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QUADRATIC SPLINES INTERPOLATING DERIVATIVES

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Abstract: There are described the algorithms for computing appropriate parameters of quadratic splines interpolating prescribed values of the first or the second derivative in the points of interpolation, which are different from the knots of the spline in general. The relations between various types of linear and quadratic splines are mentioned.

Key words: Spline functions, quadratic splines, interpolation.

MS Classification: 41A05, 41A15

1. Introduction

Quadratic spline interpolating given function values have been studied by many authors ([3], [5], [6], [7], [9]). There are very simple two-term recurrence relations between parameters of such a spline in case of coinciding knots and points of interpolation. But there are also some unpleasant features connected with such splines (error propagation without damping, unsymmetry of boundary conditions, existence questions). It was soon recognized, that some of that features escape, when separate knots of spline and points of interpolation are used (error propagation with damping - see [6]). We can also find approaches to use splines interpolating first or second derivatives in the area of "shape-preserving approximations" ([4]) or in solution of differential equations ([1]). The first results for quadratic splines interpolating the first derivative on equidistant mesh can be found in [8].

The aim of our contribution is to give more detailed results concerning the quadratic splines interpolating the first or the second derivative in case of nonequidistant or separated meshes and to mention the relations between various types of such simple splines.

2. Simple knot set

Let us have an increasing set of spline knots

$$(\Delta x) = \{x_i;\ i = 0(1)n+1\} .$$

We call a quadratic spline on the set (Δx) the function $s(x)$ fulfilling conditions

$$1^o \quad s(x) \in C^1[x_0, x_{n+1}] ; \qquad (1)$$

2^o $s(x)$ is a quadratic polynomial on every interval $[x_i, x_{i+1}]$, $i = 0(1)n$.

Let us denote $S(2,\Delta x)$ the linear space of functions fulfilling conditions 1^o, 2^o.

Statement of the problem

Given real numbers m_i, $i = 0(1)n+1$, we have to find $s \in S(2,\Delta x)$ such that $s'(x_i) = m_i$, $i = 0(1)n+1$ hold (e.g. we have to find spline interpolating the first derivatives at the knots of the spline).

Simple calculation shows that we have together 3n+3 spline parameters and only 3n+2 connecting continuity and interpolation conditions - one free parameter is also at our disposal.

Theorem 1

Quadratic spline $s \in S(2, \Delta x)$ is uniquelly determined by conditions

$$s'(x_i) = m_i\ ,\ i = 0(1)n+1 \quad \text{(conditions of interpolation)},(2)$$

$$s(x_0) = s_0 \ \text{(or } s(x_k) = s_k\ ,\ k \in \{0,1,\dots,n+1\}\ ,\ \text{initial condition).} \quad (2)$$

Proof

A spline $s \in S(2, \Delta x)$ can be written as

$$s(x) = (1-t^2)s_i + t^2 s_{i+1} + h_i t(1-t)m_i \ \text{ for } \ x \in [x_i, x_{i+1}]\ , \quad (3)$$
$$i = 0(1)n\ ,\ \text{with}\ \ h_i = x_{i+1} - x_i\ ,\ \ t = (x - x_i)/h_i\ ,$$
$$s_i = s(x_i)\ .$$

We have further

$$s'(x) = 2t(s_{i+1} - s_i)/h_i + (1-2t)m_i\ .$$

The continuity condition on $s'(x)$ in the knot $x = x_i$ leads to the recurrence relation

$$s_i - s_{i-1} = h_{i-1}(m_{i-1} + m_i)/2\ ,\quad i = 1(1)n+1\ . \quad (4)$$

Given data (2), we can use (4) to the computation of all values s_i , $i \neq k$. Than we know all parameters needed for using (3) to compute $s(x)$.

Remarks

1. The computation of values m_i from given s_i using (4) (interpolation of function values) is known to be a little unstable; in [11] the composed relation on equidistant mesh

$$m_{i+1} - m_i = 2(s_{i+1} - 2s_i + s_{i-1})/h \quad (5)$$

with s_0, s_1 given is used.

