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Abstract. We investigate the stability of some properties of locally convex Riesz spaces in connection with subspaces and quotients and also the corresponding three-space-problems. We show that in the richer structure there are more positive answers than in the category of locally convex spaces.

Keywords: locally convex Riesz space, short exact sequence, three-space-problem

Classification: 46A40, 46A04

0. Introduction

In the theory of topological vector spaces one of the widely considered questions is the question of the stability of certain properties in connection with subspaces and quotients. In a certain sense the inverse problem is the so called "three-space-problem": if in a short exact sequence $0 \to F \to E \to E/F \to 0$ of topological vector (particularly, locally convex) spaces the terms F and E/F possess a certain property (P), does the space E have to possess the same property? The problem was solved for many properties — see e.g. [2], [3], [4], [12]. The mentioned questions can be considered in the category of *ordered* topological vector spaces, too, and, as we shall see, in the richer structure there are positive answers in more cases than in the category of topological vector spaces.

In this article we shall study the problem of stability in connection with subspaces and quotients, as well as the three-space-problem for certain properties of *locally convex Riesz spaces* (abbrev. *lcRs*), i.e. locally convex lattices. For terminology in connection with ordered vector spaces we shall follow [7] and [14]. When we speak about the short exact sequence of ordered locally convex spaces

$$(*) \qquad 0 \to (F, C \cap F, t|F) \xrightarrow{j} (E, C, t) \xrightarrow{q} (E/F, C/F, t/F) \to 0,$$

we shall assume triple exactness, i.e.:

1° exactness in the vector sense, i.e. $\mathrm{Im}(j) = \mathrm{Ker}(q)$;

2° topological exactness in the sense that $j\colon F \to E$ is a topological injection, and $q\colon E \to E/F$ is a topological homomorphism;

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3° orders in F and E/F are canonical in connection with the order in E; that means that we shall assume that F is a closed l-ideal of E such that in the quotient E/F the cone $C/F = q(C)$ is proper and the order in F is given by the cone $C \cap F$ (see [13, Chapter V, Exercise 3]).

1. Dieudonné topology and lifting of order-bounded subsets

A lot of properties in the category of locally convex Riesz spaces are enabled by the so called Dieudonné topology. Namely, if (E, C, t) is the given (separated) *lcRs*, then there exist two new topologies in the space E and in its dual space E', denoted as $\sigma_s(E, E')$ and $\sigma_s(E', E)$, respectively, such that $(E, C, \sigma_s(E, E'))$ and $(E', C', \sigma_s(E', E))$ are *lcRs*. $\sigma_s(E', E)$ is the coarsest solid and locally convex topology on E' which is finer than the weak topology $\sigma(E', E)$ and $\sigma_s(E, E')$ is defined similarly. They are the topologies of uniform convergence on the families of order-bounded subsets of the spaces E and E', respectively.

It is known ([11]) that each l-ideal F of the space $(E, C, \sigma_s(E, E'))$ always carries the Dieudonné topology, i.e. $\sigma_s(E, E')|F = \sigma_s(F, F')$, but the same is not always true for the quotient ([11, Remark on p. 206]). As far as the three-space-problem is concerned, we have

Proposition 1.1. *If* $(*)$ *is a short exact sequence of lcRs and if* $t|F = \sigma_s(F, F')$ *and* $t/F = \sigma_s(E/F, F^\circ)$, *then* $t = \sigma_s(E, E')$, *i.e. the property of "having the Dieudonné topology" is three-space stable.*

Proof: First of all, observe that the topologies $\sigma_s(E, E')$ and t are comparable: $\sigma_s(E, E') \leq t$. Hence, $\sigma_s(E, E')|F \leq t|F = \sigma_s(F, F')$. As $\sigma_s(F, F')$ is the coarsest locally solid topology which is finer than $\sigma(F, F')$, we have $\sigma_s(F, F') \leq \sigma_s(E, E')|F$, too, and so $\sigma_s(E, E')|F = t|F$.

