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## Limiting behavior of global attractors for singularly perturbed beam equations with strong damping

DANIEL ŠEVČOVIČ

*Abstract.* The limiting behavior of global attractors  $\mathcal{A}_\varepsilon$  for singularly perturbed beam equations

$$\varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \varepsilon \delta \frac{\partial u}{\partial t} + A \frac{\partial u}{\partial t} + \alpha Au + g(\|u\|_{1/4}^2) A^{1/2} u = 0$$

is investigated. It is shown that for any neighborhood  $\mathcal{U}$  of  $\mathcal{A}_0$  the set  $\mathcal{A}_\varepsilon$  is included in  $\mathcal{U}$  for  $\varepsilon$  small.

*Keywords:* strongly damped beam equation, compact attractor, upper semicontinuity of global attractors

*Classification:* 35B40, 35Q20

### 1. Introduction.

Consider the following problems

$$(1.1)_\varepsilon \quad \begin{cases} \varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \varepsilon \delta \frac{\partial u}{\partial t} + A \frac{\partial u}{\partial t} + \alpha Au + g(\|u\|_{1/4}^2) A^{1/2} u = 0 \\ u(0) = u_0 \\ \frac{\partial u}{\partial t}(0) = v_0 \end{cases}$$

and

$$(1.1)_0 \quad \begin{cases} \frac{\partial u}{\partial t} + \alpha u + g(\|u\|_{1/4}^2) A^{-1/2} u = 0 \\ u(0) = u_0 \end{cases}$$

where  $g$  is an increased  $C^1$  function,  $\varepsilon > 0$  is a small parameter,  $\alpha < 0$  and  $\delta$  is a real unrestricted on the sign. Here  $A$  is a sectorial operator in  $\mathcal{L}_2(0, l)$  defined by a differential operator  $\partial^4/\partial x^4$  and the boundary conditions corresponding either to hinged ends, when

$$(1.2)_H \quad u(x) = u_{xx}(x) = 0 \quad \text{at } x = 0, l$$

or to clamped ends, when

$$(1.2)_C \quad u(x) = u_x(x) = 0 \quad \text{at } x = 0, l.$$

Let  $\{S(t); t \geq 0\}$  be a semidynamical system in a Banach space  $\mathcal{X}$  (for definition, see, for example, [H, Chapter 4]). A set  $J \subseteq \mathcal{X}$  is called *invariant* if  $S(t)J = J$  for all  $t \geq 0$ . An invariant set  $\mathcal{U} \subseteq \mathcal{X}$  is called a *global compact attractor* for the semidynamical system  $S(t)$  if it is a compact set in  $\mathcal{X}$  and  $\lim_{t \rightarrow \infty} \text{dist}(S(t)B, \mathcal{U}) = 0$  for any bounded set  $B \subseteq \mathcal{X}$ , where

$$\text{dist}(\mathcal{A}, \mathcal{B}) = \sup_{x \in \mathcal{A}} \inf_{y \in \mathcal{B}} \|x - y\|.$$

It is shown (Theorem 3.1) that, for small  $\varepsilon$ , there is a compact global attractor  $\mathcal{A}_\varepsilon \subseteq W^{2,2}(0, l) \times \mathcal{L}_2(0, l)$  for a semidynamical system generated by  $(1.1)_\varepsilon$ . For  $\varepsilon = 0$ , the problem  $(1.1)_0$  also has a compact attractor which can be naturally embedded into compact set  $\mathcal{A}_0 \subseteq W^{2,2} \times \mathcal{L}_2(0, l)$ .

Let us note that under the assumptions  $g \geq 0$  and  $\delta \geq 0$ , the dynamics of  $(1.1)_\varepsilon, \varepsilon \geq 0$ , is simple—every trajectory approaches a zero equilibrium state (see Remark 3.2). On the other hand, if  $g(0) < 0$  is sufficiently small, then the attractor  $\mathcal{A}_\varepsilon, \varepsilon \geq 0$ , contains  $2n - 1$  distinct equilibrium states (Remark 3.1) for some  $n \in \mathbb{N}$ . In this case the attractor  $\mathcal{A}_\varepsilon$  is a union of unstable manifolds for equilibrium states (see, for example, [BV, Theorem 10.1]).

The purpose of this paper is to obtain some relationships between the attractors  $\mathcal{A}_\varepsilon$  and  $\mathcal{A}_0$  for small  $\varepsilon$ . It is given in terms of upper semicontinuity of  $\mathcal{A}_0$  at  $\varepsilon = 0$  with respect to the sets  $\{\mathcal{A}_\varepsilon; \varepsilon > 0\}$ .

In this paper, the following hypotheses are needed:

$$(H1) \quad g \in C^1(\mathbb{R}^+, \mathbb{R}); g'(r) > 0 \text{ for } r \geq 0 \text{ and } \int_0^\infty g(s) ds > -\infty$$

$$(H2) \quad \alpha > 0, \delta \in \mathbb{R}.$$

We can now state our main result.

**Theorem 1.1.** *Suppose that the hypotheses (H1)-(H2) are satisfied. Then the attractor  $\mathcal{A}_0$  is upper semicontinuous at zero with respect to the sets  $\mathcal{A}_\varepsilon; \varepsilon > 0$ , i.e.*

$$\lim_{\varepsilon \rightarrow 0^+} \text{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

In other words, for any neighborhood  $\mathcal{U}$  of  $\mathcal{A}_0$ , the set  $\mathcal{A}_\varepsilon$  is included in  $\mathcal{U}$  for  $\varepsilon$  small.

