Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 35 (1996), No. 1, 7--20

Persistent URL: http://dml.cz/dmlcz/120336

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica 35 (1996) 7–20

On the Method of Esclangon

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(Received January 31, 1996)

Abstract

The effective asymptotic estimates of derivatives of solutions to dissipative nonhomogeneous linear ordinary differential equations with constant coefficients are shown to be available by means of the technique due to E. Esclangon [E]. Establishing this procedure, we compare the appropriate results with those obtained by different methods.

Key words: Esclangon's method, asymptotic estimates, nonhomogeneous equations, comparison of results.

1991 Mathematics Subject Classification: 34A30, 34D05, 34D40.

1 Introduction

In 1915, E. Esclangon published the well-know theorem (see e.g. [E], [L], [KBK]) for the linear ordinary differential equations with constant coefficients and a bounded (on the half-line) continuous nonhomogenity. This theorem says that the boundedness of solutions implies the same for their derivatives up to the order of the given equation. As we will show, the basic idea of the proof can be used, under the slight modification, as a method for the asymptotic estimates of such derivatives, provided additionally that the associated characteristic polynomial is asymptotically stable.

Under these assumptions, all the solutions of the nth-order equations, as well as their derivatives up to the nth-order, are known (see e.g. [A1], [AT],

*Supported by grant No. 311 03 001 of Palacký University, Olomouc.

[BVGN]) to be uniformly ultimately bounded by the common constants. Hence, the problem consists in the estimation of these constants; for the related results see e.g. [A1], [AT], [AV] and the references therein.

Although the obtained estimates here will be shown better than their known analogies in particular situations, they are suitable only for lower-order equations. The reason consists in a cumbersome calculation of the appropriate recurrent formulas.

Our paper is organized as follows. In Part II, Esclangon's method is presented for a general nth-order equation. Part III is devoted to its application for $n \leq 5$. The comparison with the known analogies is done in Part IV. The last Part V consists of the concludings remarks. Two supplementary sections are added not to break the context.

2 Esclangon's method

Consider the equation

$$x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} = p(t) \tag{1}$$

with positive constant coeficients a_j, $j = 1, \ldots, n$, where $p(t)$ is a continuous function on the positive half-line, by which all solutions of (1), as well as their derivatives up to the nth-order, exist for all future times (see e.g. [C]).

Assume, furthermore, that the associated characteristic polynomial, namely

$$\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}, \tag{2}$$

is asymptotically stable, i.e. $\operatorname{Re}\lambda_j < 0$, $j = 1, \ldots, n$, where λ_j are the roots of (2). This is well-known to be expressed explicitly in terms of coefficients by means of the necessary and sufficient conditions of the Routh–Hurwitz type (see e.g. [C]).

At last, let a positive constant P exist such that

$$\limsup_{t\to\infty} |p(t)| \leq P. \tag{3}$$

Under the above assumptions, all solutions of (1) as well as their derivatives up to the nth-order are uniformly ultimately bounded (see e.g. [BVGN]). We also know (see [AT]) that every solution $x(t)$ satisfies

$$\limsup_{t\to\infty} |x(t)| \leq \frac{P}{a_n}. \tag{4}$$

Hence, let $x(t)$ be a solution of (1). The following identity obviously takes place for an arbitrary positive number α_1:

$$\frac{d}{dt} e^{-\alpha_1 t}[x^{(n-1)}(t) + \lambda_{11} x^{(n-2)}(t) + \ldots + \lambda_{1,n-1}\dot{x}(t)] = \tag{5}$$
$$= e^{-\alpha_1 t}[x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_n x(t) + u_1(t)],$$

where

$$\lambda_{11} = \alpha_1 + a_1,$$
$$\lambda_{1,i+1} = \lambda_{1,i}\alpha_1 + a_{i+1} \qquad \text{for } i = 1, \dots, n-2,$$
$$u_1(t) = -(\alpha_1\lambda_{1,n-1} + a_n)x(t).$$

Because of (4) and $\lambda_{1,n-1}$ being a constant, $u_1(t)$ is bounded as well. Therefore, integrating (5) from T to ∞, we get

$$-e^{-\alpha_1 T}[x^{(n-1)}(T) + \lambda_{11}x^{(n-2)}(T) + \ldots + \lambda_{1,n-1}x(T)] = \int_T^\infty e^{-\alpha_1 t}[p(t) + u_1(t)]\,dt\,, \tag{6}$$

when using the identity $x^{(n)}(t) + \sum_{j=1}^n a_j x^{(n-j)}(t) = p(t)$ and the fact that $e^{-\alpha_1 t}x^{(k)}(t)$ vanishes at infinity for $k = 0, 1, \dots, n-1$.