On the other hand, from $\sigma_s(E, E') \leq t$ it follows that $\sigma_s(E, E')/F \leq t/F = \sigma_s(E/F, F^\circ)$. Since we always have $\sigma_s(E/F, F^\circ) \leq \sigma_s(E, E')/F$ ([11]), it follows that $\sigma_s(E, E')/F = t/F$. According to the final observation in the proof of Proposition 2.11 of [12], we obtain that $\sigma_s(E, E') = t$. □

The previous proposition suggests the following question: if $(*)$ is a short exact sequence of locally-o-convex Riesz spaces ([14, p. 155]), is the sequence

$$0 \to (F, C \cap F, (t|F)_s) \xrightarrow{j} (E, C, t_s) \xrightarrow{q} (E/F, C/F, (t/F)_s) \to 0$$

exact? The answer is negative by [11, p. 206].

For further investigations we shall need the following auxiliary assertion.

Lemma 1.2. *Let* $(F, C \cap F, t|F)$ *be a closed l-ideal of an lcRs* (E, C, t) *and let* $(*)$ *be the corresponding short exact sequence. Then the mapping q lifts order-bounded subsets with closure, i.e. for each order-bounded subset A in* E/F *there exists an order-bounded subset B in E such that* $\overline{q(B)} \supset A$.

Proof: Consider the dual spaces (E', C') and $((E/F)', (C/F)') = (F^\circ, F^\circ \cap C')$, the corresponding Dieudonné topologies $\sigma_s(E', E)$ and $\sigma_s(F^\circ, E/F)$ and the transposed mapping $q^t\colon F^\circ \to E'$. Let us prove that $q^t\colon (F^\circ, F^\circ \cap C', \sigma_s(F^\circ, E/F)) \to (E', C', \sigma_s(E', E))$ is a topological injection, i.e. that $\sigma_s(E', E)|F^\circ = \sigma_s(F^\circ, E/F)$. Since the Dieudonné topology σ_s is the topology of uniform convergence on order-intervals, it will follow that q lifts order-bounded subsets with closure.

First of all, as the q-image of each order-bounded subset of E is order-bounded in E/F, we have $\sigma_s(E', E)|F^\circ \leq \sigma_s(F^\circ, E/F)$. Conversely, $\sigma_s(F^\circ, E/F)$ is the coarsest locally solid topology which is finer than the weak topology $\sigma(F^\circ, E/F)$. Also, $\sigma(F^\circ, E/F) = \sigma(E', E)|F^\circ$ and $\sigma_s(E', E)|F^\circ$ is a locally solid topology on F° which is finer than $\sigma(F^\circ, E/F)$. Hence, $\sigma_s(F^\circ, E/F) \leq \sigma_s(E', E)|F^\circ$ and the lemma is proved. □

Remark. The previous assertion can also be proved by the given method in some cases without the assumption of local-convexity of the spaces concerned, provided sufficiently rich dual spaces exist. A similar remark applies for some of the propositions that follow, too. We shall keep to the locally convex case.

2. Spaces of order-barrelled type

It is known that the property of "being order-quasibarrelled" is preserved when passing from an *lcRs* to its arbitrary quotient, but not always when passing to its closed l-ideal ([14]), which is analogous to the corresponding situation for the property of "being quasibarrelled" among locally convex spaces. When the three-space-problem is concerned, without additional assumptions it has the negative answer in the category of locally convex spaces ([12]). However, we shall prove

Proposition 2.1. *Let* $(*)$ *be a short exact sequence of lcRs, where* F *and* E/F *are order-quasibarrelled. Then* E *is an order-quasibarrelled lcRs, too.*

Proof: Let T be a barrel in E which absorbs all order-bounded subsets. Then $T \cap F$ is a barrel in F which absorbs all order-bounded subsets, since each order-bounded subset in $(F, C \cap F)$ is also an order-bounded set in (E, C). Therefore there exists a neighbourhood U of the origin in E such that $T \supset T \cap F \supset (3U) \cap F$. Now, $\overline{q(T \cap U)}$ is a barrel in E/F which absorbs all order-bounded subsets, according to Lemma 1.2, and so, by the assumption, $\overline{q(T \cap U)}$ is a neighbourhood of the origin in E/F. Further procedure is the same as in the proof of Proposition 2.4 [12] — for the set $V = U \cap \overline{T \cap U + F}$ one can prove that it is a neighbourhood of the origin in E and that $V \subset 2T$, and so T is a neighbourhood of the origin in E, too. □

