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Abstract. A construction is given, in a variety of groups, of a "functorial center" called the endocenter. The endocenter facilitates the identification of universal multiplication groups of groups in the variety, addressing the problem of determining when combinatorial multiplication groups are universal.

Keywords: multiplication group, quasigroup, center

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The theory of quasigroup modules, or quasigroup representation theory, is equivalent to the representation theory of quotients of group algebras of certain groups associated with quasigroups; namely, the stabilizers in the so-called universal multiplication groups (cf. [Sm, p. 56] and below). Universal multiplication groups give functors from varieties of quasigroups to the variety of groups. To help identify these universal multiplication groups we offer a construction (in varieties of groups) of a subgroup we call the endocenter. This endocenter itself gives a functor from varieties of groups to the variety of abelian groups. To a certain extent, the endocenter may be regarded as a "functorial center". We also identify some universal multiplication groups, most notably in $\mathsf{HSP}\{G\}$, the variety generated by a group G. For a quasigroup Q and for any $q \in Q$, the maps

$$\begin{aligned} &R(q) : Q \to Q; \quad x \mapsto x\,q \\ \text{and} \quad &L(q) : Q \to Q; \quad x \mapsto q\,x \end{aligned}$$

are set bijections. As such, they generate a subgroup of the symmetric group $Q!$ on Q. This subgroup is the <u>(combinatorial) multiplication group</u> $\mathrm{Mlt}\,Q$ of Q; i.e. $\mathrm{Mlt}\,Q = \langle R(q), L(q) : q \in Q\rangle_{Q!}$. Unfortunately Mlt (which assigns $\mathrm{Mlt}\,Q$ to Q) does not extend suitably to homomorphisms to give a functor [Sm, p. 28]. To overcome this failure, consider the following construction.

Suppose we have a quasigroup Q and an arbitrary variety $\mathbf{V}$ of quasigroups containing Q. The category whose objects are quasigroups in $\mathbf{V}$ and whose morphisms are quasigroup homomorphisms will also be denoted by $\mathbf{V}$. As an algebraic category, $\mathbf{V}$ is complete and co-complete [HS, 13.12, 13.14]. In $\mathbf{V}$, form the coproduct of Q with $\langle x\rangle$, the free $\mathbf{V}$-algebra on one generator. Denote this coproduct by $Q * \langle x\rangle$. Since Q may be identified with its image in $Q * \langle x\rangle$ [Sm, p. 33], we can consider the subgroup of the combinatorial multiplication group of $Q * \langle x \rangle$ generated by right and left multiplications by elements of Q. This subgroup is the universal multiplication group $U(Q;\mathbf{V})$ of Q in $\mathbf{V}$; i.e. $U(Q;\mathbf{V}) = \langle R(q), L(q) : q \in Q \rangle_{(Q*\langle x \rangle)!}$.

Remarks. 1. The assignment of $U(Q;\mathbf{V})$ to Q gives the promised functor from the category $\mathbf{V}$ to the category $\mathbf{Gp}$ of all groups [Sm, p. 34].

2. $U(Q;\mathbf{V})$ is variety dependent in the sense that, for a given quasigroup Q and varieties $\mathbf{V}_1$ and $\mathbf{V}_2$ containing Q, it is not necessarily the case that $U(Q;\mathbf{V}_1) = U(Q;\mathbf{V}_2)$ [Sm, p.36].

3. If $\mathbf{V}_1 \subseteq \mathbf{V}_2$ then there is a natural group epimorphism $F : U(Q;\mathbf{V}_2) \twoheadrightarrow U(Q;\mathbf{V}_1)$ [Sm, p. 55].

4. For any variety $\mathbf{V}$ of quasigroups containing Q, there is a natural group epimorphism $H : U(Q;\mathbf{V}) \twoheadrightarrow \operatorname{Mlt} Q$ [Sm, p. 55].

