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LINEAR FORMS ON FREE MODULES OVER CERTAIN LOCAL RING

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Abstract

A real linear algebra **A** having a $\mathbb{R}$-basis $< 1, \eta, \dots, \eta^{m-1} >$ with $\eta^m = 0$ will be called the plural algebra. The linear forms on a free finite-dimensional module **M** — especially their kernel — are investigated.

Key words: Linear algebra, free module, linear form.

MS Classification: 13C99

The problem solved in the article may be formulated as follows: It is known that the kernel of linear form on a vector space is a $n-1$-dimensional subspace. Can this be suitably generalizated in the case the real vector space be replaced by a free finite-dimensional module over a certain local ring?

In the article, it will be shown that an analogic relation between linear forms on this module and its (certain) hyperplanes can be found.

1 Real plural algebra of finite order.

Definition 1.1 *Real plural algebra of order* m is every linear algebra **A** on $\mathbb{R}$ having as a vector space over $\mathbb{R}$ a basis $\{1, \eta, \eta^2, \dots, \eta^{m-1}\}$, where $\eta^m = 0$.

Definition 1.2 By a *system of projections* $\mathbf{A} \to \mathbb{R}$ it is meant a system of mappings p_k : **A** onto $\mathbb{R}$, defined for $k = 0, \dots, m-1$, as follows:

$$\forall \beta \in \mathbf{A}, \quad \beta = \sum_{i=0}^{m-1} b_i \eta^i; \quad p_k(\beta) \stackrel{def}{=} b_k .$$

Proposition 1.3 *An element $\varepsilon \in \mathbf{A}$ is a unit if and only if $p_o(\varepsilon) \neq 0$.*

Proof

1) Let $\varepsilon \in \mathbf{A}$ be a unit and let $p_o(\varepsilon) = 0$,
$p_o(\varepsilon) = 0 \Rightarrow \exists \mu \in \mathbf{A};\ \ \varepsilon = \eta\mu$. Then $1 = \varepsilon\varepsilon^{-1} = (\eta\mu)\varepsilon^{-1} = \eta(\mu\varepsilon^{-1})$. Multiplying the equality $\eta(\mu\varepsilon^{-1}) = 1$ by η^{m-1} we get $0 = \eta^{m-1}$, which contradicts D.1.1.

2) Let $p_o(\varepsilon) \neq 0$.
Let $\varepsilon = \sum_{i=0}^{m-1} e_i\eta^i$. Then $\varepsilon^{-1} = \sum_{i=0}^{m-1} f_i\eta^i$ exists if and only if the following system of equations (expressing just the fact $\sum_{i=0}^{m-1} e_i\eta^i \cdot \sum_{i=0}^{m-1} f_i\eta^i = 1$) is solvable.

$$(k) \qquad e_0 f_k + e_1 f_{k-1} + \ldots + e_k f_0 = \delta_{0k}, \qquad 0 \leq k \leq m-1.$$

It is solvable if and only if $e_o = p_o(\varepsilon) \neq 0$.

Proposition 1.4 *Let a unit $\alpha \in \mathbf{A}$ be given. Then there exists a $\beta \in \mathbf{A}$ with $\beta^2 = \alpha$ if and only if $p_o(\alpha) > 0$.*

Proof Let $\alpha = \sum_{k=0}^{m-1} a_k\eta^k$. Let us take β, $\beta = \sum_{i=0}^{m-1} b_i\eta^i$. Then

$$\beta^2 = \sum_{i+j=0}^{m-1} b_i b_j \eta^{i+j}.$$

Thus

$$\alpha = \beta^2 \Leftrightarrow \alpha = \sum_{k=0}^{m-1} a_k\eta^k = \sum_{i+j=0}^{m-1} b_i b_j \eta^{i+j},$$

which is equivalent to the system of equations:

$$\begin{array}{lll}
(0) & a_0 & = b_0^2 \\
(1) & a_1 & = 2b_0b_1 \\
(2) & a_2 & = 2b_0b_2 + b_1^2 \\
\multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
(m-1) & a_{m-1} & = 2b_0b_{m-1} + b_1b_{m-2} + \ldots + b_{m-2}b_1
\end{array}$$

With respect to the condition $p_0(\beta) = b_0 \neq 0$ (P.1.3) it is solvable if and only if $a_0 = p_0(\alpha) > 0$.

Proposition 1.5 $\mathbf{A}$ *is a local ring with the maximal ideal $\eta\mathbf{A}$. The ideals $\eta^j\mathbf{A}$, $1 \leq j \leq m$, are the all ideals in* $\mathbf{A}$.

Proof

1) $\eta\mathbf{A}$ is the only maximal ideal in $\mathbf{A}$

$\eta\mathbf{A}$ is evidently an ideal. According to P.1.3 $\mathbf{A}\backslash\eta\mathbf{A}$ consists just of units of $\mathbf{A}$. From this follows (see the consequence 1.6.(I) of theorem 1.3. in [1]) that $\mathbf{A}$ is a local ring and $\eta\mathbf{A}$ the maximal ideal of one.