2. We cannot prescribe the values of the second derivative on the boundary for spline interpolating the first derivatives (as can be done when interpolating function values). Given

$m_i = s'(x_i)$, $i = 0(1)n+1$, it follows that $s''(x_i + 0) = M_i = (m_{i+1} - m_i)/h_i$ is determined and constant in interval $[x_i, x_{i+1}]$ (the second derivative has discontinuities in knots x_i).

3. The quadratic spline described in Theorem 1 has similar unpleasing error propagation features - the error in free parameter s_k or in parameter m_i is propagated over the whole interval $[x_0, x_{n+1}]$ without damping, as can be deduced from the relation

$$s_i = s_0 + \sum_{j=0}^{i-1} h_j(m_j + m_{j+1})/2 \,. \tag{6}$$

For example, error $v_0 = s_0 - \bar{s}_0$ in the value s_0 results in $s_i - \bar{s}_i = v_0$. The isolated error $e_i = m_i - \bar{m}_i$ in the value m_i is propagated as $s_i - \bar{s}_i = h_{i-1}e_i/2$, $s_{i+1} - \bar{s}_{i+1} = e_i(h_{i-1} + h_i/2, \ldots$.

4. It is possible to use another notation for $s(x)$ instead of (3) - for example

$$s(x) = s_i + h_i c_i t + (m_i - c_i)h_i t(1 - t) \,, \quad x \in [x_i, x_{i+1}] \,, \tag{7}$$

where $c_i = (s_{i+1} - s_i)/h_i$ (the slopes).

Continuity conditions can be now written as

$$2\,c_{i-1} = m_{i-1} + m_i \,, \quad i = 1(1)n \,. \tag{8}$$

We can use it in a similar way for construction of the spline interpolating the function values or the first derivatives.

3. Mesh with separated knots and points of interpolation

3.1 Spline representation

Let us have a mesh of knots x_i and points of interpolation t_i

$$(\Delta x \Delta t) : \; x_0 \leq t_0 < x_1 < t_1 < \ldots < t_n \leq x_{n+1}$$

and denote $h_i = x_{i+1} - x_i$, $d_i = (t_i - x_i)/h_i$, $m_i = s'(t_i)$,

$i = 0(1)n$, $s_i = s(x_i)$, $i = (0(1)n+1$;

$t = (x - x_i)/h$.

We have to find now again the spline $s \in S(2, \Delta x)$ determined by the conditions

$$s'(t_i) = m_i\ ,\quad i = 0(1)n\ ,\quad m_i \text{ given real numbers.} \tag{10}$$

Counting and comparing the number of parameters and continuity conditions in the knots x_i, $i = 1(1)n$ of the spline, we recognize existence of two free parameters, which can be prescribed for unique determination of the spline. Some specific feature of the problem (10) can be found in the fact, that the case $t_i = (x_i + x_{i+1})/2$, which is the most popular and quite regular in function values interpolation, must be treated separately now.

Lemma 1

The solution of the problem (10) is given by the quadratic spline $s \in S(2, \Delta x)$, which can be written for $x \in [x_i, x_{i+1}]$

a) in case of $d_i \neq 1/2$ as

$$s(x) = A(t)s_i + B(t)s_{i+1} + h_i C(t) m_i\ ,\quad t \in [0,1]\ , \tag{11}$$

with functions $A(t) = (t^2 - 2td_i)/(2d_i - 1) + 1 = -B(t) + 1$,

$$B(t) = -t(t - 2d_i)/(2d_i - 1)\ ,$$

$$C(t) = t(t-1)/(2d_i - 1)\ ;$$

b) in case of $d_i = 1/2$, denoting $s_i' = s'(x_i)$, as

$$s(x) = s_i + s_i'(x - x_i) + (s_{i+1}' - s_i')(x - x_i)^2/(2h_i)\ , \tag{12}$$

where we have

$$m_i = (s_i' + s_{i+1}')/2\ . \tag{13}$$

Proof follows from the properties of functions A(t), B(t), C(t) and representation (11) in case $d_i \neq 1/2$. In case of $d_i = 1/2$ we recognize the Taylor formula in (12).