As an example for  $(1.1)_\varepsilon$  one can consider a problem of a transverse motion, at a small strain, in the  $x - y$  plane, of a viscoelastic beam in a viscous medium whose resistance is proportional to the velocity. The ends of the beam are fixed at the points  $x = 0$  and  $x = l + d$ , where  $d$  is a load (positive or negative) of the beam and a stress-free state of the beam occupies the interval  $[0, l]$ . Shear deformations are neglected in this model. Then the equation of the motion in  $y$ -direction is

$$(1.3) \quad \frac{\partial^2 u}{\partial t^2} + \delta \cdot \frac{\partial u}{\partial t} + \frac{\xi I}{\varrho} \cdot A \frac{\partial u}{\partial t} + \frac{EI}{\varrho} Au + \left( \frac{ESd}{l\varrho} + \frac{ES}{2l\varrho} \cdot \int_0^l u_x^2 dx \right) A^{1/2} u = 0$$

where  $E$  is the Young's modulus,  $S$  the cross-sectional area,  $\xi$  the effective viscosity,  $I$  the cross-sectional second moment of area,  $\rho$  the mass per unit length and  $\delta$  the coefficient of external damping. For details see [F], [B1], [B2] and references therein.

Put  $\varepsilon = \frac{\rho}{\xi I} > 0$ . Then the equation  $(1.1)_\varepsilon$  follows from (1.3) by a suitably rescaling the time. The limit  $\varepsilon \rightarrow 0^+$  corresponds to the case in which the effective viscosity tends to  $+\infty$ .

In recent years, many authors have studied the attractors for a singularly perturbed hyperbolic equation

$$(1.4)_\varepsilon \quad \varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u = f(u).$$

See, for example, [GT], [ChL] and other references in [HR1] and [HR2]. Hale and Rougel have shown that the attractors of  $(1.4)_\varepsilon$  converge in the Hausdorff topology towards the one corresponding to  $\varepsilon = 0$

$$(1.4)_0 \quad \frac{\partial u}{\partial t} - \Delta u = f(u).$$

Clearly, the main difference between  $(1.1)_\varepsilon$ - $(1.1)_0$  and  $(1.4)_\varepsilon$ - $(1.4)_0$  is that  $(1.4)_0$  is the quasilinear parabolic equation with an unbounded linear operator  $-\Delta$ , while the problem  $(1.1)_0$  is the quasilinear differential equation in a Hilbert space with a bounded operator  $\alpha \cdot Id$ .

The paper is organized as follows. Definitions and notations are recalled in Section 2. Following the style of Henry's lecture notes [H, Chapter 3, 4], one can obtain a local and global existence of solutions of  $(1.1)_\varepsilon$ . Section 3 deals with the existence and uniform boundedness of attractors  $\mathcal{A}_\varepsilon$ . Section 4 is devoted to the singular equation  $(1.1)_0$ . The proof of the existence of  $\mathcal{A}_0$  is given. In Section 5 we prove Theorem 1.1.

## 2. Preliminaries.

Let  $X = L_2(0, l)$  be a real Hilbert space equipped with its usual scalar product  $(\cdot, \cdot)$  and norm  $\|\cdot\|$ . Define  $A : X \rightarrow X; Au = \partial^4 u / \partial x^4$  for each  $u \in C_B^\infty(0, l)$ , where

$$C_B^\infty(0, l) = \{\Phi \in C^\infty(0, l); \Phi \text{ satisfies b.c. } B\},$$

for  $B = H$  or  $B = C$ . Let  $A$  be the self-adjoint closure in  $X$  of its restriction to  $C_B^\infty(0, l)$ . It is well known that  $A$  is a sectorial operator in  $X$  (see [H, p.19]). Therefore the fractional powers  $A^\beta$  can be defined. Let  $X^\beta$  be a Hilbert space consisting of the domain of fractional power  $A^\beta$  with the graph norm, i.e.  $\|u\|_\beta = \|A^\beta u\|$  for all  $u \in X^\beta$ . Let us note that  $X^\beta \hookrightarrow W^{4\beta, 2}(0, l)$  for  $\beta \geq 0$ . We also have  $\|u\|_\beta \leq \lambda_1^{\beta-\sigma} \|u\|_\sigma$  for any  $0 \leq \beta \leq \sigma$  and  $u \in X^\sigma$ . Recall that  $A$  has a compact resolvent  $A^{-1}$ . Therefore the imbedding  $X^\sigma \hookrightarrow X^\beta$  is compact, whenever  $0 \leq \beta < \sigma$ .

Let  $\Phi_n, j \in \mathbb{N}$ , denote the orthonormal basis of  $X$  consisting of eigenvectors of the operators  $A$ :

$$A\Phi_n = \lambda_n \Phi_n; \quad 0 < \lambda_1 < \lambda_2 < \dots; \lambda_n \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

Denote by  $\mathbb{P}_m$  the projector in  $X$  onto the space spanned by  $\{\Phi_1, \dots, \Phi_m\}$ . Clearly,

$$\|\mathbb{P}_m u\|_\beta \leq \lambda_m^{\beta-\sigma} \|\mathbb{P}_m u\|_\sigma \leq \lambda_m^{\beta-\sigma} \|u\|_\sigma \text{ for each } u \in X^\sigma \text{ and } \beta, \sigma \geq 0.$$

Let  $S(t)$  be a semidynamical system in a Banach space  $\mathcal{X}$ .

A set  $B$  *dissipates* a set  $J$  if there exists  $T = T(J) > 0$  such that  $t \geq T$  implies  $S(t)J \subseteq B$ . A semidynamical system  $S(t)$  is called *bounded dissipative* if there exists a bounded set  $B$  which dissipates all bounded sets.