2) $\eta^j\mathbf{A}$, $1 < j \leq m$ are the only ideals in $\mathbf{A}$

Let J, $J \neq \mathbf{A}$, is an ideal in $\mathbf{A}$ and let us suppose that

$$\forall j, \quad 1 < j < m; \quad J \neq \eta^j\mathbf{A}.$$

For such ideal certainly $\exists k, \quad 1 \leq k < m; \quad J \subset \eta^k\mathbf{A} \wedge J \not\subset \eta^{k+1}\mathbf{A}$.

Let $\alpha \in J$,

$$\alpha \notin \eta^{k+1}\mathbf{A} \Rightarrow \alpha = \sum_{j=0}^{m-1} a_j\eta^j, \quad a_0 = \ldots = a_{k-1} = 0, \ a_k \neq 0.$$

Thus $\varepsilon = \sum_{j=k}^{m-1} a_j\eta^{j-k}$ is a unit, $\alpha = \eta^k\varepsilon$. If $\xi \in \eta^k\mathbf{A}$ then:

$$\exists\beta \in \mathbf{A}; \qquad \xi = \eta^k\beta = (\beta\varepsilon^{-1})\alpha \Rightarrow \xi \in J \Rightarrow \ J = \eta^k\mathbf{A}$$

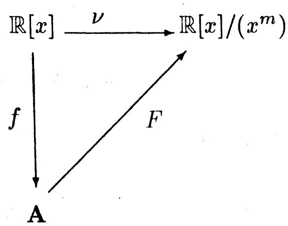
which is a contradiction.

Proposition 1.6 *The ring* $\mathbf{A}$ *is isomorphic to the factor ring of polynoms* $\mathbb{R}[x]/(x^m)$.

Proof Let us consider the mapping

$f : \mathbb{R}[x] \to \mathbf{A}, \quad f(h(x)) = h(\eta), \quad \forall h(x) \in \mathbb{R}[x]$.

Then f is clearly an epimorphismus with the kernel (x^m). Therefore following diagram commutes:



and the mapping F is an isomorphism.

Proposition 1.7 *The ring* $\mathbf{A}$ *is isomorphic to the linear algebra of matrix* $\mathcal{M}_{mm}(\mathbb{R})$ *of the form:*

$$\begin{pmatrix} b_0 & b_1 & \ldots & b_{m-1} \\ 0 & b_0 & \ldots & b_{m-2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & b_0 \end{pmatrix}$$

Proof Let us define $g: \mathbf{A} \to \mathcal{M}_{mm}(\mathbb{R})$ in the following way:

$$\alpha = \sum_{j=0}^{m-1} a_j \eta^j \Rightarrow g(\alpha) \stackrel{def}{=} (a_{ij}) \Leftrightarrow [(j < i \Rightarrow a_{ij} = 0) \wedge (j \geq i \Rightarrow a_{ij} = a_{j-i})].$$

Considered mapping is evidently the founded isomorphism $\mathbf{A} \to \mathcal{M}$.

2 Free finite-dimensional modules over the algebra A

Agreement 2.1 In the following text we denote by $\mathbf{A}$ the $\mathbb{R}$-algebra introduced in section 1. We will have a deal with the free finite-dimensional modules over the algebra $\mathbf{A}$ [1]. The capital $\mathbf{M}$ denotes always such module.

Proposition 2.2 *Let* $\{\underline{E}_1, \ldots \underline{E}_n\}$ *be some system of generators of a module* $\mathbf{M}$. *If* $\underline{U}_1, \ldots, \underline{U}_k$ *are linearly independent elements from* $\mathbf{M}$ *then:*

(1) $k \leq n$

(2) by a suitable renumbering of elements $\underline{E}_1, \ldots, \underline{E}_n$, $\{\underline{U}_1, \ldots, \underline{U}_k, \underline{E}_{k+1}, \ldots, \underline{E}_n\}$ *will be a set of generators of* $\mathbf{M}$.

Proof (by induction)

(a) $k = 1$

(1): evidently fulfiled

(2): let $\underline{U}_1$ be linearly independent, $\underline{U}_1 = \sum_{i=1}^{n} \xi_i \underline{E}_i$ $(*)$

We will show that there exists at least one unit among $\xi_1, \ldots \xi_n$. In the opposite case multiplying $(*)$ by η^{m-1} we have: $\eta^{m-1}\underline{U}_1 = \underline{o} \wedge \eta^{m-1} \neq 0 \Rightarrow \underline{U}_1$ is linearly dependent — contradiction. Let for example ξ_1 be a unit. Then from $(*)$ it follows:

$$\underline{E}_1 = \xi_1^{-1}\underline{U}_1 + \sum_{j=2}^{n}(-\xi_j \xi_1^{-1})\underline{E}_j.$$

Consequently $[\underline{U}_1, \underline{E}_2, \ldots, \underline{E}_n] = \mathbf{M}$.

(b) Let P.2.2 be fulfiled for $k - 1$.

[1] As $\mathbf{A}$ is a local ring, that $\mathbf{M}$ is an $\mathbf{A}$-space in the sence of [2].