3.2 Computation of parameters s_i

The continuity condition for $s \in S(2, \Delta x)$ at $x = x_i$ is realized implicitly in our notation $s_i = s(x_i - 0) = s(x_i + 0) = s(x_i)$.

We have further

$$s'(x) = \frac{2(t - d_i)}{h_i(2d - 1)} (t_i - s_{i+1}) + \frac{2t - 1}{2d_i - 1} m_i \qquad (14)$$

for $x \in [x_i, x_{i+1}]$.

Continuity condition for $s'(x)$ at $x = x_i$, $i = 1(1)n$ in case $d_i \neq 1/2$, $i = 0(1)n$ results in the relations

$$-\frac{1-d_{i-1}}{1-2d_{i-1}} s_{i-1} + \left[\frac{1-d_{i-1}}{1-2d_{i-1}} - \frac{p_i d_i}{1-2d_i}\right] s_i + \frac{p_i d_i}{1-2d_i} s_{i+1} =$$

$$= \frac{1}{2} h_{i-1} \left[\frac{m_{i-1}}{1-2d_{i-1}} + \frac{m_i}{1-2d_i}\right] , \qquad (15)$$

where $p_i = h_{i-1}/h_i$. Denoting further

$$a_i = (1 - d_{i-1})/(1 - 2d_{i-1}) , \quad b_i = p_i d_i/(1 - 2d_i) ,$$

$$f_i = \frac{1}{2} h_{i-1} \left(\frac{m_{i-1}}{1-2d_{i-1}} + \frac{m_i}{1-2d_i}\right) , \quad i = 1(1)n , \qquad (16)$$

we can rewrite the system of continuity conditions (15) as

$$- a_i s_{i-1} + (a_i - b_i) s_i + b_i s_{i+1} = f_i , \quad i = 1(1)n . \qquad (17)$$

In the special case of $h_i = h$, $d_i = \lambda$, $i = 1(1)n$ we obtain the result of [8].

3.2.1 Boundary conditions s_0, s_{n+1}

Choosing $s_0 = s(x_0)$, $s_{n+1} = s(x_{n+1})$ as two free spline parameters mentioned in 3.1, we recognize (17) to be the tridiagonal system of linear equations for determining parameters s_i, $i = 1(1)n$:

$$\begin{bmatrix} a_1-b_1, & b_1, & & & \\ -a_2, & a_2-b_2, & b_2 & & \\ \ddots & \ddots & \ddots & & \\ & -a_{n-1}, & a_{n-1}-b_{n-1}, & b_{n-1} \\ & & -a_n & , & a_n-b_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{n-1} \\ s_n \end{bmatrix} = \begin{bmatrix} f_1+a_1 s_0 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n-b_n s_{n+1} \end{bmatrix} . \qquad (18)$$

The determinants D_n of the matrix of system (18) fulfil recurrence relation

$$D_i = (a_i - b_i)D_{i-1} + a_i b_{i-1} D_{i-2} . \tag{19}$$

Hence,

$$(a_i - b_i)a_i b_{i-1} > 0 , \quad i = 3(1)n , \quad D_1 D_2 > 0$$

is sufficient for $D_n \neq 0$.

In case of the equidistant mesh with $h_i = h$, $d_i \neq 1/2$ we have $p_i = 1$,

$$a_i = a\,(1 - d)/(1 - 2d) , \quad a_i - b_i = 1 , \tag{20}$$

$$b_i = d/(1 - 2d) = a - 1 , \quad f_i = (m_{i-1} + m_i)(2(1 - 2d)) .$$

The matrix M_n of the system (18) and its determinant are

$$M_n = \begin{bmatrix} 1, & a-1, & & \\ -a, & 1, & a-1, & \\ & \ddots & \ddots & \ddots \\ & -a, & 1, & a-1 \end{bmatrix}, \quad \det(M_n) = \begin{cases} 1 & \text{for } n = 1 , \\ 1 + a(a - 1) & \text{for } n = 2 , \\ 1 + 2a(a - 1) & \text{for } n = 3 \\ \dots & \end{cases} .$$

The recurrence relation (19) is now

$$\det(M_i) = \det(M_{i-1}) + a(a - 1)\det(M_{i-2}) . \tag{22}$$

From positivity of $a(a - 1) = d(1 - d)/(1 - 2d)^2$ for $d \in (0,1)\setminus\{\frac{1}{2}\}$ follows, that we have $\det(M_n) > 0$ generally in our case. Let us summarize our discussion in the following theorem.