Applying the well-known second mean value theorem to the right-hand side of (6), we arrive at

$$-e^{-\alpha_1 T}[x^{(n-1)}(T) + \lambda_{11}x^{(n-2)}(T) + \cdots + \lambda_{1,n-1}x(T)] = \frac{e^{-\alpha_1 T}}{\alpha_1}[p(\xi_1) + u_1(\xi_1)], \qquad \text{where } \xi_1 \in (T, \infty).$$

Multiplying the last relation by $e^{\alpha_1 t}$, we obtain

$$x^{(n-1)}(T) + \lambda_{11}x^{(n-2)}(T) + \ldots + \lambda_{1,n-1}x(T) = -\frac{p(\xi_1) + u_1(\xi_1)}{\alpha_1}.$$

Since this equation holds for each sufficiently big T, we can rewrite it into the form

$$x^{(n-1)} + \lambda_{11}x^{(n-2)} + \cdots + \lambda_{1,n-1}x = p_1(t)\,,$$

where $p_1(t) = -\frac{u_1(\xi_1(t)) + p(\xi_1(t))}{\alpha_1}$.

Now, repeating the same manner as above to this equation, we can get the equation of the $(n-2)$th order. Hence, starting with the identity

$$\frac{d}{dt}e^{-\alpha_2 t}[x^{(n-2)}(t) + \lambda_{21}x^{(n-3)}(t) + \cdots + \lambda_{2,n-2}x(t)] = e^{-\alpha_2 t}[x^{(n-1)}(t) + \lambda_{11}x^{(n-2)}(t) + \cdots + \lambda_{1,n-1}x(t) + u_2(t)],$$

where

$$\lambda_{21} = \alpha_2 + \lambda_{11},$$
$$\lambda_{2,i+1} = \lambda_{2,i}\alpha_2 + \lambda_{1,i+1} \qquad \text{for } i = 1, \dots, n-3,$$
$$u_2(t) = -(\alpha_2\lambda_{2,n-2} + \lambda_{1,n-1})x(t),$$

we come to

$$-e^{-\alpha_2 T}[x^{(n-2)}(T) + \lambda_{21}x^{(n-3)}(T) + \cdots + \lambda_{2,n-2}x(T)] = \frac{e^{-\alpha_2 T}}{\alpha_2}[p_1(\xi_2) + u_2(\xi_2)], \qquad \text{where } \xi_2 \in (T, \infty).$$

Thus, the desired equation reads

$$x^{(n-2)} + \ldots + \lambda_{2,n-2}x = p_2(t)\,,$$

where $p_2(t) = -\frac{u_2(\xi_2(t))+p_1(\xi_2(t))}{\alpha_2}$.

Proceeding analogously, we receive after $(n-1)$ steps the equation

$$x' + \lambda_{n-1,1}x = p_{n-1}(t)\,,$$

where $p_{n-1}(t) = -\frac{u_{n-1}(\xi_{n-1}(t))+p_{n-2}(\xi_{n-1}(t))}{\alpha_{n-1}}$.

Applying (4), we have therefore

$$\limsup_{t\to\infty} |x'(t)| \le \limsup_{t\to\infty} |p_{n-1}(t)| + \lambda_{n-1,1}\frac{P}{a_n}\,,$$

where $p_{n-1}(t)$ and $\lambda_{n-1,1}$ can be derived recurrently from the above formulas. One can also obtain recurrently the asymptotic estimates for $|x^{(l)}(t)|$, where $l = 2,\ldots,n-1$, when coming back to the higher-order equations.

The above procedure can be described in form of the following algorithm, when considering the coeficients a_j as a vector $(a_1,\ldots,a_n)$ and λ_{ij} as a matrix (a_{ij}).

0th step:

$$\lambda_{0,i} := a_i, \quad i = 1,\ldots,n,$$

$$p_0 := P, \qquad y_0 := \frac{P}{a_n},$$

1st step:

FOR i FROM 1 TO $n-1$ DO

$\lambda_{i,1} := \lambda_{i-1,1} + \alpha_i,$

FOR j FROM 2 TO $n-i$ DO $\lambda_{i,j} := \lambda_{i,j-1} * \alpha_i + \lambda_{i-1,j}$ OD,

$u_i := (\alpha_i * \lambda_{i,n-i} + \lambda_{i-1,n-i+1}) * y_0,$

$p_i := \frac{(p_{i-1}+u_i)}{\alpha_i}$

OD,

2nd step:

FOR i FROM 1 TO n DO

$y_i := p_{n-i},$

FOR j FROM 1 TO i DO $y_i := y_i + \lambda_{n-i,j} * y_{i-j}$ OD,

OD,

In the vector $(y_1, \ldots, y_n)$, y_k denotes the asymptotic estimate of $|x^{(k)}(t)|$, $k = 1, \ldots, n$, obtained by Esclangon's method.