As $\underline{U}_1, \ldots, \underline{U}_k$ are linearly independent, then $\underline{U}_1, \ldots, \underline{U}_{k-1}$ are linearly independent as well. By the induction supposition we have by a suitable renumbering of $\underline{E}_i$: $[\underline{U}_1, \ldots, \underline{U}_{k-1}, \underline{E}_k, \ldots, \underline{E}_n] = \mathbf{M}$. Now

$$\underline{U}_k \in \mathbf{M} \Rightarrow \underline{U}_k = \sum_{i=1}^{k-1} \xi_i \underline{U}_i + \sum_{j=k}^{n} \xi_j \underline{E}_j \qquad (**)$$

Let us derive that there exists at least one unit among $\xi_k, \ldots, \xi_n$. Otherwise after multiplying $(**)$ by η^{m-1} we would obtain:

$$(\eta^{m-1}\xi_1)\underline{U}_1 + \cdots + (\eta^{m-1}\xi_{k-1})\underline{U}_{k-1} - \eta^{m-1}\underline{U}_k = \underline{o} \wedge \eta^{m-1} \neq 0$$

which contradicts to linear independence of $\underline{U}_1, \ldots, \underline{U}_k$.
Let for example ξ_k be a unit. Then from $(**)$ we have:

$$\underline{E}_k = (-\xi_k^{-1}\xi_1)\underline{U}_1 + \cdots + (-\xi_k^{-1}\xi_{k-1})\underline{U}_{k-1} + \\ + \xi_k^{-1}\underline{U}_k + (-\xi_k^{-1}\xi_{k+1})\underline{E}_{k+1} + \cdots + (-\xi_k^{-1}\xi_n)\underline{E}_n.$$

It follows from this that: $[\underline{U}_1, \ldots, \underline{U}_k, \underline{E}_{k+1}, \ldots, \underline{E}_n] = \mathbf{M}$, i.e. (2) is true.
From the induction supposition we get that $k-1 \leq n$.
From $(**)$ it follows that $k-1 = n$ implies the linear dependence of $\underline{U}_1, \ldots, \underline{U}_k$, which is not possible, i.e. (1).

Consequence 2.3 If the module $\mathbf{M}$ has one basis consisting of n elements then any its basis consists of the same number n elements. Any linear independent system of n elements of $\mathbf{M}$ forms a basis of $\mathbf{M}$. The number n is called the *dimension* (more precisely $\mathbf{A}$-*dimension*) of the $\mathbf{M}$. Moreover it follows from the proof of P.2.2 that a linear independence of the system $\{\underline{E}_1, \ldots, \underline{E}_n\}$ implies the linear independence of the system $\{\underline{U}_1, \ldots, \underline{U}_k, \underline{E}_{k+1}, \ldots, \underline{E}_n\}$.

Proposition 2.4 *Let* $\mathbf{M}$ *be a free n-dimensional module on* $\mathbf{A}$. *Then* $\mathbf{M}$ *is an mn-dimensional vector-space over* $\mathbb{R}$ *(m denotes —as usually— the order of* $\mathbf{A}$*).*

Proof Let $\mathcal{E} = < \underline{E}_1, \ldots, \underline{E}_n >$ be a basis of $\mathbf{A}$-module $\mathbf{M}$. Let $\underline{U} \in \mathbf{M}$, $\xi_i \in \mathbf{A}$,

$$\underline{U} = \sum_{i=1}^{n} \xi_i \underline{E}_i, \quad \xi_i = \sum_{j=0}^{m-1} x_{ij}\eta^j, \quad 1 \leq i \leq n \;\Rightarrow\; \underline{U} = \sum_{i=1}^{n}\sum_{j=0}^{m-1} x_{ij}(\eta^j \underline{E}_i).$$

I.e. $\mathbf{M}$ is evidently a vector-space over $\mathbb{R}$. It remains to prove that the system of generators

$$\mathcal{B} = < \underline{E}_1, \ldots, \underline{E}_n, \eta\underline{E}_1, \ldots, \eta\underline{E}_n, \ldots, \eta^{m-1}\underline{E}_1, \ldots, \eta^{m-1}\underline{E}_n >$$

is (over $\mathbb{R}$) linearly independent.
Let us suppose that

$$\exists\, e_{ij} \in \mathbb{R}; \qquad \sum_{i=1}^{n}\sum_{j=0}^{m-1} e_{ij}(\eta^j \underline{E}_i) = \underline{o}$$

It follows from this

$$\sum_{i=1}^{n}(\sum_{j=0}^{m-1} e_{ij}\eta^j)\underline{E}_i = \underline{o} \Rightarrow \sum_{j=0}^{m-1} e_{ij}\eta^j = 0,$$

$$\forall i,\ 1 \le i \le n \ (\text{as } \underline{E}_i \in \mathcal{E}) \Rightarrow \forall i,\ 1 \le i \le n,\ \forall j,\ 0 \le j \le m-1;\ \ e_{ij} = 0.$$

Therefore $\mathcal{B}$ is a basis of $\mathbf{M}$ as a vector-space on $\mathbb{R}$, thus card $\mathcal{B}= \dim_R \mathbf{M}= mn$.