With obvious changes one can prove

Proposition 2.2. *Let* $(*)$ *be a short exact sequence of lcRs in which* F *and* E/F *are countably-order-quasibarrelled* (*COQ* [7]). *Then* E *is COQ; in other words, the property COQ is three-space stable.*

When order-(DF) *lcRs* (i.e. COQ Riesz spaces with a fundamental sequence of order-bounded subsets) are concerned, it was without detailed proof stated in [7] that the property is preserved when passing to an arbitrary quotient. A possible proof can be as follows: let (E, C, t) be an order-(DF) *lcRs* and F its closed l-ideal. Then $(E', C', \sigma_s(E', E))$ is a Fréchet *lcRs*, and so $(F^\circ, C' \cap F^\circ, \sigma_s(F^\circ, E/F))$ is a metrizable *lcRs*, since $\sigma_s(E', E)|F^\circ = \sigma_s(F^\circ, E/F)$ (see the proof of Lemma 1.2). It follows that the quotient $(E/F, C/F)$, as a Riesz space, possesses a fundamental sequence of order-bounded subsets. Since the property of "being COQ" is inherited by quotients, the space $(E/F, C/F, t/F)$ is order-(DF), too.

When three-space stability of the mentioned property is concerned, again to the contrary of the non-ordered case ([12]), we have

Proposition 2.3. *The property of "being order-(DF)" is three-space stable in the class of locally convex Riesz spaces.*

Proof: Let $(*)$ be a short exact sequence of *lcRs* in which F and E/F are order-(DF) spaces. From the previous proposition it follows that E is COQ. Further, the spaces $(F', \sigma_s(F', F))$ and $(F^\circ, \sigma_s(F^\circ, E/F))$ are Fréchet *lcRs* and it should be mentioned that $\sigma_s(F', F) = \beta(F', F)$ (β — the strong topology). Also, $\beta(F', F) = \beta(E', E)/F^\circ$ ([12, Lemma 4.1]). Now we have

$$\sigma_s(F', F) \le \sigma_s(E', E)/F^\circ \le \beta(E', E)/F^\circ = \beta(F', F) = \sigma_s(F', F),$$

and so the following "dual sequence" is exact:

$$0 \to (F^\circ, C' \cap F^\circ, \sigma_s(F^\circ, E/F)) \to \\ \to (E', C', \sigma_s(E', E)) \to (F', C'/F^\circ, \sigma_s(F', F)) \to 0.$$

According to [5, Theorem 6], $(E', C', \sigma_s(E', E))$ is a metrizable locally convex space, which means that (E, C, t) has a fundamental sequence of order-bounded subsets. In that way, it is proved that (E, C, t) is an order-(DF) space. □

We shall finish this section with an investigation of the three-space stability for the order-bound topology, which will be, for the given *lcRs* E, denoted by t_{bE}.

Proposition 2.4. *Let $(*)$ be a short exact sequence of lcRs in which $t|F = t_{bF}$ and $t/F = t_{b(E/F)}$. Then $t = t_{bE}$, i.e. the property of "having the order-bound topology" is three-space stable.*

Proof: First of all, $t \le t_{bE}$, and so $t|F \le t_{bE}|F$. Let $U \subset E$ be an absolutely convex set which absorbs all order-bounded subsets of E. Then $U \cap F$ absorbs all order-bounded subsets of F, and so $t_{bE}|F \le t_{bF}$. Hence, $t_{bF} = t|F \le t_{bE}|F \le t_{bF}$, i.e. $t|F = t_{bE}|F$.