Since all such estimates depend on the real parameters $\alpha_1, \ldots, \alpha_{n-1}$, it is very useful to give

Lemma 1 *Each component $y_k(\alpha_1, \ldots, \alpha_{n-1})$ $k = 1, \ldots, n$, in the above vector attains its minimum on $(R^+)^{n-1}$.*

Proof Since all the functions $y_k(\alpha_1, \ldots, \alpha_{n-1})$, $k = 1, \ldots, n$, are obviously continuous on $(R^+)^{n-1}$, it is enough to show that the values of each y_k are outside some closed parallepiped bigger than the value at a certain point inside it. This is because of the well-known Weierstrass theorem implying the global maximum on the compact set.

Hence, let us construct such a parallepiped and find the desired internal point. For y_1, one could see that the term $\frac{P}{\Pi_{j=1}^{n-1}\alpha_j}$ is involved in p_{n-1} and that

$$\lambda_{n-1,1} = a_1 + \sum_{j=1}^{n-1} \alpha_j ,$$

which yields

$$P\left(\frac{1}{\Pi_{j=1}^{n-1}\alpha_j} + \sum_{j=1}^{n-1} \alpha_j\right) < y_1 .$$

Since $\lambda_{n-k,1}y^{(n-k-1)}$ belongs to y_k, while $\lambda_{n-k,1}$ includes a_1, we have furthermore

$$P\left(\frac{1}{\Pi_{j=1}^{n-1}\alpha_j} + \sum_{j=1}^{n-1} \alpha_j\right) a_1^{k-1} < y_k , \quad k = 1, \ldots, n .$$

Taking

$$c := \min_{k=1,\ldots,n} P a_1^{k-1} = P \min(1, a_1^{n-1}) ,$$

we get

$$c\left(\frac{1}{\Pi_{j=1}^{n-1}\alpha_j} + \sum_{j=1}^{n-1} \alpha_j\right) < y_k , \quad k = 1, \ldots, n ,$$

which leads for

$$f := \max_{k=1,\ldots,n} y_k(1, \ldots, 1)$$

to the inequality $f > c$, i.e. $q := \frac{f}{c} > 1$.

Defining the parallepiped as follows: $\left((\frac{1}{q})^{n-1}, q\right)^{n-1}$, we will show that for each external point we arrive at

$$y_k > f, \quad k = 1, \ldots, n-1 .$$

This is indeed true because of the two following implications:

(i) $\exists i \in \{1,\ldots,n-1\} : \alpha_i > q \Rightarrow y_k > \alpha_i c > qc = f,$

(ii) $\forall i \in \{1,\ldots,n-1\} : \alpha_i \le q, \quad \exists j \in \{1,\ldots,n-1\} :$

$$\alpha_j < \left(\frac{1}{q}\right)^{n-1} \Rightarrow y_k > \frac{c}{q^{n-2}(\frac{1}{q})^{n-1}} = f,$$

which completes the proof.

3 Applications for $n \le 5$

Although Lemma 1 affirms the solvability of the minimum problem in $(R^+)^{n-1}$, it is rather difficult even for $n \le 5$. Letting $\alpha := \alpha_1 = \ldots = \alpha_{n-1}$ $(n \le 5)$, the optimality question for α has still some meaning. Since α is positive, $y_k \in C^\infty(R^+)$, for $k = 1,\ldots,n$. Thus, the problem is related to finding the critical points of $y_k(\alpha)$ on $(0,\infty)$.

Below, we introduce at first all the possible cases solvable analytically, where $x(t)$ denotes again the solution of nth-order equation (1); observe that $n \le 4$.

$n = 2$:

$$\limsup_{t\to\infty} |x'(t)| \le 2P\left(\frac{2}{\sqrt{a_2}} + \frac{a_1}{a_2}\right).$$

$n = 3$:

$$\limsup_{t\to\infty} |x'(t)| \le \left(\left(P + \frac{(\alpha,(a_2+\alpha,(a_1+\alpha))+a_3)P}{a_3}\right)\alpha^{-1} + \right.$$
$$\left. + \frac{(\alpha,(a_1+2,\alpha)+a_2+\alpha,(a_1+\alpha))P}{a_3}\right)\alpha^{-1} + \frac{(a_1+2,\alpha)P}{a_3},$$

where

$$\alpha = \frac{\left(9,a_3+\sqrt{-a_2^3+81,a_3^2}\right)^{\frac{2}{3}} + a_2}{3,\sqrt[3]{9,a_3+\sqrt{-a_2^3+81,a_3^2}}},$$

$$\limsup_{t\to\infty} |x''(t)| \le \frac{P + \frac{(\alpha(a_2+\alpha(a_1+\alpha))+a_3)P}{a_3}}{\alpha} +$$
$$+(a_1+\alpha)\left(\left(\frac{P + \frac{\alpha(a_2+(\alpha(a_1+\alpha))+a_3)P}{a_3}}{\alpha} + \frac{(\alpha(a_1+2\alpha)+a_2+\alpha(a_1+\alpha))P}{a_3}\right)\alpha^{-1} + \right.$$
$$\left. + \frac{(a_1+2\alpha)P}{a_3}\right) + \frac{(a_2+\alpha(a_1+\alpha))P}{a_3},$$