Proposition 2.5 *Let* $\mathcal{E}=< \underline{E}_1, \ldots, \underline{E}_n >$ *be a basis of* **A**-*module* **M**. *Let us define a system of vector-spaces* $\mathbf{P}_0, \ldots, \mathbf{P}_{m-1}$ *over* $\mathbb{R}$*:*

$$\mathbf{P}_j = [\eta^j \underline{E}_1, \ldots, \eta^j \underline{E}_n], \quad 0 \le j \le m-1,$$

Considering **M** *as an* $\mathbb{R}$*-vector space, then the following statements are valid:*

(1) $\mathbf{M}= \oplus_{j=0}^{m-1}\mathbf{P}_j$

(2) $\forall \underline{X} \in \mathbf{M} \quad \exists!(\underline{X}_0, \ldots, \underline{X}_{m-1}) \in \mathbf{P}_0^m; \quad \underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j.$

Proof

1) As $\mathcal{E}$ is a basis of **A**-module **M**, then according to the proof of P.2.4

$$\mathcal{B} =< \underline{E}_1, \ldots, \underline{E}_n, \eta\underline{E}_1, \ldots, \eta\underline{E}_n, \ldots, \eta^{m-1}\underline{E}_1, \ldots, \eta^{m-1}\underline{E}_n >$$

is a basis of a vector-space **M** over $\mathbb{R}$, from which we have (1).

2) Let $\underline{X} \in \mathbf{M}$.

Then $\underline{X} = \sum_{i=1}^{n} \xi_i \underline{E}_i, \quad \xi = \sum_{j=0}^{m-1} x_{ij}\eta^j, \quad x_{ij} \in \mathbb{R}, \quad 1 \le i \le n, 0 \le j < m.$

Then $\underline{X} = \sum_{i=1}^{n}\sum_{j=0}^{m-1} x_{ij}\eta^j \underline{E}_i = \sum_j(\eta^j \sum_i x_{ij}\underline{E}_i).$

Let us put $\sum_{i=1}^{n} \xi_i \underline{E}_i = \underline{X}_j$. Then $\underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j, \quad \underline{X}_j \in \mathbf{P}_0, \quad 0 \le j < m.$

As $\mathcal{B}$ is a basis of the vector-space **M**, we get from this that the system of elements $x_{ij} \in \mathbb{R}$, $1 \le i \le n$, $0 \le j < m$, and thus also vectors $\underline{X}_j$ are unique i.e. (1).

Notation 2.6 The system of vector-spaces $\mathbf{P}_0, \ldots, \mathbf{P}_{m-1}$ is determined by a given basis of **A**-module **M**. Therefore "unique" in 2.5 (2) means unique up to selection of a basis of **M**.

3 Linear forms on modules over the algebra A

Proposition 3.1 *Let ϕ be a linear form on* **M** *(A.2.1). Then there exists exactly one system of linear forms $\phi_0, \dots, \phi_{m-1}$* **M** *into* $\mathbb{R}$ *such that:*

$$\phi = \sum_{j=0}^{m-1} \phi_j \eta^j$$

Proof

$$\underline{U} \in \mathbf{M} \Rightarrow \phi(\underline{U}) = \sum_{j=0}^{m-1} u_j \eta^j \Rightarrow p_j(\phi(\underline{U})) = u_j, \quad 0 \leq j < m.$$

Denoting $\phi_j = \phi \circ p_j$, $0 \leq j < m$, we cleary obtain a system of mappings $\phi_0, \dots, \phi_{m-1}$ satisfying the equality $\phi = \sum_{j=0}^{m-1} \phi_j \eta^j$. Exactly one such system exists for arbitrary linear form ϕ. [if $\{\phi_j\}$, $\{\psi_j\}$ are two such systems then

$$\phi = \sum_{j=0}^{m-1} \phi_j \eta^j \wedge \phi = \sum_{j=0}^{m-1} \psi_j \eta^j \Rightarrow 0 = \sum_{j=0}^{m-1} (\phi_j - \psi_j)\eta^j \Rightarrow \phi_j = \psi_j, 0 \leq j < m]$$

Due to D.1.2 it follows that $\{\phi_j\}$ is a system of linear forms **M** into $\mathbb{R}$.

Proposition 3.2 *If $\phi_0, \dots, \phi_{m-1}$ are linear forms then the mapping*

$$\phi = \sum_{j=0}^{m-1} \phi_j \eta^j$$

is a linear form **M** *into* **A** *if and only if* $\forall \underline{X} \in \mathbf{M}$:

$$\left. \begin{array}{l} \phi_0(\eta \underline{X}) = 0, \\ \qquad\qquad\qquad\qquad 1 \leq k \leq m-1 \\ \phi_k(\eta \underline{X}) = \phi_{k-1}(\underline{X}) \end{array} \right\} \qquad (*)$$

Proof

1) Let $\phi = \sum_{j=0}^{m-1} \phi_j \eta^j$ be a linear form As ϕ is a linear form **M** into **A**, then $\forall \underline{X} \in \mathbf{M}$; $\phi(\eta \underline{X}) = \eta \phi(\underline{X})$. Thus

$$\sum_{j=0}^{m-1} \phi_j(\eta \underline{X}) \eta^j = \sum_{k=0}^{m-2} \phi_k(\underline{X}) \eta^{k+1} = \sum_{j=1}^{m-1} \phi_{j-1}(\underline{X}) \eta^j,$$

we get from this

$$[\phi_j(\underline{X}) \in \mathbb{R}] : \phi_0(\eta \underline{X}) = 0, \quad \phi_j(\eta \underline{X}) = \phi_{j-1}(\underline{X}), \quad 1 \leq j \leq m-1,$$

i.e. $(*)$.