The *omega-limit set* is defined by

$$\Omega(B) = \bigcap_{t \geq 0} \text{cl} \left( \bigcup_{s \geq t} S(s)B \right) \quad (\text{the closure is taken in } \mathcal{X}).$$

In this paper, the time derivatives will be denoted by

$$\frac{\partial}{\partial t} (\cdot) = (\cdot)'$$

In order to obtain a local and global existence we rewrite (1.1) $_\varepsilon$  as a first order ordinary differential equation in the Hilbert space  $\mathcal{X} = X^{1/2} \times X$ . This is to do by letting  $v = u'$ . Then we can rewrite (1.1) $_\varepsilon$  as

$$(2.1) \quad \frac{d}{dt} \phi(t) + \mathcal{L}_\varepsilon \phi(t) + \mathcal{F}_\varepsilon(\phi(t)) = 0; \quad \phi(0) = \phi_0$$

where

$$\begin{aligned} \phi(t) &= [u(t), v(t)]; \quad \mathcal{L}_\varepsilon[u, v] = [-v, \varepsilon^{-2}A(\alpha u + v) + \varepsilon^{-1}\delta v] \\ \text{and } \mathcal{F}_\varepsilon([u, v]) &= [0, -\varepsilon^{-2}g(\|u\|_{1/4}^2)A^{1/2}u]. \end{aligned}$$

It is known [M1, Theorem 1.1] that the operator  $\mathcal{L}([u, v]) = [-v, A(\alpha u + v)]$  is sectorial in  $X^{1/2} \times X$ . Then Theorem 1.3.2 of [H] demonstrates that the operator  $\mathcal{L}_\varepsilon$  is sectorial in  $\mathcal{X}$ . The domain of  $\mathcal{L}_\varepsilon$  is

$$D(\mathcal{L}_\varepsilon) = \{[u, v] \in X^{1/2} \times X^{1/2}; \alpha u + v \in D(A)\}.$$

From now on we restrict  $\varepsilon_0$  by

$$(H3) \quad \lambda_1 - 2 \cdot \varepsilon_0 |\delta| > 0.$$

Since  $\text{Re } \sigma(A) \geq \lambda_1$ , then, by looking at the spectrum  $\sigma(\mathcal{L}_\varepsilon)$ , we see that

$$(2.2) \quad \text{Re } \sigma(\mathcal{L}_\varepsilon) > \frac{\alpha}{2} \text{ for each } \varepsilon \in (0, \varepsilon_0].$$

Since  $\mathcal{L}_\varepsilon$  is the sectorial operator, then  $-\mathcal{L}_\varepsilon$  generates an analytic semigroup  $\exp(-\mathcal{L}_\varepsilon t)$ . Let  $\omega \in (0, \alpha/2)$ . Due to the estimate (2.2), it follows that there is  $M(\varepsilon) > 0$  such that

$$(2.3) \quad \|\exp(-\mathcal{L}_\varepsilon t)\|_{\mathcal{X}} \leq M(\varepsilon) \cdot e^{-\omega t} \quad \text{for each } t \geq 0.$$

According to [H, Theorems 3.3.3, 3.3.4, 3.4.1 and 3.5.2], the local existence, uniqueness, continuous dependence on initial conditions and continuation of solutions of (2.1) immediately follow. More precisely, for each  $\Phi_0 \in \mathcal{X}$  there exists  $T = T(\Phi_0) > 0$  and a unique function  $\Phi = \Phi(t, \Phi_0)$  such that

$$\Phi \in C([0, t_1] : \mathcal{X}) \cap C_1((t_0, t_1) : \mathcal{X}) \quad \text{for each } 0 < t_0 < t_1 < T,$$

$\Phi(0) = \Phi_0, \Phi(t) \in D(L)$  for each  $t \in (0, T)$  and  $\Phi(t)$  is the solution of (2.1) on the interval of existence  $(0, T)$ .

If we take the scalar product in  $X$  of (1.1) $_\varepsilon$  with  $u'$ , we conclude that

$$(2.4) \quad \frac{1}{2} \frac{d}{dt} \left\{ \alpha \|u\|_{1/2}^2 + \varepsilon^2 \|u'\|^2 + \mathcal{G}(\|u\|_{1/4}^2) \right\} + \|u'\|_{1/2}^2 + \varepsilon \delta \|u'\|^2 = 0$$

where  $\mathcal{G}$  is the primitive of  $g$ , i.e.

$$\mathcal{G}(r) = \int_0^r g(s) ds \quad \text{for } r \geq 0.$$

Thanks to (H1) we infer the existence of  $C_0 > 0$  such that

$$(2.5) \quad g(r) \cdot r \geq \int_0^r g(s) ds \geq -C_0 \quad \text{for each } r \geq 0.$$

From (2.4) we observe that

$$(2.6) \quad \begin{aligned} & \int_0^r \|u'(s)\|_{1/2}^2 ds + \varepsilon^2 \|u'(t)\|^2 + \alpha \cdot \|u(t)\|_{1/2}^2 \leq \\ & \leq \varepsilon^2 \|u'(0)\|^2 + \alpha \cdot \|u(0)\|_{1/2}^2 + \mathcal{G}(\|u(0)\|_{1/4}^2) + C_0 \\ & \quad \text{for each } t \geq 0. \end{aligned}$$

Thus the solutions of (1.1) $_\varepsilon$  and (2.1) exist globally on  $\mathbb{R}^+$ . Hence the initial value problem (2.1) generates a semidynamical system  $\{S_\varepsilon(t); t \geq 0\}$  in  $\mathcal{X}$ , where  $S_\varepsilon(t)\Phi(0) = \Phi_\varepsilon(t, \Phi(0))$  for  $t \geq 0$ .

Since there are many estimates in this paper, we will let  $C_0, C_1, C_2, \dots$  be generic positive constants always assumed to be independent of  $\varepsilon$ .

### 3. The existence and uniform regularity of global attractors.

**Lemma 3.1.** *The semidynamical system  $S_\varepsilon$  is bounded dissipative in  $\mathcal{X}$ . More precisely, there exists a constant  $C_1 > 0$  such that for any  $\varepsilon \in (0, \varepsilon]$  and any bounded set  $B \subseteq X^{1/2} \times X$  there is  $T(\varepsilon, B) > 0$  with the property*

$$t \geq T(\varepsilon, B) \text{ implies} \\ \varepsilon^2 \|v\|^2 + \alpha \|u\|_{1/2}^2 \leq C_1 \text{ for each } (u, v) \in S_\varepsilon(t)B.$$