where

$$\alpha = -\frac{3}{16}a_1 + \frac{1}{48}\sqrt{3}\sqrt{B}\frac{1}{48}\Big(-(-162a_1^2\sqrt[3]{A}\sqrt{B} + 24\sqrt{B}A^{\frac{2}{3}} + 432\sqrt{B}a_1^2a_2-$$

$$-3744\sqrt{B}a_1a_3 - 1152\sqrt{3}\sqrt[3]{A}a_1a_2 - 2304\sqrt{3}\sqrt[3]{A}a_3 + 486\sqrt{3}\sqrt[3]{A}a_1^3)/(\sqrt[3]{A}\sqrt{B})\Big)^{\frac{1}{2}},$$

$$\begin{aligned} A &= 108a_1^2a_2^2 + 432a_1a_2a_3 + 432a_3^2 - 972a_1^3a_3 + \\ &\quad + 6\big(-162a_1^6a_23 - 1620a_1^5a_2^2a_3 - 59832a_1^4a_2a_3^2 + 82128a_1^3a_3^3 + 324a_1^4a_2^4 + \\ &\quad + 2592a_1^3a_2^3a_3 + 7776a_1^2a_2^2a_3^2 + 10368a_1a_2a_3^3 + 5184a_3^4 + 26244a_1^6a_3^2\big)^{\frac{1}{2}}, \\ B &= \frac{27a_1^2\sqrt[3]{A} + 8A^{\frac{2}{3}} + 144a_1^2a_2 - 1248a_1a_3}{\sqrt[3]{A}}. \end{aligned}$$

***n* = 4**:

$$\limsup_{t\to\infty}|x'(t)| \le$$

$$\le \Bigg(\bigg(\frac{P + \frac{(\alpha(a_3+\alpha(a_2+\alpha(a_1+\alpha)))+a_4)P}{a_4}}{\alpha} + \frac{(a_2+\alpha(a_1+\alpha)+\alpha(a_1+2\alpha))P}{a_4}\bigg)/\alpha+$$

$$+((a_1+3\alpha)\alpha + a_2 + \alpha(a_1+\alpha) + \alpha(a_1+2\alpha))P/a_4\Bigg)/\alpha + \frac{(a_1+3\alpha)P}{a_4},$$

where

$$\begin{aligned} \alpha &= \frac{1}{60}\sqrt{30}\sqrt{B} + \frac{1}{60}\sqrt[4]{30}\Big(-(-8a_2\sqrt[3]{A}\sqrt{30}\sqrt{B} - 8a_1\sqrt[3]{A}\sqrt{30}\sqrt{B} + \\ &\quad + \sqrt{30}\sqrt{B}A^{\frac{2}{3}} - 720\sqrt{30}\sqrt{B}a_4 + 4\sqrt{30}\sqrt{B}a_2^2 + 8\sqrt{30}\sqrt{B}a_2a_1 + \\ &\quad + 4\sqrt{30}\sqrt{B}a_1^2 - 360\sqrt[3]{A}a_3 - 360a_2\sqrt[3]{A})/(\sqrt[3]{A}\sqrt{B})\Big)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} A &= -4320a_4a_2 - 4320a_4a_1 + 540a_3^2 + 1080a_3a_2 + 540a_2^2 - 8a_2^3 - 24a_2^2a_1 - \\ &\quad - 24a_2a_1^2 - 8a_1^3 + 12\Big(172800a_4^2a_2a_1 + 2880a_4a_2^2a_1 + 4320a_4a_2^2a_12 + \\ &\quad + 2880a_4a_2a_1^3 - 32400a_4a_2a_3^2 - 64800a_4a_2^2a_3 - 32400a_4a_1a_3^2 - \\ &\quad - 32400a_4a_1a_2^2 - 180a_3^2a_2^2a_1 - 180a_3^2a_2a_1^2 - 360a_3a_2^3a_1 - 360a_3a_2^2a_1^2 - \\ &\quad - 120a_3a_2a_1^3 + 86400a_4^2a_2^2 + 86400a_4^2a_12 + 720a_4a_2^4 + 720a_4a_1^4 + \\ &\quad + 8100a_3^3a_2 + 12150a_3^2 - a_2^2 - 60a_3^2a_23 - 60a_3^2a_13 + 8100a_3a_2^3 - \\ &\quad - 120a_3a_2^4 + 2025a_2^4 - 32400a_4a_2^3 + 2592000a_4^3 + 2025a_3^4 - 60a_2^5 - \\ &\quad - 64800a_4a_1a_3a_2 - 180a_2^4a_1 - 180a_2^4a_1 - 180a_2^3a_1^2 - 60a_2^2a_1^3\Big)^{\frac{1}{2}}, \\ B &= \Big(4a_2\sqrt[3]{A} + 4a_1\sqrt[3]{A} + A^{\frac{2}{3}} - 720a_4 + 4a_2^2 + 8a_2a_1 + 4a_1^2\Big)/\sqrt[3]{A}. \end{aligned}$$

Now, let us complete the remaining estimates for $n = 4, 5$, when putting $\alpha = 1$.