2) Let $(*)$ be true

(a) as ϕ_j are linear forms, evidently $\forall \underline{U}, \underline{V} \in \mathbf{M}$; $\phi(\underline{U} + \underline{V}) = \phi(\underline{U}) + \phi(\underline{V})$

(b) we prove: $\forall \underline{X} \in \mathbf{M}$; $\phi(\eta \underline{X}) = \eta.\phi(\underline{X})$:

$$\phi(\eta \underline{X}) = \sum_{j=0}^{m-1} \phi_j(\eta \underline{X})\eta^j \ [(*)] =$$

$$= \sum_{j=1}^{m-1} \phi_{j-1}(\underline{X})\eta^j = \eta(\sum_{i=0}^{m-2} \phi_i(\underline{X})\eta^i) = \eta(\sum_{j=0}^{m-1} \phi_j(\underline{X})\eta^j) = \eta.\phi(\underline{X})$$

(c) we prove: $\forall \underline{X} \in \mathbf{M}$, $\forall \alpha \in \mathbf{A}$, $\alpha = \sum_{j=0}^{m-1} a_j \eta^j$; $\phi(\alpha \underline{X}) = \alpha.\phi(\underline{X})$:

$$\phi(\alpha \underline{X}) = \phi(\sum_{j=0}^{m-1} (a_j \eta^j)\underline{X}) \ [(a)] = \sum_{j=0}^{m-1} \phi(a_j \eta^j \underline{X}) =$$

$$= \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \phi_k(a_j \eta^j \underline{X})\eta^k = \sum_{j=0}^{m-1} a_j \sum_{k=0}^{m-1} \phi_k(\eta^j \underline{X})\eta^k =$$

$$= \sum_{j=0}^{m-1} a_j \phi(\eta^j \underline{X})[(b)] = (\sum_{j=0}^{m-1} a_j \eta^j)\phi(\underline{X}) = \alpha.\phi(\underline{X})$$

It follows that ϕ is a linear form.

Proposition 3.3 *Let $\phi_0, \ldots, \phi_{m-1} : \mathbf{M} \to \mathbb{R}$ be a system of linear forms such that*

$$\sum_{j=0}^{m-1} \phi_j \eta^j$$

is the linear form $\mathbf{M}$ *into* $\mathbf{A}$. *Then*

$$\forall \underline{X} \in \mathbf{M}, \quad \underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j, \ \underline{X}_j \in \mathbf{P}_0; \quad \phi_k(\underline{X}) = \sum_{j=0}^{k} \phi_{k-j}(\underline{X}_j), \ 0 \leq j \leq m-1.$$

Proof Let $\underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j$. Then

$$\phi_k(\underline{X}) = \phi_k(\underline{X}_0 + \eta \underline{X}_1 + \cdots + \eta^k \underline{X}_k + \cdots + \eta^{m-1}\underline{X}_{m-1}) =$$

$$= \phi_k(\underline{X}_0 + \eta \underline{X}_1 + \eta^k \underline{X}_k + \cdots +$$

$$+\eta^k(\eta(\underline{X}_{k+1} + \cdots + \eta^{m-k-2}\underline{X}_{m-1}))) \ [\text{P.3.2}] =$$

$$= \phi_k(\underline{X}_0) + \phi_{k-1}(\underline{X}_1) + \cdots + \phi_0(\underline{X}_k) +$$

$$+\phi_0(\eta(\underline{X}_{k+1} + \cdots + \eta^{m-k-2}\underline{X}_{m-1})) =$$

$$= \phi_k(\underline{X}_0) + \phi_{k-1}(\underline{X}_1) + \cdots + \phi_0(\underline{X}_k) + 0, \quad 0 \leq k \leq m-1.$$

Proposition 3.4 *If $\phi : \mathbf{M} \to \mathbf{A}$ is a linear form then there exists exactly one system of linear forms $f_0, \ldots, f_{m-1} : \mathbf{P}_0 \to \mathbb{R}$ such that*

$$\forall \underline{X} \in \mathbf{M}, \quad \underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j, \quad \underline{X}_j \in \mathbf{P}_0$$

the following relation is valid:

$$\phi_k(\underline{X}) = \sum_{j=0}^{k} f_{k-j}(\underline{X}_j), \quad 0 \le k \le m-1. \qquad (*)$$

where

$$\sum_{j=0}^{m-1} \phi_j \eta^j = \phi$$

Proof Putting $f_j = \phi_j / \mathbf{P}_0$, $0 \le j \le m-1$, we get (due to P.3.3) the system of linear forms $\mathbf{P}_0 \to \mathbb{R}$ fulfiling $(*)$, i.e.

$$\phi_k(\underline{X}) = \sum_{j=0}^{k} f_{k-j}(\underline{X}_j), \quad 0 \le k \le m-1.$$

We prove the unicity of this system: $\{f_j\}, \{g_j\}$ being two systems fulfiling $(*)$ and determining systems of linear forms $\mathbf{M}$ into $\mathbf{A}$ $\{\phi_j\}, \{\psi_j\}$ consecutively. From the equality $\phi = \sum_{j=0}^{m-1} \phi_j \eta^j = \sum_{j=0}^{m-1} \psi_j \eta^j$ it follows (due to P.3.1): $\phi_j = \psi_j$, $0 \le j \le m-1$. From this we arrive in equalities 3.4 $(*)$ in the form as follows $\forall \underline{X}$, $\underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j$:

(0) $k = 0: \; g_0(\underline{X}_0) = \psi_0(\underline{X}) = \phi_0(\underline{X}) = f_0(\underline{X}_0) \Rightarrow f_0 = g_0$

(1) $k = 1: \; g_1(\underline{X}_0) + g_0(\underline{X}_1) = \psi_1(\underline{X}) = \phi_1(\underline{X}) = f_1(\underline{X}_0) + f_0(\underline{X}_1)$,
due to (0) $\Rightarrow g_1 = f_1$

....