Further, from $t \le t_{bE}$ it follows that $t_{b(E/F)} = t/F \le t_{bE}/F$. On the other hand, $t_{b(E/F)}$ is the finest locally convex topology for which all the order-bounded subsets are bounded, and so $t_{bE}/F \le t_{b(E/F)}$. Thus, $t_{bE}/F = t_{b(E/F)} = t/F$. By the already mentioned final observation in the proof of Proposition 2.11 [12], $t = t_{bE}$. □

3. Distinguished, semi-reflexive and Montel locally convex Riesz spaces

It is known that in some cases order structure provides more regular behaviour of "classical" topological vector properties. A reflexive, even semi-reflexive *lcRs* is always complete ([13, p. 237] or [14, p. 173]), so that in the class of *lcRs* there is no counterpart to Komura's example of an incomplete Montel space. Properties of "being bornological" and "being quasibarrelled" are inherited by each l-ideal (solid subspace) (see [11] and [14, p. 182]), which differs from the locally convex case without order. We shall use these facts to show that each quotient of a Montel (resp. Fréchet-Montel) *lcRs* is again of the same type. This means that in the class of *lcRs* there is no counterpart to the famous Köthe-Grothendieck Fréchet-Montel space whose quotient by some closed subspace is isomorphic to l_1. In other words, Pisier's method cannot be used to construct counterexamples for non-three-space stability of certain properties ([2], [3], [4]). Also, using the same facts we shall derive some conclusions in connection with distinguished and semi-reflexive *lcRs*.

It is known that semi-reflexivity in the class of locally convex spaces is inherited by arbitrary closed subspaces, but not by arbitrary quotients, even in the case of Fréchet spaces. However, we have

Proposition 3.1. *Let E be a Fréchet lcRs and F its closed l-ideal. Then F and E/F are semi-reflexive if and only if E is semi-reflexive.*

Proof: Taking into account Proposition 4.3 of [12], we only have to prove that semi-reflexivity of E implies semi-reflexivity of E/F.

From the semi-reflexivity of E it follows that $E'_\tau = E'_\beta$ is a barrelled locally convex space, i.e. $(E', C', \beta(E', E)) = (E', C', \tau(E', E))$ is a barrelled ([7], [14]) and also bornological *lcRs* (these two properties are in this case equivalent and equivalent to the property of being quasibarrelled ([6])). Furthermore, F° equipped with the topology inherited from $E'_\tau = E'_\beta$ is bornological ([8]), and so quasibarrelled *lcRs* ([14, p. 182]). Also,

$$\begin{aligned}\sigma(F^\circ, E/F) \le \sigma_s(F^\circ, E/F) = \sigma_s(E', E)|F^\circ \le \tau(E', E)|F^\circ \\ = \beta(E', E)|F^\circ \le \tau(F^\circ, E/F) \le \beta(F^\circ, E/F),\end{aligned}$$

and so $(F^\circ, C' \cap F^\circ, \tau(F^\circ, E/F))$ is a bornological (quasibarrelled) *lcRs*, hence a barrelled *lcRs* (because the quotient E/F is Fréchet ([6])). Therefore the space E/F is semi-reflexive. □

Proposition 3.2. *Each quotient E/F of a Montel (Fréchet-Montel) lcRs is Montel (Fréchet-Montel) lcRs.*

Proof: Since E/F is (in both cases) a barrelled *lcRs*, to show that it is Montel it is enough to prove that topologies $\beta(F^\circ, E/F)$ and $c(F^\circ, E/F)$ (topology of uniform convergence on compact absolutely convex subsets of E/F) are equal on F°. Since $E'_c = E'_\beta$, $(E', C', \beta(E', E))$ is a barrelled (and so quasibarrelled) *lcRs*. According to [14, p. 182], $(F^\circ, C' \cap F^\circ, \beta(E', E)|F^\circ)$ is a quasibarrelled *lcRs*. Furthermore, $c(E', E)|F^\circ \leq c(F^\circ, E/F) \leq \tau(F^\circ, E/F) \leq \beta(F^\circ, E/F)$ ([9, 22.2]),

$$c(E', E)|F^\circ = \tau(E', E)|F^\circ = \beta(E', E)|F^\circ \leq \tau(F^\circ, E/F) \leq \beta^*(F^\circ, E/F)$$

and

$$\begin{aligned}(F^\circ, C' \cap F^\circ, \beta(E', E)|F^\circ) &= (F^\circ, C' \cap F^\circ, \tau(F^\circ, E/F)) \\ &= (F^\circ, C' \cap F^\circ, \beta^*(F^\circ, E/F))\end{aligned}$$

and so $c(F^\circ, E/F) = \beta^*(F^\circ, E/F)$, where β^* is the topology of uniform convergence on strongly bounded subsets. Since weakly and strongly bounded subsets in E/F (which is barrelled) are the same, we have that $\beta^*(F^\circ, E/F) = \beta(F^\circ, E/F)$ and the proposition is proved. □

Remark. From [14, p. 187] we know that an l-ideal of a barrelled (resp. order-quasibarrelled) *lcRs* need not be of the same kind.