PROOF: Define a functional  $V_\varepsilon : \mathcal{X} \rightarrow \mathbb{R}$  by

$$V_\varepsilon(\Phi, \Psi) = \frac{1}{2} \left\{ \alpha \|\Phi\|_{1/2}^2 + \varepsilon^2 \|\Psi\|^2 + \mathcal{G}(\|\Phi\|_{1/4}^2) \right\} + b\varepsilon^2(\Phi, \Psi)$$

where  $b$  is a positive real satisfying

$$0 < b < \min \left\{ \alpha, \frac{\sqrt{\alpha\lambda_1}}{2\varepsilon_0}; (\lambda_1 - \varepsilon_0|\delta|) \left( \frac{\lambda_1}{\alpha} + \varepsilon_0^2 + \frac{\varepsilon_0^2\delta^2}{\alpha\lambda_1} \right)^{-1} \right\}.$$

From (2.4) we obtain

$$\begin{aligned} \frac{d}{dt} V_\varepsilon(u_\varepsilon, u'_\varepsilon) &= -\|u'_\varepsilon\|_{1/2}^2 - \varepsilon\delta \|u'_\varepsilon\|^2 + b\varepsilon^2 \|u'_\varepsilon\|^2 - b \cdot (Au'_\varepsilon, u_\varepsilon) - \\ &\quad - b\alpha \cdot (Au_\varepsilon, u_\varepsilon) - b\varepsilon\delta \cdot (u'_\varepsilon, u_\varepsilon) - b \cdot g(\|u_\varepsilon\|_{1/4}^2) \cdot \|u_\varepsilon\|_{1/4}^2 \leq \\ &\leq -\|u'_\varepsilon\|_{1/2}^2 - (\varepsilon\delta - b\varepsilon^2) \cdot \|u'_\varepsilon\|^2 - b\alpha \cdot \|u_\varepsilon\|_{1/2}^2 - b \cdot (A^{1/2}u'_\varepsilon, A^{1/2}u_\varepsilon) - \\ &\quad - b\varepsilon\delta \cdot (u'_\varepsilon, u_\varepsilon) + bC_0. \end{aligned}$$

Then we deduce from the Young's inequality

$$|(\Phi, \Psi)| \leq (r^2 \|\Phi\|^2 + r^{-2} \|\Psi\|^2)/2$$

that

$$\begin{aligned} \frac{d}{dt} V_\varepsilon(u_\varepsilon, u'_\varepsilon) &\leq -\|u'_\varepsilon\|_{1/2}^2 - (\varepsilon\delta - b\varepsilon^2) \cdot \|u'_\varepsilon\|^2 - b\alpha \cdot \|u_\varepsilon\|_{1/2}^2 + bC_0 + \\ &\quad + b \cdot (r^2 \|u'_\varepsilon\|_{1/2}^2 + r^{-2} \|u_\varepsilon\|_{1/2}^2)/2 + b\varepsilon|\delta| \cdot (s^2 \|u'_\varepsilon\|^2 + s^{-2} \|u_\varepsilon\|^2)/2. \end{aligned}$$

Put  $r^2 = 2/\alpha$  and  $s^2 = \frac{2\varepsilon|\delta|}{\alpha\lambda_1}$ . Then

$$\begin{aligned} \frac{d}{dt} V_\varepsilon(u_\varepsilon, u'_\varepsilon) &\leq -\left(1 - \frac{b}{\alpha}\right) \cdot \|u'_\varepsilon\|_{1/2}^2 - (\varepsilon\delta - b\varepsilon^2 - b \frac{\varepsilon^2\delta^2}{\alpha \cdot \lambda_1}) \cdot \|u'_\varepsilon\|^2 - \\ &\quad - b(\alpha - \alpha/4 - \alpha/4) \cdot \|u_\varepsilon\|_{1/2}^2 + bC_0 \leq \\ &\leq -\left(\lambda_1(1 - \frac{b}{\alpha}) + \varepsilon\delta - b\varepsilon^2 - b \frac{\varepsilon^2\delta^2}{\alpha \cdot \lambda_1}\right) \cdot \|u'_\varepsilon\|^2 - b \cdot \frac{\alpha}{2} \cdot \|u_\varepsilon\|_{1/2}^2 + bC_0. \end{aligned}$$

Since  $b \cdot \left( \frac{\lambda_1}{\alpha} + \varepsilon_0^2 + \frac{\varepsilon_0^2 \delta^2}{\alpha \lambda_1} \right) < \lambda_1 - \varepsilon_0 |\delta|$  and  $b < \alpha$ , one can easily show that there are constants  $C_2, C_3 > 0$  such that

$$(3.1) \quad \frac{d}{dt} V_\varepsilon(u_\varepsilon, u'_\varepsilon) \leq -C_2(\|u'_\varepsilon\|^2 + \|u_\varepsilon\|_{1/2}^2) + C_3.$$

Let us introduce a function

$$y_\varepsilon(t) = V_\varepsilon(u_\varepsilon(t), u'_\varepsilon(t)) + C_3.$$

Thanks to the inequality

$$b\varepsilon^2(u'_\varepsilon, u_\varepsilon) \leq \frac{\varepsilon^2}{2} \cdot \|u'_\varepsilon\|^2 + \frac{\varepsilon^2 b^2}{2 \cdot \lambda_1} \cdot \|u_\varepsilon\|_{1/2}^2$$

we have

$$0 \leq y_\varepsilon(t) \leq \alpha \cdot \|u_\varepsilon(t)\|_{1/2}^2 + \varepsilon^2 \|u'_\varepsilon(t)\|^2 + \frac{1}{2} \mathcal{G}(\|u_\varepsilon(t)\|_{1/4}^2) + C_3.$$

Since  $g$  increases on  $\mathbb{R}^+$ , there exists an increasing function  $\vartheta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$  such that

$$0 \leq y_\varepsilon(t) \leq \vartheta(\|u_\varepsilon(t)\|_{1/2}^2 + \|u'_\varepsilon(t)\|^2)$$

and  $\vartheta'(r) \geq \sigma > 0$  for each  $r \geq 0$ .