$n = 4$:

$$\limsup_{t\to\infty}|x''(t)| \le \frac{2(3a_4+18a_1+18+6a_2+3a_3+a_1a_4+4a_1^2+2a_1a_2+a_1a_3)}{a_4}P,$$

$$\begin{aligned}\limsup_{t\to\infty}|x'''(t)| \le\ & 2(5a_4+48a_1+16a_2+5a_3+26a_1^2+a_1^2a_4+4a_1^3+a_2a_4+2a_2^2+\\ &+26+2a_1^2a_2+a_1^2a_3+5a_1a_4+14a_1a_2+5a_1a_3+a_2a_3)P/a_4\ .\end{aligned}$$

$n = 5$:

$$\limsup_{t\to\infty}|x'(t)| \le \frac{2(a_5+8a_1+15+4a_2+2a_3+a_4)}{a_5}P\ ,$$

$$\begin{aligned}\limsup_{t\to\infty}|x''(t)| \le\ & 2(4a_5+46a_1+56+16a_2+8a_3+4a_4+\\ &+a_1a_5+8a_1^2+4a_1a_2+2a_1a_3+a_1a_4)P/a_5\ ,\end{aligned}$$

$$\begin{aligned}\limsup_{t\to\infty}|x'''(t)| \le\ & 2(4a_1^2a_2+12a_4+12a_5+2a_1^2a_3+a_1^2a_4+8a_1a_5+40a_1a_2+\\ &+16a_1a_3+8a_1a_4+2a_2a_3+a_2a_4+162+206a_1+62a_2+\\ &+24a_3+78a_1^2+a_1^2a_5+8a_1^3+a_2a_5+4a_2^2)P/a_5\ ,\end{aligned}$$

$$\begin{aligned}\limsup_{t\to\infty}|x^{(IV)}(t)| \le\ & 2(18a_5+4a_1^3a_2+2a_1^3a_3+a_1^3a_4+10a_1^2a_5+56a_1^2a_2+\\ &+20a_1^2a_3+10a_1^2a_4+16a_2a_3+6a_2a_4+26a_1a_4+26a_1a_5+\\ &+180a_1a_2+60a_1a_3+2a_1a_2a_5+8a_1a_2^2+6a_2a_5+a_3a_4+24a_2^2+\\ &+a_3a_5+2a_3^2+4a_1a_2a_3+2a_1a_2a_4+494a_1+154a_2+50a_3+\\ &+18a_4+346a_1^2+94a_1^3+a_1^3a_5+8a_1^4+234)P/a_5\ .\end{aligned}$$

4 Comparison with the analogies

The following analogical theorems have been obtained, under the above assumptions, for the asymptotic estimates of solutions of (1) and thier derivatives up to the $(n-1)$th order.

Theorem 1 [A1] *Every solution $x(t)$ of (1) satisfies*

$$\limsup_{t\to\infty}\sum_{i=0}^{n-1}|x^{(i)}(t)| \le P\sum_{i=0}^{n-1}\frac{2^i\|A\|^i}{\hat{\lambda}^{i+1}} \tag{7}$$

with $\|A\| = \max(1+a_1,\ldots,1+a_{n-1},a_n)$ and $\hat{\lambda} = \min_{j=1,\ldots,n}|Re\lambda_j|$, where λ_j are the roots of (2).

Theorem 2 [AT] *Every solution $x(t)$ of (1) safistfies*

$$\limsup_{t\to\infty} |x^{(k)}(t)| \le \frac{2^k P}{a_n}\Lambda^k, \quad k = 0, 1, \dots, n-1, \tag{8}$$

with $\Lambda = \max_{j=1,\dots,n} |\lambda_j|$, where λ_j are the roots of (2).

Remark 1 The spectral radius Λ in (8) satisfies (see [P, pp. 30–33]) $\Lambda \le \min[\max(a_1+1, \dots, a_{n-1}+1, a_n), \max(1, a_1+\dots+a_n), \max(a_1, \frac{a_2}{a_1}, \dots, \frac{a_n}{a_{n-1}})]$.