Theorem 2

Under boundary conditions $s(x_0) = s_0$, $s(x_{n+1}) = s_{n+1}$ with given numbers s_0, s_{n+1} there exists a unique quadratic spline $s(x)$ interpolating prescribed values of the first derivatives $m_i = s'(t_i)$ on the mesh $(\Delta \times \Delta t)$

a) on equidistant mesh with $h_i = h$, $d_i = d \neq 1/2$, $i = 0(1)n$;

b) on general mesh ($\Delta x \Delta t$) with $D_1 D_2 > 0$, $(a_i - b_i)a_i b_{i-1} > 0$, $i = 3(1)n$, $d_i \neq 1/2$.

The values $s_i = s(x_i)$ can be computed from the system (18) for the use of spline representation (11).

3.2.2 Another types of boundary conditions

There is possible to prescribe the boundary conditions on the first derivatives $s'(x_0) = s'_0$, $s'(x_{n+1}) = s'_{n+1}$ for the spline fulfilling conditions (10). Suppose $d_0, d_n \neq 1/2$; then using (14) we obtain

$$s'_0 = -\frac{2d_0}{h_0(2d_0 - 1)}(s_0 - s_1) - \frac{m_0}{2d_0 - 1} , \tag{23}$$

$$s'_{n+1} = \frac{2(1 - d_n)}{h_n(1 - 2d_n)}(s_{n+1} - s_n) - \frac{m_n}{1 - 2d_n} .$$

Denote

$$a_0 = -b_0 = d_0/(1 - 2d_0) , \quad a_{n+1} = -b_{n+1} = (1 - d_n)/(1 - 2d_n); \tag{24}$$

than we can join (23) with (17) to obtain the needed system of linear equations

$$\begin{bmatrix} a_0, & b_0, & & & \\ -a_1, & a_1 - b_1, & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_n, & a_n - b_n, & b_n \\ & & & -a_{n+1}, & -b_{n+1} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \\ s_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}h_0(s'_0 + m_0/(2d_0 - 1)) \\ f_1 \\ \vdots \\ f_n \\ \frac{1}{2}h_n(s'_{n+1} + m_n/(2d_n - 1)) \end{bmatrix} .$$

According to the defining relations (24) we have

$$a_0 + b_0 = 0, \quad -a_{n+1} - b_{n+1} = 0 ; \quad -a_i + (a_i - b_i) + b_i = 0, \quad i = 1(1)n .$$

It indicates singularity of the matrix of the system (25) and we couldn't obtain the solution with every s'_0, s'_{n+1} .

Similarly, we obtain the system with singular matrix when we try to determine the parameters s_i for the spline with boundary conditions $s''(x_0) = M_0$, $s''(x_{n+1}) = M_{n+1}$.

It could be seen from (14) that

$$s''(x) = \frac{2}{h_i(2d_i - 1)} \frac{s_i - s_{i+1}}{h_i} + \frac{2}{2d_i - 1} \frac{m_i}{h_i} \quad \text{for } x \in [x_i, x_{i+1}] \ .$$

With $i = 0,n$ we obtain the equations of boundary conditions with zero sum of coefficients.

Similar problems are involved in boundary conditions

$$s''(t_0) = M_0 \ , \quad s''(t_n) = M_n \ ,$$

or when the first (resp. the second) derivatives on the boundary are approximated by some numerical differentiation formula using the values s_i .