Then we can rewrite (3.1) as an ordinary differential inequality

$$\frac{d}{dt} y_\varepsilon \leq -C_2 \vartheta^{-1}(y_\varepsilon) + C_3.$$

An obvious contradiction argument gives us either  $0 \leq y_\varepsilon(t) \leq \vartheta(C_3/C_2)$  for each  $t \geq 0$  or there is  $T(\varepsilon, y_\varepsilon(0)) > 0$  such that  $0 \leq y_\varepsilon(t) \leq \vartheta(C_3/C_2) + 1$  for each  $t \geq T(\varepsilon, y_\varepsilon(0))$ . Due to the assumption on  $b$ , it follows that

$$y_\varepsilon(t) \geq \frac{1}{4}(\alpha \|u_\varepsilon(t)\|_{1/2}^2 + \varepsilon^2 \|u'_\varepsilon(t)\|^2) + C_3 - C_0/2.$$

Thus Lemma 3.1 is proved.  $\square$

Consider a solution  $w_\varepsilon$  of the following linear strongly damped evolution equation

$$\varepsilon^2 w_\varepsilon'' + Aw'_\varepsilon + \alpha \cdot Aw_\varepsilon + \varepsilon \delta \cdot w'_\varepsilon + h_\varepsilon = 0$$

where

$$(3.2) \quad h_\varepsilon \in \mathcal{L}_p(\mathbb{R}^+; X) \quad \text{for } p = 2 \text{ or } p = \infty.$$

**Lemma 3.2.** *Assume  $p = 2$  or  $p = \infty$ . Then there are constants  $C_4, C_5, a > 0$  such that*

$$\begin{aligned} & \varepsilon^2 \|\mathbb{P}_m w'_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m w_\varepsilon(t)\|_1^2 \leq \\ & \leq C_4(\varepsilon^2 \|\mathbb{P}_m w'_\varepsilon(0)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m w_\varepsilon(0)\|_1^2) \cdot e^{-2at} + C_5 \|h_\varepsilon\|_{\mathcal{L}_p(\mathbb{R}^+; X)}^2 \\ & \text{for each } t \geq 0; \varepsilon \in (0, \varepsilon_0] \text{ and } m \in \mathbb{N}. \end{aligned}$$

PROOF: Put  $y(t) = \mathbb{P}_m w_\varepsilon(t)$ . Clearly,  $y(t), y'(t) \in D(A)$  for each  $t \geq 0$ . Let us introduce a substitution

$$z = y' + a \cdot y$$

where  $a$  is a positive real satisfying

$$0 < a < \min \left\{ \frac{\alpha}{2}; \frac{\lambda_1 - 2|\delta|\varepsilon_0}{4\varepsilon_0^2}; \frac{\alpha\lambda_1}{4\varepsilon_0} \left( \frac{\varepsilon_0\alpha}{2} + |\delta| \right)^{-1} \right\}.$$

Then

$$(3.3) \quad \varepsilon^2 z' + (A - a\varepsilon^2 + \delta\varepsilon)z + ((\alpha - a)A + a^2\varepsilon^2 - a\delta\varepsilon)y + \mathbb{P}_m h_\varepsilon = 0$$

Take the scalar product in  $X$  of (3.3) with  $Az$  to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \varepsilon^2 \|z\|_{1/2}^2 + (\alpha - a) \|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon) \|y\|_{1/2}^2 \right\} + \\ & + \|z\|_1^2 + (\delta\varepsilon - a\varepsilon^2) \|z\|_{1/2}^2 + a \cdot \left\{ (\alpha - a) \|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon) \|y\|_{1/2}^2 \right\} = \\ & = -(\mathbb{P}_m h_\varepsilon, Az) \leq \frac{1}{2} \cdot \|\mathbb{P}_m h_\varepsilon\|^2 + \frac{1}{2} \cdot \|z\|_1^2. \end{aligned}$$

From the assumption  $a < \frac{\lambda_1 - 2|\delta|\varepsilon_0}{4\varepsilon_0^2}$  we have

$$\theta'(t) + 2a\theta(t) \leq \|h_\varepsilon(t)\|^2 \quad \text{for } t \geq 0$$

where

$$\theta(t) = \varepsilon^2 \|z\|_{1/2}^2 + (\alpha - a) \|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon) \|y\|_{1/2}^2.$$

Therefore

$$\begin{aligned} \theta(t) & \leq \theta(0) \cdot e^{-2at} + \int_0^t e^{-2a(t-s)} \|h_\varepsilon(s)\|^2 ds \leq \\ & \leq \theta(0) \cdot e^{-2at} + C'_5 \|h_\varepsilon\|_{\mathcal{L}_p(\mathbb{R}^+; X)}^2. \end{aligned}$$

Since  $a < \frac{\alpha}{2}$  and  $\frac{a\varepsilon_0}{\lambda_1} \left( \frac{\varepsilon_0\alpha}{2} + |\delta| \right) < \frac{\alpha}{4}$ , then

$$\begin{aligned} & (\alpha - a) \|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon) \|y\|_{1/2}^2 \geq \\ & \geq \frac{\alpha}{2} \cdot \|y\|_1^2 - a\varepsilon_0 \left( \frac{\varepsilon_0\alpha}{2} + |\delta| \right) \cdot \|y\|_{1/2}^2 \geq \frac{\alpha}{4} \cdot \|y\|_1^2. \end{aligned}$$

Then one can easily show that there are  $C_4, C_5 > 0$  such that

$$\begin{aligned} & \varepsilon^2 \|y'(t)\|_{1/2}^2 + \alpha \cdot \|y(t)\|_1^2 \leq \\ & \leq C_4 (\varepsilon^2 \|y'(0)\|_{1/2}^2 + \alpha \cdot \|y(0)\|_1^2) \cdot e^{-2at} + C_5 \|h_\varepsilon\|_{\mathcal{L}_p(\mathbb{R}^+; X)}^2 \end{aligned}$$

as claimed.  $\square$

The solution of (2.1) is given by the variation of constants by the formula

$$S_\varepsilon(t)\Phi_0 = \exp(-\mathcal{L}_\varepsilon t)\Phi_0 + \mathcal{U}_\varepsilon(t)\Phi_0$$

$$\text{where } \mathcal{U}_\varepsilon(t)\Phi_0 = \int_0^t \exp(-\mathcal{L}_\varepsilon(t-s)) \left[ 0, -\varepsilon^{-2}g(\|u_\varepsilon(s)\|_{1/4}^2)A^{1/2}u_\varepsilon(s) \right] ds.$$