Theorem 3 [AT] *Assume additionally that all the roots λ_j, $j = 1, \dots, n$, of (2) are real (and subsequently negative). Then every solution $x(t)$ of (1) satisfies ($a_0 = 1$)*

$$\limsup_{t\to\infty} |x^{(k)}(t)| \le \frac{2^k a_k P}{\binom{n}{k} a_n}, \quad k = 0, 1, \dots, n-1. \tag{9}$$

Theorem 4 [AT] *For $n \ge 5$, assume additionally that the coefficients a_j in the polynomials*

$$\lambda^{n-p} + \sum_{j=1}^{n-p} a_j \lambda^{n-j-p}, \qquad \textit{where } p = 1, \dots, n-4, \tag{10}$$

obey successively the Routh–Hurwitz conditions. Then every solution $x(t)$ of (1) satisfies

$$\limsup_{t\to\infty} |x^{(k)}(t)| \le \frac{2^k P}{a_{n-k}}, \quad k = 0, 1, \dots, n-1. \tag{11}$$

Lemma 2 [AT] *If all the roots of (2) are negative, then (cf. (9) and (11) ($a_0 = 1$))*

$$\frac{2^k P}{a_{n-k}} \le \frac{2^k a_k P}{\binom{n}{k} a_n} \qquad \textit{for } k = 0, 1, \dots, n-1.$$

In view of Lemma 2, Theorem 3 becomes actual only for $n \ge 5$, because then the Routh–Hurwitz structure of coefficients in polynomials (10) is not anymore invariant, in general (see Appendix I and [A2]), under the "shift" for $p = 1, \dots, n-4$.

Lemma 3 [K] *The whole family of necessary and sufficient conditions for the negativity of all the roots of the polynomial*

$$\lambda^5 + \sum_{j=1}^{5} a_j \lambda^{5-j}$$

reads as follows:

$$\begin{aligned} &a_1 a_4 - 25 a_5 \ge 0, \;\; 4a_1^2 - 10 a_2 \ge 0, \\ &A_0 \ge 0, \;\; A_2 \ge 0, \;\; B_0 \ge 0, \;\; B_1 \ge 0, \;\; C_0 \ge 0, \end{aligned} \tag{12}$$

where the constants A_0, A_2, B_0, B_1, C_0 are defined in Appendix II.

For higher-degree polynomials, the situation becomes much worse.

Theorem 5 [C, Chapter II, Th. 3.11.1] *Assuming additionally that*

$$\lim_{t\to\infty} p(t) = P\,,$$

every solution $x(t)$ of (1) satisfies

$$\lim_{t\to\infty} x(t) = \frac{P}{a_n} \quad \textit{and} \quad \lim_{t\to\infty} x^{(l)}(t) = 0 \qquad \textit{for } l = 1, \dots, n-1.$$

Remark 2 In view of Theorem 5, the estimates (8), (9) and (11) are sharpest for $k = 0$.

Now, let us demonstrate the power of the foregoing theorems in two examples.

Example 1 Consider

$$x^{(V)} + 15x^{(IV)} + 85x''' + 225x'' + 274x' + 120x = p(t)\,, \tag{13}$$

where $p(t)$ fulfils (3). The roots of the associated characteristic polynomial are: $-1, -2, -3, -4, -5$. One can check (see Appendix I) that the additional assumptions of Theorem 4 are satisfied.

Denoting the constants estimating the kth-order derivatives of solutions to (1) by D_k, $k = 0, 1, \dots, 4$, respectively, we have the following table:

without factor P	D_0	D_1	D_2	D_3	D_4
(11) in Th. 4	$\frac{1}{120}$	0.00730	0.01778	0.09412	1.06667
(9) in Th. 3	$\frac{1}{120}$	0.05000	0.28333	1.50000	7.30667
(8) in Th. 2	$\frac{1}{120}$	0.08333	0.83333	8.33333	83.33333
Chapter 3 *)	$\frac{1}{120}$	4.6748	149.84	4340.9	100682
(7) in Th. 1	$D_0 + D_1 + D_2 + D_3 + D_4 = 9.1673\ 10^{10}$				

*) $\alpha := \alpha_1 = \ldots = \alpha_4$ **has been optimalized numerically**

Although the results obtained by means of the Esclangon method are for (13) the second worst,the next example says something different.

Example 2 Consider

$$x'' + 0.2x' + 9.01x = p(t)\,, \tag{14}$$

where $p(t)$ again fulfils (3). The roots of the associated characteristic polynomial are: $-0.1 + 3i$, $-0.1 - 3i$. The following table shows that the Esclangon method gives here the second best estimates.

without factor P	D_0	D_1
(11) in Th. 4	0.11098	10
(8) in Th. 2	0.11098	0.66962
Chapter 3	0.11098	1.37699
(7) in Th. 1	$D_0 + D_1 = 1812$	

Without an explicit knowlege of the spectral radius Λ (i.e. when applying the inequalities in Remark 1), the result obtained by means of Esclangon's method is, however, the best of all.