Remark 3 can be phrased as: "The smaller the variety, the smaller the universal multiplication group". Remark 4 can be phrased as: "A universal multiplication group can be no smaller than the combinatorial multiplication group". Since the smallest variety containing Q is just $\mathsf{HSP}\{Q\}$, it would be natural to ask whether $U(Q;\mathsf{HSP}\{Q\}) \cong \operatorname{Mlt} Q$, i.e. whether the combinatorial multiplication group is universal. Since lack of associativity leads to complications, we will concentrate on the "easy" case of groups. Thus, from now on G will denote a group and $\mathbf{V}$ an arbitrary variety of groups containing G. In particular, $\mathbf{V}$ could be $\mathsf{HSP}\{G\}$ but it is not required to be so. Theorem 5 below gives a sufficient condition for $U(G;\mathsf{HSP}\{G\}) \cong \operatorname{Mlt} G$. On the other hand, Theorems 6 and 7 furnish examples of groups with $U(G;\mathsf{HSP}\{G\}) \ncong \operatorname{Mlt} G$.

For a group G, the combinatorial multiplication group $\operatorname{Mlt} G$ is given by the exact sequence

$$1 \to Z(G) \xrightarrow{\Delta} G \times G \xrightarrow{F} \operatorname{Mlt} G \to 1,$$

where Δ is the diagonal embedding given by $\Delta : Z(G) \to G \times G; z \mapsto (z,z)$, and where F is the group epimorphism given by $F : G \times G \twoheadrightarrow \operatorname{Mlt} G; (g_1, g_2) \mapsto L(g_1^{-1}) R(g_2)$. Thus,

$$\operatorname{Mlt} G \cong G \times G / \widehat{Z}, \tag{1}$$

where $\widehat{Z} = Z(G)\Delta$. Next, we define the group epimorphism $T : G \times G \to U(G;\mathbf{V})$; $(g_1, g_2) \mapsto L(g_1^{-1}) R(g_2)$. Clearly

$$U(G;\mathbf{V}) \cong G \times G / \operatorname{Ker} T. \tag{2}$$

The map T will play a prominent role throughout, as will its kernel, $\operatorname{Ker} T$. By (1) and (2) it is clear that:

$$\text{If } \operatorname{Ker} T = \widehat{Z}, \text{ then } U(G;\mathbf{V}) \cong \operatorname{Mlt} G. \tag{3}$$

Thus, we note that since G embeds naturally in $G * \langle x \rangle$, it is always the case that

$$\operatorname{Ker} T \leq \widehat{Z}. \tag{4}$$

This discussion leads to two results:

Proposition 1. *If G is an abelian group and $\mathbf{V}$ is any variety of abelian groups containing G, then $\operatorname{Ker} T = \widehat{Z}$ (and hence $U(G;\mathbf{V}) \cong \operatorname{Mlt} G$ by (3)).*

Proposition 2. *If G is a group such that $Z(G) = 1$ and $\mathbf{V}$ is any variety of groups containing G, then $\operatorname{Ker} T = \widehat{Z}$ (and hence $U(G;\mathbf{V}) \cong \operatorname{Mlt} G$ by (3)).*

In the study of these universal multiplication groups (of groups), attention focusses on the behavior of the subgroup $\operatorname{Ker} T$. If $\operatorname{Ker} T = \widehat{Z}$ then we have seen that $U(G;\mathbf{V}) \cong \operatorname{Mlt} G$. If $\operatorname{Ker} T < \widehat{Z}$, and if G satisfies suitable finiteness conditions (most trivially, if G is finite), then we will see that $U(G;\mathbf{V}) \ncong \operatorname{Mlt} G$. An intrinsic description of $\operatorname{Ker} T$ would clearly be beneficial. Towards that end we offer the following

Definition. The endocenter, $Z(G;\mathbf{V})$, of a group G in a variety $\mathbf{V}$ of groups is defined to be:

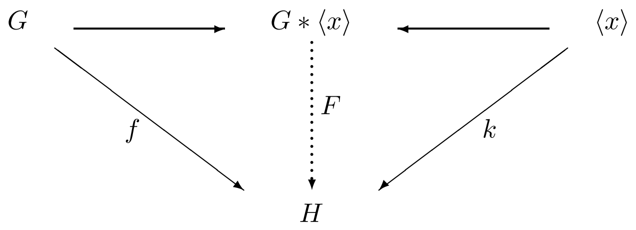
$$Z(G;\mathbf{V}) = \bigcap_{G \le H \in \mathbf{V}} Z(H).$$

The relevance of this definition to representation theory, especially to the study of universal multiplication groups, is seen in

Theorem 3. $Z(G;\mathbf{V})\Delta = \operatorname{Ker} T$.