$(m-1)$ $k = m-1:$
$g_{m-1}(\underline{X}_0) + g_{m-2}(\underline{X}_1) + \cdots + g_0(\underline{X}_{m-2}) = \psi_{m-1}(\underline{X}) =$
$= \phi_{m-1}(\underline{X}) = f_{m-1}(\underline{X}_0) + f_{m-2}(\underline{X}_1) + \cdots + f_0(\underline{X}_{m-1})$,
due to $(0), (1), \ldots, (m-1) \Rightarrow \; g_{m-1} = f_{m-1}$.

Thus $f_j = g_j$, $0 \le j \le m-1$ and the unicity of the system is proved.

Proposition 3.5 *If $\{f_j\}_{j=0}^{m-1}$ is a system of linear forms $\mathbf{P}_0$ into $\mathbf{A}$ and let $\{\phi_k\}_{k=0}^{m-1}$ be the system of linear forms $\mathbf{M}$ into $\mathbb{R}$ defined as follows:*

$\forall \underline{X} \in \mathbf{M},\ \underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j;$

$$\phi_k(\underline{X}) \stackrel{def}{=} \sum_{j=0}^{k} f_{k-j}(\underline{X}_j), \qquad 0 \le k \le m-1 \qquad (**)$$

then the mapping $\phi = \sum_{k=0}^{m-1} \phi_k \eta^k$ *is the linear form* $\mathbf{M} \to \mathbf{A}$ *determined uniquely by the system* $\{f_j\}$.

Proof If $f_0, \ldots, f_{m-1} : \mathbf{P} \to \mathbb{R}$ are linear forms then the system of mappings $\{\phi_k\}$ defined by $(**)$ is evidently the system of linear forms $\mathbf{M}$ into $\mathbb{R}$. Hands the supposition is correct. It is necessary to show that the mapping ϕ is the linear form $\mathbf{M} \to \mathbf{A}$. According to P.3.2 it is sufficient to show linear forms defined by $(**)$ have the property 3.2.$(*)$ i.e.:
$\forall \underline{X} \in \mathbf{M};$

(1) $\phi_0(\eta \underline{X}) = 0,$

(2) $\phi_k(\eta \underline{X}) = \phi_{k-1}(\underline{X}), \quad 1 \le k \le m-1.$

Let $\underline{X} = \sum_{j=0}^{m-1} \eta^j \underline{X}_j$.
Then obviously $(\eta \underline{X})_j = \underline{X}_{j-1}, 1 \le j < m$ and $(\eta \underline{X})_0 = 0$.
So we have:
(1) $\phi_0(\eta \underline{X})[(**)] = f_0((\eta \underline{X})_0) = f_0(\underline{o}) = 0$
(2) $\phi_k(\eta \underline{X})[(**)] = \sum_{j=0}^{k} f_{k-j}((\eta \underline{X})_j)\ \ [f_k(\underline{o}) = 0] = \sum_{j=1}^{k} f_{k-j}((\eta \underline{X})_j) =$
$= \sum_{j=1}^{k} f_{k-j}(\underline{X}_{j-1}) = [j-1 = h] = \sum_{h=0}^{k-1} f_{(k-1)-h}(\underline{X}_h) = \phi_{k-1}(\underline{X}).$
From this $\phi = \sum_{j=0}^{m-1} \phi_j \eta^j$, $\{\phi_j\}$ defined by $(**)$ is the linear form. Due to P.3.1 the unicity is evident.

Definition 3.6 A linear form ϕ $\mathbf{M}$ into $\mathbf{A}$ is called a *linear form of order k* $(0 \le k \le m)$ if:

(1) $\forall X \in \mathbf{M};\ \ \phi(\underline{X}) \in \eta^k \mathbf{A},$

(2) $\exists \underline{Y} \in \mathbf{M};\ \ \phi(\underline{Y}) \notin \eta^{k+1} \mathbf{A}.$

In the special case $k = 0$ the linear form is called the *epiform*.