Put  $[w_\varepsilon(t), w'_\varepsilon(t)] = \mathcal{U}_\varepsilon(t)[u_0, v_0]$ . Clearly,  $w_\varepsilon$  is a solution of the linear strongly damped evolution equation

$$\begin{aligned} \varepsilon^2 w''_\varepsilon(t) + Aw'_\varepsilon(t) + \alpha Aw_\varepsilon(t) + \varepsilon \delta w'_\varepsilon(t) + h_\varepsilon(t) &= 0 \\ w'_\varepsilon(0) = w_\varepsilon(0) &= 0 \end{aligned}$$

where  $h_\varepsilon(t) = g(\|u_\varepsilon(t)\|_{1/4}^2)A^{1/2}u_\varepsilon(t)$  and  $u_\varepsilon$  is a solution of (1.1) $_\varepsilon$  satisfying the initial conditions

$$u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = v_0.$$

**Lemma 3.3.** *Let  $\varepsilon \in (0, \varepsilon_0]$  be fixed. Then the set  $K_\varepsilon = \bigcup_{t \geq 0} \mathcal{U}_\varepsilon(t)B$  is bounded in  $X^1 \times X^{1/2}$  for any bounded set  $B \subseteq X^{1/2} \times X$ .*

PROOF: Let  $B$  be a bounded set in  $X^{1/2} \times X$ , i.e. there is  $M_1 > 0$  such that

$$\varepsilon^2 \|v\|^2 + \alpha \|u\|_{1/2}^2 + \mathcal{G}(\|u\|_{1/4}^2) \leq M_1 \quad \text{for each } (u, v) \in B.$$

Let  $(u_0, v_0) \in B$  and  $u_\varepsilon$  be a solution of (1.1) $_\varepsilon$  which satisfies the initial data  $u_\varepsilon(0) = u_0, u'_\varepsilon(0) = v_0$ . From (2.6) we have

$$\varepsilon^2 \|u'_\varepsilon(t)\|^2 + \alpha \|u(t)\|_{1/2}^2 \leq M_1 + C_0 = M'_1 \quad \text{for each } t \geq 0.$$

Therefore there exists  $M_2 > 0$  such that

$$\|h_\varepsilon\|_{\mathcal{L}_\infty(\mathbb{R}^+; X)}^2 \leq M_2.$$

Thanks to Lemma 3.2 (with  $p = \infty$ ) we have

$$\varepsilon^2 \|\mathbb{P}_m w'_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m w_\varepsilon(t)\|_1^2 \leq C_5 M_2 \quad \text{for each } t \geq 0 \text{ and } m \in \mathbb{N}.$$

Letting  $m \rightarrow \infty$ , we conclude that

$$\varepsilon^2 \|w'_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|w_\varepsilon(t)\|_1^2 \leq C_5 M_2 = M_3 \quad \text{for each } t \geq 0.$$

Then the arbitrariness of  $(u_0, v_0) \in B$  implies the assertion of Lemma 3.3.  $\square$

**Theorem 3.1.** *Let  $\varepsilon \in (0, \varepsilon_0]$  be fixed. Then there exists a compact global attractor  $\mathcal{A}_\varepsilon$  for  $S_\varepsilon$ . Moreover,  $\mathcal{A}_\varepsilon$  is bounded in  $X^1 \times X^{1/2}$ .*

PROOF: In order to exploit the general results of [GT], we have to show that  $S_\varepsilon$  is bounded dissipative and for any bounded set  $B \subseteq X^{1/2} \times X$  there is a compact set  $K_\varepsilon^B$  which attracts  $B$ , i.e.

$$\lim_{t \rightarrow \infty} \text{dist}(S_\varepsilon(t)B, K_\varepsilon^B) = 0.$$

Clearly, by Lemma 3.1,  $S_\varepsilon$  is bounded dissipative, i.e. there exists a bounded set  $B_\varepsilon$  which dissipates all bounded sets of  $X^{1/2} \times X$ .

Let  $B$  be any bounded set in  $X^{1/2} \times X$ . From Lemma 3.3 we have that

$$K_\varepsilon^B = \bigcup_{t \geq 0} \mathcal{U}_\varepsilon(t)B \quad \text{is bounded in } X^1 \times X^{1/2}.$$

Therefore  $K_\varepsilon^B$  is compact in  $X^{1/2} \times X$ . Since

$$\begin{aligned} \text{dist}(S_\varepsilon(t)B, K_\varepsilon^B) &\leq \sup_{\Phi \in B} \|\exp(-\mathcal{L}_\varepsilon t)\Phi\|_{\mathcal{X}} \leq M(\varepsilon) \exp(-\omega t) \cdot \sup_{\Phi \in B} \|\Phi\|_{\mathcal{X}} \\ &\quad \text{where } \omega \in (0, \frac{\alpha}{2}), \end{aligned}$$

then

$$\lim_{t \rightarrow \infty} \text{dist}(S_\varepsilon(t)B, K_\varepsilon^B) = 0.$$

According to [GT, Proposition 3.1]  $\mathcal{A}_\varepsilon = \Omega(B_\varepsilon)$  is a compact global attractor for  $S_\varepsilon$ . Furthermore, since  $\Omega(B_\varepsilon)$  is the bounded and invariant set then we see that

$$\text{dist}(\Omega(B_\varepsilon), K_\varepsilon^{\Omega(B_\varepsilon)}) = 0.$$

Thus  $\mathcal{A}_\varepsilon = \Omega(B_\varepsilon) \subseteq K_\varepsilon^{\Omega(B_\varepsilon)}$ . Hence  $\mathcal{A}_\varepsilon$  is bounded in  $X^1 \times X^{1/2}$ .  $\square$

