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Addition theorems and D-spaces

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Abstract. It is proved that if a regular space X is the union of a finite family of metrizable subspaces then X is a D-space in the sense of E. van Douwen. It follows that if a regular space X of countable extent is the union of a finite collection of metrizable subspaces then X is Lindelöf. The proofs are based on a principal result of this paper: every space with a point-countable base is a D-space. Some other new results on the properties of spaces which are unions of a finite collection of nice subspaces are obtained.

Keywords: D-space, point-countable base, extent, metrizable space, Lindelöf space

Classification: 54D20, 54F99

1. Introduction

Briefly, the aim of this article is twofold: to expand our knowledge on D-spaces and, on this basis, to obtain new addition theorems for metrizable and paracompact spaces.

How complex can be the structure of a (Tychonoff) space which is the union of two (of a finite family, of a countable family) of "nice" subspaces? This is a most natural question, it is especially important to know an answer to it when we are constructing concrete spaces with a certain combination of properties. In particular, how "bad" can be a space which is the union of two metrizable subspaces? How "bad" can be a σ-metrizable space, that is, a space which is the union of a countable family of metrizable subspaces? What can we say about spaces which are unions of a finite collection of paracompact subspaces?

Quite a few interesting facts in this direction are already known for some time. A very delicate result was established by Howard Wicke and John Worrell: each σ-metrizable countably compact space is compact (see [18], [19], [20]). For a series of strong general addition theorems involving countable unions of not necessarily metrizable spaces from certain classes, see [17]. A. Ostaszewski proved another astonishing theorem: every regular locally countably compact Hausdorff σ-metrizable space is sequential (in ZFC!) [15]. The Alexandroff compactification $A(\omega_1)$ of the uncountable discrete space ω_1 is the union of two metrizable (in fact, discrete) subspaces, while $A(\omega_1)$ is not first countable and, therefore, not metrizable. Of course, it is essential here that the metrizable spaces we consider are

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not necessarily separable, since every pseudocompact space which is the union of a countable family of separable metrizable subspaces is compact and metrizable. M. Ismail and A. Szymanski introduced the notion of the metrizability number $m(X)$ of a topological space as the smallest cardinal number k (finite or infinite) such that X can be represented as a union of k many metrizable subspaces (see [12], [13]). In particular, they proved, by an elegant argument, that every locally compact Hausdorff space X with the finite metrizability number contains an open dense metrizable subspace. Interesting results on the metrizability number of powers of spaces were obtained in [3]. In particular, it was established in [3] that if X is a regular Lindelöf space, n a positive integer, and X^n can be represented as the union of n metrizable subspaces, then X is metrizable.

A natural question that arises after one learns that every countably compact σ-metrizable space is compact is whether every σ-metrizable space of countable extent is Lindelöf. A negative answer to this question is provided by an amazing example constructed (in ZFC) by E. van Douwen and H.H. Wicke [11]. However, the answer to the following question remained unknown:

Question A. Suppose that X is the union of a finite family of metrizable spaces and the extent of X is countable. Is then X Lindelöf?

This question is at the origins of this paper. The key idea in our approach to addition theorems is to use in their proofs the not so well known notion of a D-space, introduced by E. van Douwen (see [10]). Though, at the first glance, the notion of a D-space seems to be quite a bit exotic, we show it to be instrumental in obtaining addition theorems.

One of our principal results is the theorem that every regular space X which is the union of a finite collection of metrizable spaces is a D-space (Theorem 5). The proof of this fact is based on another central result of this paper: every space with a point-countable base is a D-space (Theorem 2). This combination of results leads to a positive solution of Question A.

We also discuss a weaker and more "handy" version of the notion of a D-space, the notion of an aD-space related to the notion of an irreducible space introduced by R. Arens and J. Dugundji in [1] and studied in a more systematic way by J. Boone [4] and U. Christian [9]. Using the notion of aD-space, we establish that if a regular space X is the union of a finite collection of paracompact subspaces and the extent of X is countable, then X is Lindelöf (Corollary 16). However, we do not know whether every regular space X, which is the union of two paracompact D-subspaces, is a D-space.

In Section 3 we prove some addition theorems under weaker separation axioms, with somewhat weaker conclusions. The arguments here are based on a different approach.