Proposition 3.7 *If* ϕ *is a linear form of order k then there exists at least one epiform* χ *such that*

$$\phi = \eta^k \chi.$$

Proof Let ϕ be a linear form of order k. We get clearly from this:

$$\phi_0 \equiv \phi_1 \equiv \ldots \equiv \phi_{k-1} \equiv 0 \ \wedge \ \exists \underline{Y} \in \mathbf{M}; \quad \phi_k(\underline{Y}) \neq 0.$$

Let us denote $\phi^* = \phi_k + \cdots + \eta^{m-k-1}\phi_{m-1}$. Then $\phi = \eta^k \phi^*$, though ϕ^* is not a linear form from $\mathbf{M}$ to $\mathbf{A}$ generally. According to P.3.4 there is the system $\{f_j\}$ of linear forms $\mathbf{P}_0$ into $\mathbb{R}$ fulfilling 3.4 $(*)$ for the linear form ϕ. Since ϕ is the form of order k from 3.4 $(*)$ we have:

$$f_0 \equiv f_1 \equiv \ldots \equiv f_{k-1} \equiv 0.$$

Let us define the system $\{h_j\}_{j=0}^{m-1}$ of linear forms $\mathbf{P}_o$ into $\mathbb{R}$ as follows:

$$h_0 = f_k, \quad h_1 = f_{k+1}, \ldots, h_{m-k-1} = f_{m-1}. \tag{$*$}$$

and linear forms $h_{m-k}, \ldots, h_{m-1}$ are arbitrary.
According to P.3.5 to the $\{h_j\}$ we can construct the system $\{\chi_j\}$ by means of 3.5 $(**)$ for which $\chi = \sum_{j=0}^{m-1} \chi_j \eta^j$ is the linear form. And due to $(*)$ we get:

$$\phi_k(\underline{X}) = f_k(\underline{X}_0) + f_{k-1}(\underline{X}_1) + \cdots + f_0(\underline{X}_k) = f_k(\underline{X}_0) = h_0(\underline{X}_0) = \chi_0(\underline{X})$$
$$\phi_{k+1}(\underline{X}) = f_{k+1}(\underline{X}_0) + f_k(\underline{X}_1) + 0 = h_1(\underline{X}_0) + h_0(\underline{X}_1) = \chi_1(\underline{X})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\phi_{m-1}(\underline{X}) = f_{m-1}(\underline{X}_0) + \cdots + f_k(\underline{X}_{m-k-1}) + 0 =$$
$$= h_{m-k-1}(\underline{X}_0) + \cdots + h_0(\underline{X}_{m-k-1}) = \chi_{m-k-1}(\underline{X}).$$

Thus $\eta^k \chi = \phi$ and since $\exists \underline{Y} \in \mathbf{M}$; $\phi_k(\underline{Y}) = \chi_0(\underline{Y}) \neq 0$, χ is the epiform.

4 Kernels of linear forms

Definition 4.1 Let $\mathbf{M}$ be a n-dimensional $\mathbf{A}$-module (by C.2.1).
A free $(n-1)$-dimensional submodule of $\mathbf{M}$ is called a *hyperplane* of the $\mathbf{M}$.

Theorem 4.2 *If ϕ is an epiform then there exists exactly one hyperplane $\mathcal{N}$ of the $\mathbf{M}$ such that*

$$\mathcal{N} = Ker\ \phi.$$

Proof Let $\mathcal{E}= \{\underline{E}_1, \ldots, \underline{E}_n\}$ be a basis of the $\mathbf{A}$-module $\mathbf{M}$. $\underline{X} = \sum_{i=1}^n \xi_i \underline{E}_i$ is a vector from $\mathbf{M}$. Let us put

$$\phi(\underline{E}_i) = \alpha_i, \quad 1 \leq i \leq n.$$

Then $\phi(\underline{X}) = \sum_{i=1}^n \xi_i \alpha_i$. As ϕ is an epiform there exists an α_j, $1 \leq j \leq n$, being a unit. We may suppose that α_n is a unit. We will construct vectors $\underline{V}_1, \ldots, \underline{V}_{n-1}$ as follows:

$$\forall j, \quad 1 \leq j \leq n; \quad \underline{V}_j = \alpha_n \underline{E}_j - \alpha_j \underline{E}_n.$$

Evidently each of them turns the form ϕ to zero. Let us prove their linear independence over **A**: Let us suppose that $\exists \beta_j \in \mathbf{A}, \quad 1 \leq j \leq n-1$;

$$\sum_{j=1}^{n-1} \beta_j \underline{V}_j = \underline{o} \Rightarrow \sum_{j=1}^{n-1} \beta_j (\alpha_n \underline{E}_j - \alpha_j \underline{E}_n) = \underline{o} \Rightarrow$$

$$\Rightarrow \alpha_n^{-1} (\sum_{j=1}^{n-1} (\alpha_n (\beta_j \underline{E}_j) - \underline{E}_n (\beta_j \alpha_j))) = \underline{o} \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{n-1} \beta_j \underline{E}_j + (-\alpha_n^{-1} (\sum_{j=1}^{n-1} \beta_j \alpha_j)) \underline{E}_n = \underline{o} \Rightarrow$$

$$\Rightarrow \beta_1 = \ldots = \beta_{n-1} = 0.$$

Let us put

$$\mathcal{N} = [\underline{V}_1, \ldots, \underline{V}_{n-1}]$$

We show $\mathcal{N} = \mathrm{Ker}\ \phi$.