Proof: First note that $Z(G;\mathbf{V}) \le Z(G * \langle x \rangle)$ since $G * \langle x \rangle \in \mathbf{V}$ and $G \le G * \langle x \rangle$. This means that if $g \in Z(G;\mathbf{V})$, then for every $t \in G * \langle x \rangle$ we have $g^{-1}tg = t$, i.e. $(g,g) \in \operatorname{Ker} T$. Therefore, $Z(G;\mathbf{V})\Delta \le \operatorname{Ker} T$.

Conversely, if $(g,g) \in \operatorname{Ker} T$ and $H \in \mathbf{V}$ with $G \le H$ we need to show that $g \in Z(H)$. So given $h \in H$, we need to show $g^{-1}hg = h$. If we let $f : G \to H$ be the inclusion map, and $k : \langle x \rangle \to H$ be determined by mapping $x \mapsto h$, then since $G * \langle x \rangle$ is a $\mathbf{V}$-coproduct, there exists a unique group homomorphism $F : G * \langle x \rangle \to H$ such that the following diagram commutes:



Since $(g,g) \in \operatorname{Ker} T$, we have $g^{-1}xg = x$. Thus,

$$\begin{aligned} F(g^{-1}xg) &= F(x), \quad \text{which implies} \\ F(g^{-1})F(x)F(g) &= F(x), \quad \text{which implies} \\ f(g^{-1})k(x)f(g) &= k(x), \quad \text{and so} \\ g^{-1}hg &= h, \end{aligned}$$

as desired. Therefore, $\operatorname{Ker} T \le Z(G;\mathbf{V})\Delta$; and hence, $\operatorname{Ker} T = Z(G;\mathbf{V})\Delta$. □

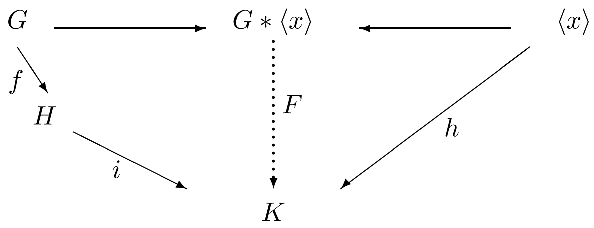
Remark. In light of Theorem 3, we can recast (3) in the following form:

$$\text{If } \; Z(G;\mathbf{V}) = Z(G), \quad \text{then } \; U(G;\mathbf{V}) \cong \operatorname{Mlt} G. \tag{5}$$

The usual center of a group is not a functorial construction. By contrast, the endocenter is natural:

Theorem 4. *$Z(\ ;\mathbf{V})$ is a functor from $\mathbf{V}$ to $\mathbf{Gp}$.*

PROOF: Given a group homomorphism $f : G \to H$, define $Z(f;\mathbf{V})$ to be the restriction of f to $Z(G;\mathbf{V})$. So if $g \in Z(G;\mathbf{V})$, we must show that $f(g) \in Z(H;\mathbf{V})$, i.e. we must show that for a group $K \in \mathbf{V}$ with $H \leq K$ we have $f(g) \in Z(K)$. Hence, given $k \in K$, we must show that $f(g)^{-1}kf(g) = k$. Towards that end, define $h : \langle x \rangle \to K$ to be the unique group homomorphism determined by mapping $x \mapsto k$. Let $i : H \to K$ be the inclusion map. Since $G * \langle x \rangle$ is a $\mathbf{V}$-coproduct, there exists a unique group homomorphism $F : G * \langle x \rangle \to K$ such that the following diagram commutes:



Now $g \in Z(G;\mathbf{V})$ implies that $g \in (G * \langle x \rangle)$, so that

$$\begin{aligned} g^{-1}xg &= x, && \text{which implies} \\ F(g^{-1}xg) &= F(x), && \text{which implies} \\ F(g^{-1})F(x)F(g) &= F(x), && \text{which implies} \\ f(g^{-1})h(x)f(g) &= h(x), && \text{which implies} \\ f(g)^{-1}kf(g) &= k. \end{aligned}$$

Thus $f(g) \in Z(K)$, and hence $f(g) \in Z(H;\mathbf{V})$. It is now easy to check that $Z(f;\mathbf{V}) : Z(G;\mathbf{V}) \to Z(H;\mathbf{V})$ is a group homomorphism and that $Z(\ ;\mathbf{V})$ is a functor. □

Corollary. *$Z(G;\mathbf{V})$ is fully invariant in G.*

PROOF: Suppose $f : G \to G$ is a group endomorphism. By functorality, $Z(f;\mathbf{V})$ is a group homomorphism from $Z(G;\mathbf{V})$ to $Z(G;\mathbf{V})$. But $Z(f;\mathbf{V}) = f\,|_{Z(G;\mathbf{V})}$, so that f maps $Z(G;\mathbf{V})$ to $Z(G;\mathbf{V})$. □

Anticipating the next theorem, we recall the definition of a verbal subgroup: a subgroup H of a group G is verbal if there exists a set W of words such that $H = \langle w(g_1, \dots) : g_i \in G, w \in W \rangle$ [Ne, p. 5]. In the event that $\mathbf{V} = \mathsf{HSP}\{G\}$, Propositions 1 and 2 are special cases of

Theorem 5. *If the center $Z(G)$ of a group G is verbal, then $Z(G;\mathsf{HSP}\{G\}) = Z(G)$. Thus, by (5), $U(G;\mathsf{HSP}\{G\}) \cong \mathrm{Mlt}\, G$.*

Proof: Since $Z(G)$ is a verbal subgroup, there exists a set W of words such that $Z(G) = \langle w(g_1,\dots) : g_i \in G, w \in W\rangle$. Thus, for every $w \in W$,

$$[y, w(x_1,\dots)] = 1 \qquad (6)$$

is an identity in G. By Birkhoff's Theorem (6) is an identity in every group H in $\mathsf{HSP}\{G\}$, in particular in those H for which $G \le H$. So, given $g \in Z(G)$, since $g = w_g(g_1,\dots)$ for some $g_i \in G, w_g \in W$, and since $[y, w_g(x_1,\dots)] = 1$ is an identity in H, we know that $[y,g] = [y, w_g(g_1,\dots)] = 1$ for every $y \in H$. Thus, $g \in Z(H)$, i.e. $g \in Z(G;\mathsf{HSP}\{G\})$. Hence, $Z(G) \le Z(G;\mathsf{HSP}\{G\})$ and we have $Z(G) = Z(G;\mathsf{HSP}\{G\})$, as desired. □

Many familiar groups have verbal centers. For instance abelian groups, simple groups, free groups, symmetric groups, and dihedral groups all have verbal centers. Such groups constitute a fairly large class of groups, and in light of Cayley's theorem and the fact that every group is the homomorphic image of a free group, one might be tempted to think that perhaps $U(G;\mathsf{HSP}\{G\}) \cong \mathrm{Mlt}\, G$ for every group G. Before dispelling this notion, we recall the definition of Hopfian: a group G is said to be Hopfian if it is not isomorphic to a proper quotient of itself [Rb, p. 159].

Theorem 6. *If G is a group such that:*
- (a) $1 < Z(G) < G$;
- (b) $\mathsf{HSP}\{G\} = \mathbf{Gp}$; and
- (c) $G \times G$ is Hopfian,

then $\mathrm{Mlt}\, G \not\cong U(G;\mathsf{HSP}\{G\})$.