3.2.3 Mesh with $d_i = 1/2$

Let us consider the case $d_i = 1/2$, $i = 0(1)n$, $m_i = s'(t_i)$ given. The continuity relations are now

$$s_{i+1} - s_i = h_i(s_i' + s_{i+1}')/2 = h_i m_i \ , \quad s_j = s(x_j), \ s'(x_j) = s_j' \ . \qquad (27)$$

Given $s_0 = s(x_0)$ as the first free parameter, it is possible recursively calculate the values s_i :

$$s_{i+1} = s_i + h_i m_i \ , \quad i = 0(1)n \ .$$

But the spline $s \in S(2,\Delta x)$ is not uniquely determined by the parameters s_i, m_i only. Given $s_0' = s'(x_0)$ as the second free parameter, we can find remaining values s_i using (27) as

$$s_{i+1}' = 2m_i - s_1' \ , \quad i = 0(1)n \ .$$

Now the spline $s \in S(2,\Delta x)$ is completely determined. More genera ly, it could be possible to choose the parameters s_k, s_j' and use (27) for determining other values s_i, s_i'. It is not possible to determine spline $s \in S(2,\Delta x)$ with the free parameters (s_k, s_j), or (s_k', s_j').

Remark

For the given spline $s(x)$ let us denote $s'(x) = s_1(x)$ the first degree spline (polygon). The problem (2) can be reformulated now as to find $s_1(x)$ given by interpolation conditions $s_1(t_i) = s'(t_i) = m_i$. It is now easy to see, that given $s_1(x_0) = s_0'$, the spline $s_1(x)$ is uniquely determined. For exact determination of $s(x) = \int s_1(x)dx$ we have to prescribe yet the integration constant - the value $s(x)$ in some point ($s(x_0)$ in case 3.2.2). We can see as well, that it is not possible generally to prescribe two values $s_1(x_0) = s'(x_0)$, $s_1(x_{n+1}) = s'(x_{n+1})$ (see 3.2.2). Another results of 3.2.3 could be interpreted similarly.

3.3 Error propagation

3.3.1 Errors in boundary values

Let us have equidistant mesh with $d_i = d = (t_i - x_i)/h \neq 1/2$, and consider splines s, $\bar{s} \in S(2,\Delta x)$ determined by the same conditions of interpolation derivatives at points t_i, $i = 0(1)n$, but by different boundary conditions: the spline s by boundary values s_0, s_{n+1} and $\bar{s}$ by values $\bar{s}_0, \bar{s}_{n+1}$. The difference $\tilde{s} = s - \bar{s}$ belongs to $S(2,\Delta x)$ and fulfils the interpolation conditions $s'(t_i) = 0$, $i = 0(1)n$ and boundary conditions

$$\tilde{s}_0 = e_0 = s_0 - \bar{s}_0 , \qquad \tilde{s}_{n+1} = e_{n+1} = s_{n+1} - \bar{s}_{n+1} .$$

The system of continuity and boundary conditions can be written now (see 3.2.1) as

$$-a\tilde{s}_{i-1} + \tilde{s}_i + (a-1)\tilde{s}_{i+1} = 0 , \qquad \tilde{s}_0 = e_0 , \quad \tilde{s}_{n+1} = e_{n+1} . \tag{28}$$

We can consider it also as boundary value problem for the second order difference equation and to solve it explicitly. The characteristic equation has roots $r_1 = 1$, $r_2 = 1 - 1/d$.

$$\begin{aligned} &\text{For } d \in (0,1/2) \text{ there is } r_2 \in (-\infty, -1) , \\ &\text{for } d \in (1/2,1) \text{ we have } r_2 \in (-1,0) . \end{aligned} \tag{29}$$

General solution of equation (28) could be now written as

$$s_j = c_1 + c_2(1 - 1/d)^j .$$

Constants c_1, c_2 are determined by boundary conditions e_0, e_{n+1} :

$$c_1 = (e_0 r_2^{n+1} - e_{n+1})/(r_2^{n+1} - 1) , \quad c_2 = (e_{n+1} - e_0)/(r_2^{n+1} - 1) .$$