5 Conclusion

In spite of difficulties related to applications of Esclangon's method, we could see that it can give comparatively very good estimates, especially at presence of complex roots of (2). The algorithm presented in the second chapter can be employed numerically in general, when putting e.g. (as in the original paper [E]) $\alpha_1 := \ldots = \alpha_{n-1} = 1$. Thus, we have to our disposal at least the complementary tool to those introduced in form of theorems in Part IV.

Appendix I

The Routh–Hurwitz conditions for the coefficients of (2) take the following form, when $n = 3, 4, 5$.

$n = 3$:

$$a_1a_2 - a_3 > 0\,,$$

$n = 4$:

$$a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0\,,$$

$n = 5$:

$$a_3a_4 - a_2a_5 > 0, \qquad a_4(a_2a_3 + a_5 - a_1a_4) - a_2^2a_5 > 0,$$

$$a_4(a_1a_2a_3 + a_1a_5 - a_1^2a_4 - a_3^2) - a_5(a_1a_2^2 - a_2a_3 + a_5 - a_1a_4) > 0.$$

One can therefore easily check that the asymptotic stability of the associated characterictic polynomial to (13) in Example 1 implies the same for the "shifted" one, namely

$$\lambda^4 + 15\lambda^3 + 85\lambda^2 + 225\lambda + 274\,. \tag{15}$$

Indeed. The appropriate inequalities for (15) read $15015 > 0$ and $174600 > 0$.

One the other hand, the asymptotic stability of, for example, the polynomial

$$\lambda^5 + \lambda^4 + 4\lambda^3 + 3\lambda^2 + 3.5\lambda + 1 \tag{16}$$

does not imply the same for

$$\lambda^4 + \lambda^3 + 4\lambda^2 + 3\lambda + 3.5\,. \tag{17}$$

Indeed. The appropriate inequalities for (16) are

$$6.5 > 0,\ 17.25 > 0,\ 0.25 > 0,$$

while the one for (17), $-0.5 > 0$, is false.

For $n \leq 4$, the Routh–Hurwitz structure of coefficients in polynomials (10) can be shown invariant under the "shift" for $p = 1, \ldots, n-1$ (see [A2]).

Appendix II

For n=5, the negativity (and so reality) of all the roots of the characteristic polynomial (2) can be obtained (see e.g. (12) in Lemma 3), on the basis of the well-known Sturm theorem (see e.g. [HM]), by means of the following Sturmian functions:

$$\begin{aligned}
F_0(\lambda) &= \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5\,,\\
F_1(\lambda) &= 5\lambda^4 + 4a_1\lambda^3 + 3a_2\lambda^2 + 2a_3\lambda + a_4\,,\\
F_2(\lambda) &= (4a_1^2 - 10a_2)\lambda^3 + (3a_1a_2 - 15a_3)\lambda^2 + (2a_1a_3 - 20a_4)\lambda +\\
&\quad + a_1a_4 - 25a_5\,,\\
F_3(\lambda) &= A_0\lambda^2 + A_1\lambda + A_2\,,\\
F_4(\lambda) &= B_0\lambda + B_1\,,\\
F_5(\lambda) &= C_0\,,
\end{aligned}$$

where

$$A_0 = 3a_1^2a_2^2 - 12a_2^3 - 8a_1^3a_3 + 38a_1a_2a_3 - 45a_3^2 - 16a_1^2a_4 + 40a_2a_4\,,$$

$$\begin{aligned}
A_1 = {} & 2a_1^2a_2a_3 - 8a_2^{2a_3} + 6a_1a_3^2 - 12a_1^3a_4 + 42a_1a_2a_4 - 60a_3a_4 -\\
& - 20a_1^2a_5 + 50a_2a_5\,,
\end{aligned}$$

$$A_2 = a_1^2a_2a_4 - 4a_2^2a_4 + 3a_1a_3a_4 - 16a_1^{3a_5} + 55a_1a_2a_5 - 75a_3a_5\,,$$

$$\begin{aligned}
B_0 = {} & 2(-2a_1^2 + 5a_2)^2(a_1^2a_2^2a_3^2 - 4a_2^3a_3^2 - 4a_1^3a_3^3 + 18a_1a_2a_3^3 - 27a_3^4 -\\
& - 3a_1^2a_2^3a_4 + 12a_2^4a_4 + 14a_1^3a_2a_3a_4 - 62a_1a_2^2a_3a_4 - 6a_1^2a_3^2a_4 +\\
& + 117a_2a_3^2a_4 - 18a_1^4a_4^2 + 97a_1^2a_2a_4^2 - 88a_2^2a_4^2 - 132a_1a_3a_4^2 +\\
& + 160a_4^3 - 66a_1^2a_2a_3a_5 - 40a_2^2a_3a_5 + 120a_1a_3^2a_5 - 28a_1^3a_4a_5 +\\
& + 130a_1a_2a_4a_5 - 300a_3a_4a_5 - 50a_1^2a_5^2 + 125a_2a_5^2)\,,
\end{aligned}$$