In terminology and notation we follow [6]. However, we define the extent $e(X)$ in a slightly different manner than in standard texts and in [6]. Recall that a subset A of a space X is said to be *discrete in* X (*locally finite in* X) if every point $x \in X$ has an open neighborhood Ox containing not more than one element (only finitely many elements) of A. The *extent* $e(X)$ *of a space* X is the smallest infinite cardinal number τ such that $|A| \leq \tau$, for every subset A of X which is locally finite in X. Note that this definition obviously coincides with the usual definition of the extent of X for all T_1-spaces. Indeed, if X is a T_1-space, then a subset $A \subset X$ is locally finite in X if and only if A is discrete in itself and closed in X.

2. The main results and their proofs

A neighborhood assignment on a topological space X is a mapping ϕ of X into the topology $\mathcal{T}$ of X such that $x \in \phi(x)$, for each $x \in X$. A space X is called *a D-space* if, for every neighborhood assignment ϕ on X, there exists a locally finite in X subset A of X such that the family $\phi(A)$ covers X. One of the principal properties of D-spaces is that the extent coincides with the Lindelöf number in such spaces. In particular, every countably compact D-space is compact and every D-space with countable extent is Lindelöf. These facts make the notion of a D-space a useful tool in studying covering properties.

It is known that all metrizable spaces and even all Moore spaces are D-spaces [5]. A much more general result was recently obtained by R.Z. Buzyakova: every strong Σ-space is a D-space (see the definition and some further references in [7]). However, it is still an open problem (a fascinating one) whether every regular Lindelöf space is a D-space (it is even unknown whether every hereditarily Lindelöf regular T_1-space is a D-space). This problem was posed by E. van Douwen (see [10]). He also asked whether there exists a subparacompact or metacompact space which is not a D-space [10]. These questions are still open. Lemma 1 below will help to expand further our knowledge on D-spaces.

Lemma 1. *Let X be a topological space and ϕ an arbitrary neighborhood assignment. Suppose that the family $\phi(X)$ is point-countable. Then there exists a locally finite in X subset $D \subset X$ such that $X = \bigcup_{d \in D} \phi(d)$.*

PROOF: For each $x \in X$ denote by Φ_x the set of all elements $U \in \phi(X)$ such that $x \in U$. Enumerate Φ_x by natural numbers (recall that each Φ_x is countable). We also enumerate (well order) $X = \{x_\alpha : \alpha < |X|\}$. By transfinite induction, we will define countable subsets D_α of X the union of which will be D: $D = \bigcup\{D_\alpha : \alpha < |X|\}$.

Step 0. Let $D_0 = \emptyset$.

Suppose that for each $\beta < \alpha$, D_β is already defined.

Step α. We will define D_α by induction. During our inductive definition, once we chose a d_n at step n, we will need to return to d_n infinitely many times. To ensure that d_n is considered as many times as we need, we agree to return to it at each step p^n, where p is some prime number.

Sub-step 1. Take the first d_1 in X such that

$$d_1 \notin W = \bigcup\{\phi(d) : d \in D_\beta \text{ for some } \beta < \alpha\}.$$

If no such d_1 exists, put $D_\alpha = \emptyset$ and stop both the external and internal inductions.

Sub-step n. If n is divisible by at least two distinct primes, let d_n be the first in X such that $d_n \notin W \cup \phi(d_1) \cup \cdots \cup \phi(d_{n-1})$.

If $n = p^m$ for some prime p and an integer $m > 1$, take the first $U \in \Phi_{d_m}$ satisfying the following requirement.

Requirement (α, n): *there exists* $d_n \in X \setminus (W \cup \phi(d_1) \cup \cdots \cup \phi(d_{n-1}))$ *such that* $U = \phi(d_n)$.

Whether such a d_n exists or not, move to the next sub-induction step.

Let D_α be the set of all d_n's defined in the above sub-induction process.

Put $D = \bigcup\{D_\alpha : \alpha < |X|\}$. Let us show that $X = \bigcup_{d \in D} \phi(d)$. Let $x_\alpha \in X$. We claim that our construction ensures that

$$x_\alpha \in \bigcup\{\phi(d) : d \in \bigcup\{D_\gamma : \gamma \leq \alpha\}\}.$$

Indeed, arguing by transfinite induction, assume that the above formula is satisfied when we replace α with any smaller ordinal β. Then, if $x_\alpha \notin \phi(d)$ for each $d \in D$ chosen before Step α, we obviously have $x_\alpha = d_1 \in D_\alpha$ (see Sub-step 1).