When the property of distinguishedness is concerned, let us remark the following: In the class of Fréchet locally convex spaces ([3]) there exists a short exact sequence of even reflexive spaces, with the mapping q which does not lift bounded sets. On the other hand, if q lifts bounded sets, then the property of "being distinguished" is three-space stable ([2]). However, in the class of Fréchet *lcRs* distinguishedness of the middle term in a short exact sequence (provided F and E/F are distinguished) is *equivalent* with the lifting of bounded sets of the mapping q. Remark that by a result from [1], for Fréchet locally convex spaces, the properties of "lifting of bounded sets" and "lifting of bounded sets with closure" are equivalent.

In contrast to the Fréchet locally convex spaces, in the class of Fréchet *lcRs* we can prove

Proposition 3.3. *In the class of Fréchet lcRs the property of "being distinguished" is inherited by every closed l-ideal and quotient.*

Proof: Let E in the sequence $(*)$ be a Fréchet *lcRs*. As in the proof of Proposition 3.1 (or 3.2), $(F^\circ, C' \cap F^\circ, E'_\beta|F^\circ)$ is a quasibarrelled *lcRs* and also barrelled, i.e. $\beta(E', E)|F^\circ = \beta(F^\circ, E/F)$. It is a consequence of the fact that in the dual space of an arbitrary barrelled space, a topology which is between the weak and the strong topology, is barrelled if (and only if) it is quasibarrelled, and in that case it is equal to the strong topology. Thus, E/F is a distinguished Fréchet *lcRs*. From the equality of topologies $\beta(E', E)|F^\circ$ and $\beta(F^\circ, E/F)$ it follows that q lifts bounded sets (with closure). Furthermore, if the space E is distinguished, then, on the base of [10, 26.12], the dual sequence

$$0 \to (E/F)'_\beta \to E'_\beta \to F'_\beta \to 0$$

is exact, and so F'_β is a barrelled space, i.e. F is distinguished. □

4. Order-distinguished and order-semi-reflexive spaces

Let (E,C,t) be an *lcRs* and $(E',C',\sigma_s(E',E))$ the corresponding dual *lcRs* equipped with the Dieudonné topology. The topology $\sigma_s(E',E)$ is in general not compatible with the duality $\langle E,E'\rangle$. If E is not an l-ideal in E'^b, then $(E',C',\sigma_s(E',E))' \neq E$. Therefore we shall say that (E,C,t) is an *order-semi-reflexive lcRs* if $E''_{|\sigma|} = (E',C',\sigma_s(E',E))' = E$. Similarly, we say that (E,C,t) is *order-distinguished* if each $\sigma(E''_{|\sigma|},E')$-bounded set is contained in the closure of some order-interval from E, i.e.

$$(\forall\, A \in \mathcal{B}(E''_{|\sigma|},E'))(\exists x \in C\cap E)\, A \subset \overline{[-x,x]}^{\sigma(E''_{|\sigma|},E')}.$$

Let us state some properties of the introduced classes of spaces.

1° If E is an order-distinguished *lcRs*, then each bounded subset A of E is order-bounded. In fact,

$$A = A\cap E \subset \overline{[-x,x]}^{\sigma(E''_{|\sigma|},E')} \cap E = \overline{[-x,x]}^{E} = [-x,x].$$

It follows that each order-distinguished *lcRs* is distinguished.

2° There exist distinguished spaces (even among Banach lattices) which are not order-distinguished. An example can be the space c_0 with canonical order. Namely, it is known that order-intervals in c_0 are compact disks, and so $\sigma(l_1,c_0) \le \sigma_s(l_1,c_0) \le c(l_1,c_0) \le \tau(l_1,c_0)$. If c_0 were order-distinguished, it would follow that $\sigma_s(l_1,c_0) = \tau(l_1,c_0) = \beta(l_1,c_0)$, i.e. in c_0 bounded subsets would be order-bounded, and so each bounded set would be compact, and c_0 would be finite-dimensional.