**Remark 3.1.** In the general case (under the hypotheses H1-H3) the attractor  $\mathcal{A}_\varepsilon$ ,  $\varepsilon > 0$ , does not reduce to a single point. Indeed, one can consider the case in which

$$-\alpha \sqrt{\lambda_{n+1}} < g(0) \leq -\alpha \sqrt{\lambda_n}$$

where  $0 < \lambda_1 < \lambda_2 < \dots$  are eigenvalues of  $A$  and  $\Phi_k$ ,  $k \geq 1$ , are corresponding orthonormal eigenvectors. Since we assume

$$\int_0^\infty g(s) ds > -\infty \quad \text{and } g \text{ is an increasing function,}$$

the domain of  $g^{-1}$  (the inverse function of  $g$ ) contains a subinterval  $[g(0), 0)$ . Hence

$$w_k^\pm = \left[ \pm \left( g^{-1}(-\alpha \cdot (\lambda_k)^{1/2}) / \lambda_k^{1/2} \right)^{1/2} \cdot \Phi_k, 0 \right] \quad k = 1, 2, \dots, n$$

are non-zero equilibrium states for (2.1),  $\varepsilon > 0$ , which are contained in  $\mathcal{A}_\varepsilon$ .

**Remark 3.2.** If we restrict  $g, \delta$  by  $\delta > -\lambda_1$  and  $g(s) = \beta + k \cdot s$ , where  $k > 0$  and  $\beta > -\alpha\sqrt{\lambda_1}$  then it is known ([B2, Theorem 6]) that every solution of (1.1) $_\varepsilon$ ,  $\varepsilon > 0$ , and its time derivative decay to zero, as  $t \rightarrow +\infty$ . Due to (4.1) it follows that every solution of (1.1) $_0$  also decays to zero. Hence, under the above assumption on  $\delta$  and  $g$ , the dynamics of (2.1),  $\varepsilon > 0$  is very simple—each trajectory approaches a zero equilibrium state.

From the invariance property of  $\mathcal{A}_\varepsilon$  and Lemma 3.1, we infer the following

**Corollary 3.1.**

$$\varepsilon^2 \|v\|^2 + \alpha \cdot \|u\|_{1/2}^2 \leq C_1 \quad \text{for each } \varepsilon \in (0, \varepsilon_0] \text{ and } (u, v) \in \mathcal{A}_\varepsilon.$$

The following lemma gives us the uniform estimate of  $X^1 \times X^{1/2}$ —norm of  $\mathcal{A}_\varepsilon$ , for  $\varepsilon \in (0, \varepsilon_0]$ .

**Lemma 3.4.** *There is  $C_6 > 0$  such that*

$$\begin{aligned} \varepsilon^2 \|u''_\varepsilon(t)\|_{1/2}^2 + \|u'_\varepsilon(t)\|_1^2 + \|u_\varepsilon(t)\|_1^2 &\leq C_6 \\ \text{for each } \varepsilon \in (0, \varepsilon_0], t \in \mathbb{R} \text{ and any orbit} \\ \{(u_\varepsilon(t), u'_\varepsilon(t)); t \in \mathbb{R}\} &\subseteq \mathcal{A}_\varepsilon. \end{aligned}$$

**PROOF:** Let  $m \in \mathbb{N}$  be an arbitrary integer. We take the projection  $\mathbb{P}_m$  of (1.1) $_\varepsilon$  to obtain

$$\varepsilon^2 \mathbb{P}_m u''_\varepsilon + \varepsilon \delta \mathbb{P}_m u'_\varepsilon + A \mathbb{P}_m u'_\varepsilon + \alpha A \mathbb{P}_m u_\varepsilon + g(\|u_\varepsilon\|_{1/4}^2) A^{1/2} \mathbb{P}_m u_\varepsilon = 0.$$

Put  $w_\varepsilon(t) = \mathbb{P}_m u'_\varepsilon(t)$ . Then  $w_\varepsilon$  satisfies the linear strongly damped equation

$$\varepsilon^2 w''_\varepsilon + \varepsilon \delta w'_\varepsilon + A w'_\varepsilon + \alpha A w_\varepsilon + h_\varepsilon = 0$$

where

$$\begin{aligned} h_\varepsilon(t) &= 2g'(\|u_\varepsilon(t)\|_{1/4}^2) \cdot (A^{1/2} u'_\varepsilon(t)), u_\varepsilon(t)) A^{1/2} \mathbb{P}_m u_\varepsilon(t) + \\ &\quad + g(\|u_\varepsilon(t)\|_{1/4}^2) A^{1/2} \mathbb{P}_m u'_\varepsilon(t). \end{aligned}$$

From Corollary 3.1 and (2.6) we infer the existence of  $C_7 > 0$  such that

$$\|h_\varepsilon\|_{\mathcal{L}_2(\mathbb{R}^+; X)} \leq C_7 \quad \text{for each } \varepsilon \in (0, \varepsilon_0].$$

Obviously, we can choose  $C_7$  to be independent of  $\varepsilon$  and  $m \in \mathbb{N}$ .

Recall that  $\mathbb{P}_m w_\varepsilon = w_\varepsilon$ . Then by Lemma 3.2, we have

$$\begin{aligned} \varepsilon^2 \|w'_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|w_\varepsilon(t)\|_1^2 &\leq \\ \leq C_4 (\varepsilon^2 \|w'_\varepsilon(0)\|_{1/2}^2 + \alpha \cdot \|w_\varepsilon(0)\|_1^2) \cdot e^{-2at} &+ C_5 \cdot C_7. \end{aligned}$$

Clearly,

$$\|w_\varepsilon(0)\|_1^2 = \|\mathbb{P}_m u'_\varepsilon(0)\|_1^2 \leq \lambda_m^2 \cdot \|u'_\varepsilon(0)\|^2$$

and

$$\begin{aligned} & \|w'_\varepsilon(0)\|_{1/2} = \|\mathbb{P}_m u''_\varepsilon(0)\|_{1/2} = \\ & = \varepsilon^{-2} \|\mathbb{P}_m(\varepsilon \delta u'_\varepsilon(0) + Au'_\varepsilon(0) + \alpha Au_\varepsilon(0) + g(\|u_\varepsilon(0)\|_{1/4}^2) A^{1/2} u_\varepsilon(0))\|_{1/2} \leq \\ & \leq \varepsilon^{-2} \{ \lambda_m^{3/2} \|u'_\varepsilon(0)\| + \alpha \cdot \lambda_m \|u_\varepsilon(0)\|_{1/2} + \varepsilon |\delta| \lambda_m^{1/2} \|u'_\varepsilon(0)\| + \\ & \quad + \lambda_m^{1/2} |g(\|u_\varepsilon(0)\|_{1/4}^2)| \cdot \|u_\varepsilon(0)\|_{1/2} \}. \end{aligned}$$