Let us show that D is locally finite in X. Take any $x \in X$. Take the first α such that $x \in \bigcup_{d \in D_\alpha} \phi(d)$. Let n be the first sub-step of Step α such that $x \in \phi(d_n)$. Then, by our construction, $\phi(d_n)$ separates x from all d's in D chosen after Sub-step n of Step α. Since there are only finitely many of d's chosen at Step α before d_n, we need only to show that x can be separated from $\bigcup_{\beta<\alpha} D_\beta$. Let us show that $\phi(x)$ does not intersect $\bigcup_{\beta<\alpha} D_\beta$. Assume the contrary. Then there exists $d \in D_\beta$, where $\beta < \alpha$, such that $d \in \phi(x)$. Then $\phi(x) \in \Phi_d$. The element d is selected in our construction at some Sub-step m of Step β. Then at each Sub-step p^m, where p is a prime, $\phi(x)$ satisfies *Requirement* (β, p^m). And, at some Step p^m, $\phi(x)$ must be the first in Φ_d that satisfies the requirement. And therefore, x must be covered at Step β. The latter contradicts to the fact that $\alpha > \beta$ is the first step at which x is covered by some $\phi(d)$. Thus, $\phi(x)$ separates x from all d's chosen before Step α. □

Theorem 2. *Every space X with a point-countable base $\mathcal{B}$ is a D-space.*

Proof: Let ϕ be any neighborhood assignment on X. Since $\mathcal{B}$ is a base of X, for each $x \in X$ we can fix $\psi(x) \in \mathcal{B}$ such that $x \in \psi(x) \subset \phi(x)$. Then ψ is also a neighborhood assignment on X, and the family $\psi(X)$ is point-countable, since $\psi(X)$ is contained in $\mathcal{B}$. By Lemma 1, there exists a locally finite in X subset A of X such that $\psi(A)$ covers X. Then, clearly, $\phi(A)$ also covers X. □

Quite a few corollaries can be derived from Theorem 2.

Corollary 3. *If a regular space X is the union of a countable family γ of dense metrizable subspaces, then X is a D-space.*

PROOF: Indeed, each $Y \in \gamma$ has a σ-disjoint base $\mathcal{B}_Y$. For each $V \in \mathcal{B}_Y$ we fix an open subset $U(V)$ of X such that $U(V) \cap Y = V$. For any disjoint elements V_1 and V_2 of $\mathcal{B}_Y$ the sets $U(V_1)$ and $U(V_2)$ are disjoint, since Y is dense in X. Therefore, the family $\mathcal{P}_Y = \{U(V) : V \in \mathcal{B}_Y\}$ is σ-disjoint. Since X is regular, the family $\mathcal{P}_Y$ contains a base of X at y, for every $y \in Y$. It follows that the family $\mathcal{P} = \bigcup\{\mathcal{P}_Y : Y \in \gamma\}$ is a σ-disjoint base of X. It remains to apply Theorem 2. □

Corollary 4. *If a space X is the union of a countable family of open metrizable subspaces, then X is a D-space.*

PROOF: Clearly, X has a σ-disjoint base. Hence, X is a D-space, by Theorem 2. □

Note that Theorem 2 also explains a result of A.S. Mischenko that every countably compact space with a point-countable base is compact. Indeed, we now can say that this happens because every countably compact D-space is compact. In connection with Corollary 4, note that a locally metrizable space need not be a D-space, since there exists a countably compact locally metrizable normal space which is not compact (take ω_1, for example).

Theorem 5. *Suppose $X = \bigcup\{X_i : i = 1, \dots, n\}$, for some $n \in \omega$, where X is regular and X_i has a σ-disjoint base, for each $i = 1, \dots, n$. Then X is a D-space.*

To prove this statement, we need two technical results.

Lemma 6. *Suppose $X = \bigcup\{X_i : i = 1, \dots, n\}$, for some $n \in \omega$, and let $Y_i = \overline{X_1} \cap \overline{X_i} \cap (X_1 \cup X_i)$, for each $i = 2, \dots, n$. Then the set $Z_1 = \bigcup\{Y_i : i = 2, \dots, n\}$ is closed in X.*

PROOF: Take any $y \in \overline{Z_1}$. Then $y \in \overline{Y_i}$, for some i, where $2 \le i \le n$, which implies that $y \in \overline{X_1}$ and $y \in \overline{X_i}$. Also $y \in X_k$, for some k, where $1 \le k \le n$. Now we have to consider two cases.
Case 1: $k = 1$. Then $y \in Y_i = \overline{X_1} \cap \overline{X_i} \cap (X_1 \cup X_i) \subset Z_1$.
Case 2: $2 \le k \le n$. Then $y \in \overline{X_1} \cap \overline{X_k} \cap (X_1 \cup X_k) = Y_k \subset Z_1$.
Hence, $y \in Z_1$ and Z_1 is closed in X. □

The next fact was noticed by E. Michael and M.E. Rudin. It was established in the proof of Theorem 1.1 in [14].