The particular solution of the problem (28) is therefore

$$\tilde{s}_j = \frac{1}{r^{n+1} - 1} \left[(e_0 r_2^{n+1} - e_{n+1}) + (e_{n+1} - e_0) r^j \right] . \tag{30}$$

For large n we can write it with respect to (29) approximatively

$$\begin{aligned} &\tilde{s}_j \sim e_0 + (e_{n+1} - e_0) r_2^{j-n-1} \quad \text{for } d \in (0,1/2) \quad (|r_2| > 1) , \\ &\tilde{s}_j \sim e_{n+1} + (e_0 - e_{n+1}) r_2^j \quad \text{for } d \in (1/2,1) \quad (|r_2| < 1) . \end{aligned} \tag{31}$$

3.3.2 Isolated errors in derivatives

Let us suppose that some isolated error of magnitude e occures in the value m_k : $\bar{m}_k = m_k - e$. Then, by (16), it causes the errors in the right hand sides f_i with indices k, k+1 only :

$$\bar{f}_k = f_k + \tfrac{1}{2} eh/(1 - 2d) = f_k - E , \quad \bar{f}_{k+1} = f_{k+1} - E , \quad (E = \tfrac{1}{2} eh / (1 - 2d)) .$$

Denoting $\tilde{s} = s - \bar{s}$ the difference of splines differing in the values $m_k, \bar{m}_k$ only, the system (18) looks now

$$\begin{aligned} &- a\tilde{s}_{i-1} + \tilde{s}_i + (a - 1)\tilde{s}_{i+1} = \tilde{f}_i , \quad \tilde{f}_i = \begin{cases} 0 & \text{for } i \neq k,k+1 \\ E & \text{for } i = k,k+1 , \end{cases} \\ &\tilde{s}_0 = \tilde{s}_{n+1} = 0 . \end{aligned} \tag{32}$$

The solution of this boundary value problem can be written as

$$\tilde{s}_j = \sum_{i=1}^{n} g_{ji} \tilde{f}_i , \tag{33}$$

where g_{ji} are values of the Green s function of the problem (see [2]) defined as

$$g_{ji} = \begin{cases} -(1-r_2^j)(1-r_2^{n-i})d/(1-r_2^n) & \text{for } j \le i , \\ (1-r_2^{-i})(1-r_2^{j-i})d/(1-r_2^{-n}) & \text{for } j \ge i . \end{cases}$$

So we have for example

$$\tilde{s}_k = -dE(1-r_2^k)(1-r_2^{n+1})/(1-r_2^{n+1}) + dE(1-r_2^{-n-1})(/(1-r_2^{-n-1})$$

with $\tilde{s}_k = O(E)$ for $d \in (0,1)\setminus \frac{1}{2}$.

Therefore we can see that in both cases 3.3.1, 3.3.2 we have no damping of errors in data m_i or in boundary conditions among splines $s \in S(2,\Delta x)$ interpolating the first derivatives.

4. Quadratic splines interpolating the second derivatives

Let us have a mesh $(\Delta x \Delta t)$ and denote $s_i = s(t_i)$, $M_i = s''(t_i)$, $i = 0(1)n$ for $s \in S(2,\Delta x)$. These quantities satisfy (see [5]) continuity relations

$$a_i M_{i-1} + b_i M_i + c_i M_{i+1} = f_i , \quad i = 1(1)n-1 , \tag{34}$$

where $h_i = x_{i+1} - x_i$, $k_i = t_{i+1} - t_i$,

$$a_i = [(x_i - t_{i-1})/k_{i-1}]^2 k_{i-1}/(k_{i-1}+k_i) ,$$

$$b_i = (t_i - x_i)[1 + (x_i - t_{i-1})/k_{i-1}] + \\ +(x_{i+1} - t_i)[1+(t_{i+1}-x_{i+1})/k_i]/(k_{i-1}+k_i),$$

$$c_i = [(t_{i+1} - x_{i+1})/k_i]^2 k_i/(k_{i-1}+k_i) ,$$

$$f = 2[(s_{i+1} - s_i)/k_i + (s_i - s_{i-1})/k_{i-1}]/(k_{i-1}+k_i) = \\ = 2[t_{i-1}, t_i, t_{i+1}]s$$

(a_i, b_i, c_i are determined by the mesh only; f_i depends also on the data s_i; the symbol for divided difference is used).