$$
\begin{aligned}
B_1 = {} & (-2a_1^2 + 5a_2)^2 (a_1^2 a_2^2 a_3 a_4 - 4a_2^3 a_3 a_4 - 4a_1^3 a_3^2 a_4 + 18a_1 a_2 a_3^2 a_4 - 27a_3^3 a_4 + \\
& + 3a_1^3 a_2 a_4^2 - 12a_1 a_2^2 a_4^2 - 7a_1^2 a_3 a_4^2 + 48a_2 a_3 a_4^2 - 16a_1 a_4^3 - 9a_1^2 a_2^3 a_5 + \\
& + 36a_2^4 a_5 + 32a_1^3 a_2 a_3 a_5 - 146a_1 a_2^2 a_3 a_5 + 4a_1^2 a_3^2 a_5 + 195a_2 a_3^2 a_5 - \\
& - 48a_1^4 a_4 a_5 + 266a_1^2 a_2 a_4 a_5 - 260a_2^2 a_4 a_5 - 290a_1 a_3 a_4 a_5 + 400a_4^2 a_5 - \\
& - 80a_1^3 a_5^2 + 275a_1 a_2 a_5^2 - 375a_3 a_5^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
C_0 = {} & (-2a_1^2 + 5a_2)^4 (3a_1^2 a_2^2 - 12a_2^3 - 8a_1^3 a_3 + 38a_1 a_2 a_3 - 45a_3^2 - \\
& - 16a_1^2 a_4 + 40a_2 a_4)^2 (a_1^2 a_2^2 a_3^2 a_4^2 - 4a_2^3 a_3^2 a_4^2 - 4a_1^3 a_3^3 a_4^2 + 18a_1 a_2 a_3^3 a_4^2 - \\
& - 27a_3^4 a_4^2 - 4a_1^2 a_2^3 a_4^3 + 16a_2^4 a_4^3 + 18a_1^3 a_2 a_3 a_4^3 - 80a_1 a_2^2 a_3 a_4^3 - 6a_1^2 a_3^2 a_4^3 + \\
& + 144a_2 a_3^2 a_4^3 - 27a_1^4 a_4^4 + 144a_1^2 a_2 a_4^4 - 128a_2^2 a_4^4 - 192a_1 a_3 a_4^4 + \\
& + 256a_4^5 - 4a_1^2 a_2^2 a_3^3 a_5 + 16a_2^3 a_3^3 a_5 + 16a_1^3 a_3^4 a_5 - 72a_1 a_2 a_3^4 a_5 + 108a_3^5 a_5 + \\
& + 18a_1^2 a_2^3 a_3 a_4 a_5 - 72a_2^4 a_3 a_4 a_5 - 80a_1^3 a_2 a_3^2 a_4 a_5 + 356a_1 a_2^2 a_3^2 a_4 a_5 + 24a_1^2 a_3^3 a_4 a_5 - \\
& - 630a_2 a_3^3 a_4 a_5 - 6a_1^3 a_2^2 a_4^2 a_5 + 24a_1 a_2^3 a_4^2 a_5 + 144a_1^4 a_3 a_4^2 a_5 - 746a_1^2 a_2 a_3 a_4^2 a_5 + \\
& + 560a_2^2 a_3 a_4^2 a_5 + 1020a_1 a_3^2 a_4^2 a_5 - 36a_1^3 a_4^3 a_5 + 160a_1 a_2 a_4^3 a_5 - 1600a_3 a_4^3 a_5 - \\
& - 27a_1^2 a_2^4 a_5^2 + 108a_2^5 a_5^2 + 144a_1^3 a_2^2 a_3 a_5^2 - 630a_1 a_2^3 a_3 a_5^2 - 128a_1^4 a_3^2 a_5^2 + \\
& + 560a_1^2 a_2 a_3^2 a_5^2 + 825a_2^2 a_3^2 a_5^2 - 900a_1 a_3^3 a_5^2 - 192a_1^4 a_2 a_4 a_5^2 + 1020a_1^2 a_2^2 a_4 a_5^2 - \\
& - 900a_2^3 a_4 a_5^2 + 160a_1^3 a_3 a_4 a_5^2 - 2050a_1 a_2 a_3 a_4 a_5^2 + 2250a_3^2 a_4 a_5^2 - 50a_1^2 a_4^2 a_5^2 + \\
& + 2000a_2 a_4^2 a_5^2 + 256a_1^5 a_5^3 - 1600a_1^3 a_2 a_5^3 + 2250a_1 a_2^2 a_5^3 + 2000a_1^2 a_3 a_5^3 - \\
& - 3750a_2 a_3 a_5^3 - 2500a_1 a_4 a_5^3 + 3125a_5^4)\,.
\end{aligned}
$$

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