Lemma 7. *If $X = Y \cup Z$ where each of the subspaces Y and Z has a σ-disjoint base (in itself) and X is regular, then the subspace $\overline{Y} \cap \overline{Z}$ also has a σ-disjoint base.*

Proof of Theorem 5: We argue by induction. For $n = 1$ the statement is true, since every space with a point-countable base is a D-space, by Theorem 2. Assume now that for less than n summands the assertion holds. For any i and j such that $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$ put $Y_{i,j} = \overline{X_i} \cap \overline{X_j} \cap (X_i \cup X_j)$. By Lemma 6, the set $Z_j = \bigcup\{Y_{i,j} : i \neq j, 1 \leq i \leq n\}$ is closed in X. By Lemma 7, each $Y_{i,j}$ is a space with a σ-disjoint base. Therefore, the space Z_j is the union of less than n spaces with a σ-disjoint base. By the inductive assumption, it follows that Z_j is a D-space, for each $j = 1, \dots, n$. Therefore, since each Z_j is closed in X, the subspace $Z = \bigcup\{Z_j : j = 1, \dots, n\}$ of X is a D-space.

The family $\mu = \{V_i : 1 \leq i \leq n\}$, where $V_i = X_i \setminus Z$, is a disjoint family of open subsets of X. Indeed, $X \setminus Z$ is open in X, and no point x of V_i can belong to the closure of V_j for $i \neq j$, since otherwise x would belong to $Y_{i,j}$ which is contained in Z. Therefore, $X \setminus Z$ has a σ-disjoint base and is a D-space. It follows that X is a D-space, as the union of an open D-space and a closed D-space. □

Corollary 8. *If a regular space X is the union of a finite family of metrizable subspaces, then X is a D-space.*

Proposition 9. *There exists a Tychonoff σ-metrizable space which is not a D-space.*

Proof: Take the space Γ constructed by E. van Douwen and H.H. Wicke in [11]. Though it is not explicitly mentioned there, it is clear from the list of properties of Γ given in [11, Section 1] that Γ is the union of a countable family of discrete subspaces (not closed in Γ). Thus, Γ is σ-metrizable. The extent of Γ is countable (such spaces are called ω_1-compact). It follows that Γ is not a D-space, since otherwise Γ would have been Lindelöf. Notice, that the space Γ has, in addition, many other nice properties; in particular, it is locally compact, locally countable, separable, first countable, submetrizable, realcompact, has the diagonal G_δ, and is Tychonoff. On the other hand, the space Γ is not countably metacompact [11].

Recall that a space X is *linearly Lindelöf* if, for every uncountable subset A of X of regular cardinality, there exists a point of complete accumulation in X. It is known that every Lindelöf space is linearly Lindelöf and the extent of arbitrary linearly Lindelöf space is countable; neither one of these implications can be reversed. The space Γ is not linearly Lindelöf, since it was shown in [2] that every locally metrizable linearly Lindelöf Tychonoff space is Lindelöf. Note that Γ is locally metrizable since it is locally compact and can be mapped by a one-to-one continuous mapping onto the usual real line. Now it is natural to pose the following question:

Problem 10. Is every Tychonoff (regular) σ-metrizable linearly Lindelöf space Lindelöf?

For a positive answer to this question it would be enough to prove that every regular σ-metrizable space is countably paracompact (see, for example, [2]); however, the space Γ is a counterexample to this conjecture, since it is not even countably metacompact.

P. de Caux [8] constructed a consistent example of a collectionwise normal σ-discrete T_1-space S of the countable extent such that S is not Lindelöf. The space S is also not linearly Lindelöf (since it is locally metrizable, see [2]) and not countably metacompact (see also [16]).

Problem 11. Is there a σ-discrete linearly Lindelöf Dowker space?

Note that the space Γ in [11] is not normal and it is not clear whether a normal space with all other properties of Γ can be constructed in ZFC alone. Observe also, that there exists a locally countable locally compact pseudocompact Hausdorff space X which is the union of two discrete subspaces and is not subparacompact and not metacompact — see Example 4.5 in [6]. This space X is a D-space, by Corollary 8. Hence, not every D-space is subparacompact or metacompact.