1) Let $\underline{X} \in \mathcal{N} \Rightarrow \underline{X} = \sum_{i=1}^{n-1} \xi_i \underline{V}_i \Rightarrow \phi(\underline{X}) = 0 \Rightarrow \underline{X} \in \mathrm{Ker}\ \phi$

2) Let $\underline{X} \in \mathrm{Ker}\ \phi$. The $\underline{X}$ has the expression $\underline{X} = \sum_{i=1}^{n} \xi_i \underline{E}_i$.
We get

$$\phi(\underline{X}) = \sum_{i=1}^{n} \xi_i \alpha_i \ \wedge \ \phi(\underline{X}) = 0 \ \wedge \ (\forall \xi_i \ \exists \lambda_i; \quad \xi_i = \alpha_n \lambda_i) \Rightarrow$$
$$\Rightarrow 0 = \sum_{i=1}^{n} \xi_i \alpha_i = \alpha_n \sum_{i=1}^{n-1} \lambda_i \alpha_i + \alpha_n \xi_n \Rightarrow \sum_{i=1}^{n-1} \lambda_i \alpha_i + \xi_n = 0,$$

i.e. $\xi_n = -\sum_{i=1}^{n-1} \lambda_i \alpha_i$.

Then of course:

$$\underline{X} = \sum_{i=1}^{n} \xi_i \underline{E}_i = \sum_{i=1}^{n-1} (\alpha_n \lambda_i) \underline{E}_i - (\sum_{i=1}^{n-1} \lambda_i \alpha_i) \underline{E}_n =$$
$$= \sum_{i=1}^{n-1} \lambda_i (\alpha_n \underline{E}_i - \alpha_i \underline{E}_n) = \sum_{i=1}^{n-1} \lambda_i \underline{V}_i \Rightarrow \underline{X} \in \mathcal{N}.$$

Theorem 4.3 *If $\mathcal{N}$ is an hyperplane of module* **M** *then there exists an epiform ϕ such that*

$$Ker\ \phi = \mathcal{N}.$$

Proof Let $\{\underline{V}_1, \ldots, \underline{V}_{n-1}\}$ be a basis of $\mathcal{N}$. Then (by P.2.2) there is $\underline{V}_n \in \mathbf{M}$ such that $\{\underline{V}_1, \ldots, \underline{V}_{n-1}, \underline{V}_n\}$ is a basis of $\mathbf{M}$. Take $\underline{X} \in \mathbf{M}$, $\underline{X} = \sum_{i=1}^{n} \xi_i \underline{V}_i$. Let us define a mapping $\phi : \mathbf{M} \rightarrow \mathbf{A}$ by the relation $\phi(\underline{X}) = \xi_n$. Then ϕ evidently is the epiform. $[\phi(\underline{V}_n) = 1]$ with Ker $\phi = \mathcal{N}$.

Consequence 4.4 Let $\mathcal{N} \subseteq \mathbf{M}$. $\mathcal{N}$ is a hyperplane of $\mathbf{M}$ if and only if there exists the epiform ϕ such that

$$\mathcal{N} = \text{Ker } \phi$$

Theorem 4.5 *Let ϕ, ψ be epiforms. Then Ker ϕ = Ker ψ if and only if there exists a unit $\varepsilon \in \mathbf{A}$ such that $\phi = \varepsilon\psi$.*

Proof

1) Let $\phi = \varepsilon\psi$ where ε is a unit, then obviously Ker ϕ =Ker ψ.

2) Let Ker ϕ = Ker ψ =$\mathcal{N}$ where $\mathcal{N}$ is (by T.4.2) the hyperplane of $\mathbf{M}$. Let $\{\underline{V}_1, \ldots, \underline{V}_{n-1}\}$ be a basis of $\mathcal{N}$. Then there exists a $\underline{E}_n \in \mathbf{M}$ such that $[\underline{V}_1, \ldots, \underline{V}_{n-1}, \underline{E}_n] = \mathbf{M}$ (by P.2.2).
Then $\forall \underline{X} \in \underline{M}$:

$$\underline{X} = \sum_{i=1}^{n} \xi_i \underline{V}_i \Rightarrow \phi(\underline{X}) = \xi_n \phi(\underline{E}_n) \text{ and } \psi(\underline{X}) = \xi_n \psi(\underline{E}_n).$$

As ϕ, ψ are epiforms $\phi(\underline{E}_n), \psi(\underline{E}_n)$ are units and therefore we can find ε for which $\phi(\underline{E}_n) = \varepsilon.\psi(\underline{E}_n)$ and thus $\phi = \varepsilon\psi$.

Theorem 4.6 *If $\chi : \mathbf{M} \rightarrow \mathbf{A}$ is a linear form of order k then there exists a hyperplane $\mathcal{N}$ of $\mathbf{M}$ such that*

$$Ker\, \chi = \{\underline{X} \in \mathbf{M};\ \eta^k \underline{X} \in \mathcal{N}$$

Proof If χ is a form of order k then there is an epiform ϕ such that $\chi = \eta^k \phi$ (by P.3.7). Then by T.4.2 there exists $\mathcal{N} \subseteq \mathbf{M}$, $\mathcal{N}$=Ker ϕ.

1) $\eta^k \underline{X} \in \mathcal{N} \Rightarrow \chi(\underline{X}) = \eta^k \phi(\underline{X}) = \phi(\eta^k \underline{X}) = 0 \Rightarrow \underline{X} \in$ Ker χ

2) Let $\underline{X} \in$ Ker $\chi \Rightarrow \chi(\underline{X}) = 0 \Rightarrow \eta^k \phi(\underline{X}) = 0 \Rightarrow \phi(\eta^k \underline{X}) = 0 \Rightarrow \eta^k \underline{X} \in \mathcal{N}$.

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