{if } z \in D(s) - \{y\}, \\ y^* & \text{if } z = y. \end{cases}$$

Then

$$l(\langle s(\cdot), h\rangle)(y) > \langle y^*, h\rangle = l(\langle t'(\cdot), h\rangle)(y),$$

which is again a contradiction with lu^d-minimality of F by Proposition 2.2. □

For the proof of Theorem 3.2 we use the following lemma.

Lemma 3.1 [11, Lemma 5.1.2] *Let X be a Banach space, A be an open nonempty subset of X, F be a w^*-cusco from A into subsets of X^*. Then F is locally bounded on A.*

Theorem 3.2 *Let X be a Banach space, A be a nonempty open subset of X and F be a set-valued map from A into subsets of X^* which is a minimal w^*-cusco. If F possesses a densely defined monotone selection, then F is monotone.*

Proof Assume that $F : A \to 2^{X^*}$ is a minimal w^*-cusco and that s is its densely defined monotone selection. Now let us assume that F is not monotone on A, i.e. there are elements $x, y \in A$ and $x^* \in F(x), y^* \in F(y)$ such that

$$\langle y^* - x^*, y - x \rangle < 0.$$

Then setting $h = y - x$, we see that there are $\varepsilon > 0$, and a ball $B(h, \delta)$ such that

$$\langle y^* - x^*, h' \rangle < -\varepsilon \qquad \forall h' \in B(h, \delta).$$

Let us consider two sequences $\{x_n\} \subset A \cap D(s)$ and $\{y_n\} \subset A \cap D(s)$ such that

$$\begin{aligned} \lim_{n\to\infty} x_n = x \quad &\text{and} \quad \lim_{n\to\infty} \langle s(x_n), h \rangle = \limsup_{z\to x} \langle s(z), h \rangle, \\ \lim_{n\to\infty} y_n = y \quad &\text{and} \quad \lim_{n\to\infty} \langle s(y_n), h \rangle = \liminf_{z\to y} \langle s(z), h \rangle. \end{aligned}$$

We observe that $h_n := y_n - x_n \to h$ and that for almost every $n \in \mathbf{N}$, it holds $h_n = y_n - x_n \in B(h, \delta)$. Thus from this and from monotonicity of s we derive

$$\begin{aligned} &\langle -y^*, h_n \rangle > \langle -x^*, h_n \rangle + \varepsilon, \quad \text{for almost every } n \in \mathbf{N}, \\ &\langle s(y_n), h_n \rangle \geq \langle s(x_n), h_n \rangle, \qquad \forall n \in \mathbf{N}. \end{aligned}$$

Adding up these last two inequalities we get for almost every $n \in \mathbf{N}$ that

$$\langle s(y_n) - y^*, h_n \rangle > \langle s(x_n) - x^*, h_n \rangle + \varepsilon.$$

From this it follows that for almost every $n \in \mathbf{N}$, it holds

$$\begin{aligned} &\langle s(y_n) - y^*, h_n - h \rangle + \langle s(y_n), h \rangle - \langle y^*, h \rangle \\ &> \langle s(x_n) - x^*, h_n - h \rangle + \langle s(x_n), h \rangle - \langle x^*, h \rangle + \varepsilon. \end{aligned}$$

Letting $n \to \infty$ and using Lemma 3.1 we derive

$$\begin{aligned} \liminf_{z\to y} \langle s(z), h \rangle - \langle y^*, h \rangle &\geq \limsup_{z\to x} \langle s(z), h \rangle - \langle x^*, h \rangle + \varepsilon \\ &> \limsup_{z\to x} \langle s(z), h \rangle - \langle x^*, h \rangle. \end{aligned}$$

Now consider the case when $\limsup_{z\to x} \langle s(z), h \rangle - \langle x^*, h \rangle < 0$ and define a selection t of F as follows

$$t(z) = \begin{cases} s(z) & \text{if } z \in D(s) - \{x\}, \\ x^* & \text{if } z = x. \end{cases}$$

Then

$$u(\langle s(\cdot), h\rangle)(x) < \langle x^*, h\rangle = u(\langle t(\cdot), h\rangle)(x),$$

which is a contradiction with Proposition 2.2. Finally assume that

$$\limsup_{z \to x} \langle s(z), h\rangle - \langle x^*, h\rangle \geq 0,$$

then

$$\liminf_{z \to y} \langle s(z), h\rangle - \langle y^*, h\rangle > 0.$$

Comsider a selection t' of F given by

$$t'(z) = \begin{cases} s(z) & \text{if } z \in D(s) - \{y\}, \\ y^* & \text{if } z = y. \end{cases}$$

Then

$$l(\langle s(\cdot), h\rangle)(y) > \langle y^*, h\rangle = l(\langle t'(\cdot), h\rangle)(y),$$

which is again a contradiction with lu^d-minimality of F by Proposition 2.2. □

Corollary 3.1 *Let X be a Banach space, A be an open nonempty subset of X, F be a set-valued map from A into subsets of X^* which possesses densely defined monotone selection. Then the following are equivalent,*

(i) F is a maximal monotone map,

(ii) F is a minimal w^*-cusco.

Proof The implication (i) $\Rightarrow$ (ii) follows from [10, Theorem 7.9]. The converse is a consequence of Theorem 3.2 and [10, Lemma 7.7]. □

Corollary 3.2 *Let X be a Banach space, A be a nonempty open convex subset of X, f be a locally Lipschitz function such that its Clarke subdifferential is a minimal w^*-cusco. Then f is convex on A if and only if the map $x \to \partial f(x)$ possesses a densely defined monotone selection.*

Proof The proof follows immediately from Theorem 3.2 and Proposition 1.2. □

We will provide still another one-dimensional example.

Example 3.1 Let I be an open interval in $\mathbf{R}$ and $F : I \to 2^{\mathbf{R}}$ be a minimal cusco map on I given by $F(x) = [\alpha(x), \beta(x)]$. If we suppose that there is a densely defined and non-decreasing function s on I satisfying $\alpha(x) \leq s(x) \leq \beta(x)$ for every $x \in D(s)$, then F can be represented as a subdifferential of some continuous convex function on I.

Proof It suffices to use Corollary 3.1 and the fact that each maximal monotone map on the real line is subdifferential of some lsc convex function. □

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