Sometimes it is quite difficult to verify whether a space is a D-space. This is witnessed, in particular, by a number of open problems on D-spaces, such as whether every regular Lindelöf space is a D-space. We will now consider a property formally weaker than that of being a D-space. This property is much easier to verify, and it is still strong enough to imply compactness for countably compact spaces.

Let us say that a space X is *an aD-space* if for each closed subset F of X and each open covering γ of X there exist a locally finite in F subset A of F and a mapping ϕ of A into γ such that $a \in \phi(a)$, for each $a \in A$, and the family $\phi(A) = \{\phi(a) : a \in A\}$ covers F. A similar but weaker property was considered in [5]; it turned out to be almost equivalent to irreducibility of spaces introduced in [1]. It is easily proved by a standard argument that every paracompact space is an aD-space. For more general statements and connections to other covering properties, see [4] and [5].

The next statement is our basic addition result on aD-spaces.

Theorem 12. *Suppose that X is a regular space and $X = Y \cup Z$, where Y is a paracompact subspace of X and Z is an aD-space. Then X is an aD-space.*

To prove Theorem 12, we need the next two easy to prove statements (the first of which is used in the proof of the other one).

Proposition 13. *Every closed subspace of an aD-space is an aD-space.*

Lemma 14. *If $X = Y \cup Z$, where Y and Z are aD-spaces and Y is closed in X, then X is also an aD-space.*

Proof of Theorem 12: By Lemma 14, we can assume that Y is dense in X. Take any open covering γ of X, and let F be any closed subset of X. Let us verify the definition of an aD-space with regards to these γ and F. First, observe that, in view of Proposition 13, we can also assume that $F = X$.

Since X is regular, we can find an open covering γ_1 such that the family of closures of elements of γ_1 refines γ. Since Y is paracompact and Y is dense in X, there exists a family η of open subsets of X such that:

1) η is locally finite at each point of Y;

2) η covers Y (and, probably, something else); and

3) η refines γ_1.

Let H be the set of all points of X at which the family η is not locally finite. Clearly, H is closed in X, and $H \cap Y = \emptyset$, by condition 1). Proposition 13 implies that H is an aD-space. Since γ_1 covers H, we can find a locally finite in H subset A of H and a mapping $\psi : A \to \gamma_1$ such that $\psi(A)$ covers H. Then $W = \bigcup \psi(A)$ is an open subset of X containing H.

Take the family η_0 of all elements V of η such that $\overline{V} \cap (X \setminus W) \neq \emptyset$. Let us show that

$$X \setminus W \subset \bigcup \{\overline{V} : V \in \eta_0\}.$$

Take any $x_0 \in X \setminus W$. Then x_0 is not in H, that is, η is locally finite at x_0. Since Y is dense in X and η covers Y, it follows that $x_0 \in \overline{\bigcup \eta}$. Therefore, since η is locally finite at x_0, there exists $V_0 \in \eta$ such that $x_0 \in \overline{V_0}$. Then $x_0 \in \overline{V_0} \cap (X \setminus W)$ which implies that $V_0 \in \eta_0$ and, hence, $X \setminus W \subset \bigcup \{\overline{V} : V \in \eta_0\}$.

Put $\xi = \{(X \setminus W) \cap \overline{V} : V \in \eta_0\}$. Clearly, ξ is a locally finite covering of the space $X \setminus W$. Since we can select a minimal subcovering of ξ [1], we can assume that ξ itself is minimal. Take any $P \in \xi$. By minimality of ξ, we can fix $x_P \in P$ such that x_P does not belong to any other element of ξ. We can also select $U_P \in \gamma$ such that $P \subset U_P$, since η refines γ_1 and the closure of any element of γ_1 is contained in some element of γ. Put $\phi(x_P) = U_P$, for each $P \in \xi$. The set $B = \{x_P : P \in \xi\}$ is locally finite in $X \setminus W$, since ξ is locally finite in $X \setminus W$. Therefore, B is locally finite in X, since W is open in X. It is also clear that $\phi(B) \supset X \setminus W$. Now for $x \in A \cup B$ let $f(x) = \psi(x)$, if $x \in A$, and $f(A \cup B) = \phi(x)$, if $x \in B$. Note that A and B are disjoint and $A \cup B$ is a locally finite subset of X. Obviously, $f(x)$ covers X. Thus, X is an aD-space. □

Theorem 15. *If a regular space X is the union of a finite collection of paracompact subspaces, then X is an aD-space.*

PROOF: This follows by induction from Theorem 12. □

Corollary 16. *If X is a regular space of countable extent and X is the union of a finite family of paracompact spaces, then X is Lindelöf.*

PROOF: This follows from Theorem 15, since obviously every aD-space of countable extent is Lindelöf.