3° (E,C,t) is order-distinguished if and only if $(E',C',\sigma_s(E',E))$ is order-quasibarrelled, i.e. barrelled. This dual characterization of order-distinguished spaces can be easily checked.

4° The notions "order-distinguished" and "order-semi-reflexive" depend only on the dual pair $\langle E,E'\rangle$, as well as the notions distinguished and semi-reflexive in the class of locally convex spaces.

Passing to the behaviour of the introduced properties in connection with the short exact sequence $(*)$, we shall prove

Proposition 4.1. 1° *In the short exact sequence $(*)$ of Fréchet lcRs let E be order-distinguished. Then E/F is also order-distinguished, while F need not have this property.*

2° *If the spaces F and E/F in the short exact sequence $(*)$ of Fréchet lcRs are order-distinguished, then E has the same property; hence, order-distinguishedness is a three-space stable property in the class of Fréchet lcRs.*

PROOF: 1° Firstly, observe that the space m is an order-distinguished Fréchet *lcRs*, because its bounded subsets are order-bounded, while its l-ideal c_0 is not, as already mentioned.

Let E be an order-distinguished Fréchet *lcRs* and F its closed l-ideal. As stated before, $\beta(E',E) = \sigma_s(E',E)$ and $(E', \sigma_s(E',E))$ is a barrelled, which means here also bornological, locally convex space. Since $\sigma_s(E',E)|F^\circ = \sigma_s(F^\circ, E/F)$ (see the proof of Lemma 1.2), the space $(F^\circ, C' \cap F^\circ, \sigma_s(F^\circ, E/F))$ is bornological, too, and so it is a quasibarrelled *lcRs*, and it also has to be barrelled.

2° Let now the spaces F and E/F in the short exact sequence $(*)$ of Fréchet *lcRs* be order-distinguished, which means also distinguished. Then in their duals the Dieudonné and the strong topologies coincide. But then the mapping q lifts bounded sets because a subset of E/F is bounded if and only if it is order-bounded. Thus, E is a distinguished space ([2]) and the sequence

$$0 \to (E/F)'_\beta \to E'_\beta \to F'_\beta \to 0$$

is exact ([10, 26.12]), i.e. the sequence

$$0 \to (E/F)'_s \to E'_\beta \to F'_s \to 0$$

is exact. Taking into account Proposition 1.1 and the remark [12, p. 23], we obtain that $E'_\beta = E'_s$, and so the space E is order-distinguished. □

Observe that the already mentioned example of spaces c_0 and m shows that in the class of *lcRs* there is no matching result to [10, 26.12]:

$$0 \to c_0 \xrightarrow{j} m \xrightarrow{q} m/c_0 \to 0$$

is a short exact sequence of Fréchet *lcRs* (with norm-topologies and canonical orders), q lifts order-bounded subsets, but the dual sequence

$$0 \to ((m/c_0)', \sigma_s((m/c_0)', m/c_0)) \to (m', \sigma_s(m', m)) \to (c_0', \sigma_s(c_0', c_0)) \to 0$$

is not exact.

Finally, for order-semi-reflexive spaces we have:

Proposition 4.2. *If (E, C, t) is an order-semi-reflexive lcRs, then every closed l-ideal F in it and every quotient E/F are of the same kind.*

Proof: The assertion about l-ideals follows from the fact that each order-interval in F is also an order-interval in E, which means weakly compact in E and also weakly compact in F (as a locally convex space). Since, by order-semi-reflexivity of E, $\sigma(E',E) \le \sigma_s(E',E) \le \tau(E',E)$, we obtain that

$$\sigma(F^\circ, E/F) \le \sigma_s(F^\circ, E/F) = \sigma_s(E',E)|F^\circ \le \tau(E',E)|F^\circ \le \tau(F^\circ, E/F),$$

i.e. E/F is an order-semi-reflexive *lcRs*. □

We do not know whether order-semi-reflexivity is a three-space stable property.

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