4.1 With M_i, $i = 0(1)n$ and $s_0 = s(t_0)$, $s_1 = s(t_1)$ given, we can calculate all values s_i, $i = 2(1)n$ with the help of three-term recurrence relation (34). The value $m_i = s'(t_i)$ can be then calculated from the relations (see 5)

$$m_i k_i = s_{i+1} - s_i - [(x_{i+1} - t_i)(t_{i+1} - x_{i+1} + k_i)M_i -$$
$$- (t_{i+1} - x_{i+1})^2 M_{i+1}]/2 \quad . \qquad (35)$$

On an equidistant mesh with $h_i = h$, $t_i = (x_i + x_{i+1})/2$ we have simply

$$m_i = (s_{i+1} - s_i)/h - h(3M_i - M_{i+1})/8 \quad . \qquad (36)$$

In both cases we can then use the Taylor representation of the spline s :

$$s(x) = s_i + m_i(x - t_i) + \frac{1}{2}M_i(x - t_i)^2 \quad . \qquad (37)$$

4.2 When we prescribe boundary conditions s_0, s_n as free parameters, the system (34) can be then written as system of linear equations with symmetric regular matrix for unknown values s_i :

$$\frac{1}{k_{i-1}} s_{i-1} - (\frac{1}{k_{i-1}} + \frac{1}{k_i})s_i + \frac{1}{k_i} s_{i+1} =$$
$$= \frac{1}{2}(k_{i-1} + k_i)(a_i M_{i-1} + b_i M_i + c_i M_{i+1}) \, . \qquad (38)$$

In case of the equidistant mesh mentioned above it is

$$s_{i-1} - 2s_i + s_{i+1} = \frac{1}{8} h^2 (M_{i-1} + 6M_i + M_{i+1}) \, , \quad i = 1(1)n-1 \, . \qquad (39)$$

In both cases, we have uniquely determined parameters of such spline $s \in S(2, \Delta x)$ for any M_i, $i = 0(1)n$. Its first derivatives $m_i = s'(t_i)$ can be found using (35) or (36), values s(x) through (37).

We can similarly make an analysis of error propagation in system (38) in case of equidistant mesh. For example, when the value s_0, s_n are perturbed by errors e_0, e_n, the error e_j in value s_j is given by

$$e_j = e_0 + j(e_n - e_0)/n \, . \qquad (40)$$

It means that an isolated error is propageted with very slow damping with growing distance from the place of perturbation.

4.3 When the boundary conditions $s(x_0)$, $s(x_{n+1})$ are given on the mesh with $x_0 < t_0$, $t_n < x_{n+1}$, we can apply

$$s(t_0) = s(x_0) - m_0(x_0 - t_0) - \frac{1}{2} M_0(x_0 - t_0)^2$$

with m_0 given by (35). In this way we obtain relation between $s(t_0)$ and M_0, M_1, which we add to the system (38) or (39) as the first equation. Similarly we obtain the last equation from the condition on $s(x_{n+1})$; alltogether we have now n+1 equations for n+1 unknowns values $s_i = s(t_i)$, $i = 0(1)n$.

But the zero sum of coefficients in all rows of the matrix of the system indicates its singularity. It means, we have no solution of this system for general data M_i and boundary conditions of this kind - such a spline $s \in S(2, \Delta x)$ need not exist !

We can also try some another alternative for free parameters:

- $s(x_0)$, $s'(x_0)$ (more generally: $s(x_k)$, $s'(x_k)$);
- $s(x_j)$, $s'(x_k)$;
- $s(t_j)$, $s'(t_k)$.

For example, by given value of $s'(x_k)$ the polygon $s_1(x) = s'(x)$ with $M_i = s_1'(t_i)$ is uniquely determined. Integrating $s_1(x)$, the constant of integration is determined by any value $s(x_j)$. For calculation of s_i, m_i we can then use (34), (35).

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