Proof: Here we use a fact proved in [Sm, p.35]. Namely, $U(G;\mathbf{Gp}) \cong G \times G$. So suppose on the contrary that $U(G;\mathsf{HSP}\{G\}) \cong \mathrm{Mlt}\, G$. Then

$$\begin{aligned} G \times G &\cong U(G;\mathbf{Gp}) \\ &= U(G;\mathsf{HSP}\{G\}) && \text{[by (b)]} \\ &\cong \mathrm{Mlt}\, G && \text{[by assumption]} \\ &\cong G \times G/\widehat{Z} && \text{by (1).} \end{aligned}$$

This contradicts the Hopfian property of $G\times G$. Therefore, $U(G;\mathsf{HSP}\{G\}) \not\cong \mathrm{Mlt}\, G$. □

To see that there are groups which satisfy the hypotheses of Theorem 6, consider the following

Example. Let $G = \langle x,y,z : [x,z] = [y,z] = 1\rangle$; i.e. G is the direct product of the free group $\langle x,y\rangle$ on two generators with the free (abelian) group $\langle z\rangle$ on one generator. We note that:

(a) $1 < Z(G) < G$ (since $Z(G) = \langle z \rangle$).
(b) $\mathsf{HSP}\{G\} = \mathbf{Gp}$ (since $\langle x, y \rangle$ is clearly a homomorphic image of G, and $\mathsf{HSP}\{\langle x, y \rangle\} = \mathbf{Gp}$ [MKS, p. 413]). And
(c) $G \times G$ is Hopfian (since G is residually finite [MKS, pp. 116, 152] and finitely generated, so too is $G \times G$; and thus $G \times G$ is also Hopfian [MKS, p. 415]).

Applying Theorem 6 yields $U(G; \mathsf{HSP}\{G\}) \ncong \mathrm{Mlt}\, G$.

Clearly, groups satisfying the hypotheses of Theorem 6 belong to a restricted class. For instance, such groups must be infinite. The following theorem provides finite groups for which the combinatorial multiplication group is not universal.

Theorem 7. *If G is a group such that $Z(G)$ is not fully invariant, then $Z(G; \mathbf{V}) < Z(G)$. Suppose further that for normal subgroups N_1, N_2 of G, the proper containment $N_1 < N_2$ implies that $G \times G/N_1 \ncong G \times G/N_2$. Then $U(G; \mathbf{V}) \ncong \mathrm{Mlt}\, G$.*

Proof: By the corollary to Theorem 4, $Z(G; \mathbf{V})$ is fully invariant in G. Since we are assuming that $Z(G)$ is not fully invariant, and since $Z(G; \mathbf{V}) \leq Z(G)$, we have that $Z(G; \mathbf{V}) < Z(G)$ as desired. The final statement follows from the first with $N_1 = Z(G; \mathbf{V})$ and $N_2 = Z(G)$. □

Example. The group $G = A_4 \times Z_2$ (the direct product of the alternating group of order 12 with the cyclic group of order two) has center that is not fully invariant [Rb, p. 30]. Being finite, it also satisfies the further hypothesis of the theorem. Thus, $U(G; \mathsf{HSP}\{G\}) \ncong \mathrm{Mlt}\, G$.

Corollary. *If G is a group with center that is cyclic of prime order, but not fully invariant, and if $\mathbf{V}$ is any variety of groups containing G, then $Z(G; \mathbf{V}) = 1$. Thus, by (2) and Theorem 3, $U(G; \mathbf{V}) \cong G \times G$.*

Example. Let $G = \langle a, b, c : a^2 = b^2 = c^2 = 1, [a, c] = [b, c] = 1 \rangle$. Then G is a group with simple, non-fully invariant center $Z(G) = Z_2$ (the cyclic group of order two). Hence $U(G; \mathsf{HSP}\{G\}) \cong G \times G \ncong \mathrm{Mlt}\, G$.

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