□

3. An alternative addition theorem and some open questions

Some of the above results depend on regularity or Hausdorffness of the spaces considered. It is not clear whether we can completely get rid of these restrictions. We present a few results in this direction below. If we restrict ourselves to Tychonoff spaces, all these results are already contained in the theorems proved above. However, the separation axioms we use are much weaker.

Lemma 17. *Suppose X is a paracompact space and A is a subset of X of regular cardinality. Then either there exists a point of complete accumulation for A in X or there exists a subset B of A such that B is locally finite in X and $|B| = |A|$.*

PROOF: Assume that none of the points of X is a point of complete accumulation for A. Then there exists an open covering γ of X such that $|U \cap A| < |A|$, for each $U \in \gamma$. Since X is paracompact, we can refine γ by a locally finite open covering η. The subfamily $\xi = \{V \in \eta : V \cap A \neq \emptyset\}$ has the same cardinality as the set A, since the cardinality of A is regular. For each $V \in \xi$ we pick a point $x_V \in V \cap A$. Since the family ξ is locally finite in X, the set $B = \{x_V : V \in \xi\}$ of all selected points is locally finite in X. We also have: $|B| = |\xi| = |A|$, since the family ξ is point-finite. □

The next lemma is well known [6] and very easy to prove.

Lemma 18. *Every paracompact space of countable extent is Lindelöf.*

Theorem 19. *Suppose that X is a space of countable extent such that $X = Y \cup Z$, where Y and Z are paracompact spaces. Then X is linearly Lindelöf.*

PROOF: Take any uncountable subset A of X such that $|A|$ is regular. We have to show that there exists a point of complete accumulation for A in X. Assume the contrary. Clearly, at least one of the sets $A \cap Y$ and $A \cap Z$ has the same cardinality as A. Thus, we can also assume that $A \subset Y$. By Lemma 17, there exists a locally finite in Y subset B of Y such that B is contained in A and $|B| = |A|$. Let C be the set of all points of X at which the set B is not locally finite. Clearly, $C \subset Z$ and C is closed in X (and, therefore, closed in Z). Since Z is paracompact and the extent of X is countable, it follows that C is a paracompact space of countable extent. Hence, by Lemma 18, C is Lindelöf.

For each $z \in C$, fix an open neighborhood Oz of z in X such that $|Oz \cap B| < |A|$ (this is possible by the assumption). Since C is Lindelöf, there exists a countable subfamily η of the family $\{Oz : z \in C\}$ such that C is covered by η. Put $W = \bigcup \eta$. Clearly, W is an open subset of X, $C \subset W$ and $|W \cap B| < |B|$, since $|B| = |A|$ is a regular cardinal. It follows that $B \setminus (W \cap B)$ is an uncountable subset of X which is locally finite in X, — a contradiction with $e(X) = \omega$. □

Corollary 20. *Suppose that X is a space of countable extent such that $X = Y \cup Z$, where Y and Z are metrizable spaces. Then X is linearly Lindelöf.*

Again, the space Γ constructed by van Douwen and Wicke (see Section 2) shows that Corollary 20 does not extend to σ-metrizable spaces of countable extent.

Problem 21. Can Theorem 19 and Corollary 20 be extended to finite unions of spaces?

Problem 22. Can the conclusion in Theorem 19 be strengthened to the conclusion that X is Lindelöf?

The next delicate question was communicated to the authors by M.V. Matveev.

Problem 23. Suppose that X is a compact Hausdorff space and let $C_p(X)$ be the space of real-valued continuous functions on X in the topology of pointwise convergence. Is then $C_p(X)$ a D-space? Is every subspace of $C_p(X)$ a D-space?

We modify this question as follows:

Problem 24. Is the space $C_p(X)$ of real-valued continuous functions on arbitrary compact Hausdorff space X in the topology of pointwise convergence an aD-space? Is every subspace Y of $C_p(X)$ an aD-space when X is compact?

The answer to the next question remains unknown:

Problem 25. Is every countably metacompact σ-metrizable (σ-discrete) space a D-space? An aD-space?

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