Therefore there exists  $M(m) > 0$  and an increasing function  $\rho: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , which is independent of  $\varepsilon$ , such that

$$(3.4) \quad \begin{aligned} & \varepsilon^2 \|w'_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|w_\varepsilon(t)\|_1^2 \leq \\ & \leq \varepsilon^{-4} \cdot M(m) \cdot \rho(\varepsilon^2 \|u'_\varepsilon(0)\|^2 + \alpha \cdot \|u_\varepsilon(0)\|_{1/2}^2) \cdot e^{-2at} + C_5 \cdot C_7. \end{aligned}$$

Let  $T \geq 0$ . We set  $(\bar{u}_\varepsilon(t), \bar{u}'_\varepsilon(t)) = (u_\varepsilon(t - T), u'_\varepsilon(t - T))$  for each  $t \in \mathbb{R}$ . Using the invariance property of  $\mathcal{A}_\varepsilon$ , we have

$$((\bar{u}_\varepsilon(t), \bar{u}'_\varepsilon(t)); t \in \mathbb{R}) \subseteq \mathcal{A}_\varepsilon.$$

Then, from (3.4), we obtain

$$\begin{aligned} & \varepsilon^2 \|\mathbb{P}_m u''_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m u'_\varepsilon(t)\|_1^2 = \\ & = \varepsilon^2 \|\mathbb{P}_m \bar{u}''_\varepsilon(t + T)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m \bar{u}'_\varepsilon(t + T)\|_1^2 \leq \\ & \leq \varepsilon^{-4} M(m) \rho(\varepsilon^2 \|\bar{u}'_\varepsilon(0)\|^2 + \alpha \cdot \|\bar{u}_\varepsilon(0)\|_{1/2}^2) \cdot e^{-2a(t+T)} + C_5 \cdot C_7 \leq \\ & \leq \varepsilon^{-4} \cdot M(m) \cdot \rho(C_1) \cdot e^{-2a(t+T)} + C_5 \cdot C_7. \end{aligned}$$

Then, by letting  $T \rightarrow \infty$ , we obtain

$$\varepsilon^2 \|\mathbb{P}_m u''_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m u'_\varepsilon(t)\|_1^2 \leq 1 + C_5 \cdot C_7.$$

Since  $m \in \mathbb{N}$  was an arbitrary integer then

$$\varepsilon^2 \|u''_\varepsilon(t)\|_{1/2}^2 + \alpha \cdot \|u'_\varepsilon(t)\|_1^2 \leq 1 + C_5 \cdot C_7 \quad \text{for each } t \in \mathbb{R}.$$

According to the equation (1.1) $_\varepsilon$  we have

$$\begin{aligned} \alpha \cdot \|u_\varepsilon(t)\|_1 & \leq \|u'_\varepsilon(t)\|_1 + \varepsilon^2 \|u''_\varepsilon(t)\| + \varepsilon |\delta| \cdot \|u'_\varepsilon(t)\| + \\ & \quad + |g(\|u_\varepsilon(t)\|_{1/4}^2)| \cdot \|u_\varepsilon(t)\|_{1/2}. \end{aligned}$$

Then, with regard to Corollary 3.1, one can easily find the constant  $C_6 > 0$ , as claimed.  $\square$

#### 4. Existence of a global attractor for the equation (1.1)<sub>0</sub>.

We now turn our attention to the limiting equation (1.1)<sub>0</sub>.

$$Au' + \alpha Au + g(\|u\|_{1/4}^2)A^{1/2}u = 0$$

which is equivalent ( $0 \in \rho(A)$ ) to the differential equation in  $X^{1/2}$

$$u' + \alpha u + g(\|u\|_{1/4}^2)A^{-1/2}u = 0.$$

According to the assumption on  $g$ , a local existence uniqueness and continuation of solutions of (1.1)<sub>0</sub> immediately follow from the theory of semilinear abstract evolution equations. See, for example, [H, Theorem 3.3.3, 3.3.4, 3.4.1 and 3.5.2].

We first give some a priori estimates of solutions of (1.1)<sub>0</sub>. Take the scalar product in  $X^{1/2}$  with  $u$  to obtain

$$(4.1) \quad \frac{1}{2} \frac{d}{dt} \|u(t)\|_{1/2}^2 + \alpha \cdot \|u(t)\|_{1/2}^2 + g(\|u(t)\|_{1/4}^2) \cdot \|u(t)\|_{1/4}^2 = 0.$$

Thanks to (2.5) we have

$$(4.2) \quad \|u(t)\|_{1/2}^2 \leq e^{-2\alpha t} \|u(0)\|_{1/2}^2 + \frac{C_0}{\alpha} \cdot (1 - e^{-2\alpha t}).$$

Hence the solution  $u(t)$  exists on  $\mathbb{R}^+$ . We set  $S_0(t)u_0 = u(t)$ , where  $u(t)$  is a solution of (1.1)<sub>0</sub> with  $u(0) = u_0$ . Then, from (4.2), we have that  $S_0$  is the bounded dissipative semidynamical system in  $X^{1/2}$ . Recall that the variation of constants formula gives

$$S_0(t)u_0 = e^{-\alpha t}u_0 + \mathcal{U}_0(t)u_0$$

where

$$\mathcal{U}_0(t)u_0 = \int_0^t e^{-\alpha(t-s)} g(\|u(s)\|_{1/4}^2) A^{-1/2}u(s) ds.$$

From (4.2) one can show that

$$\bigcup_{t \geq 0} \mathcal{U}_0(t)B \text{ is bounded in } X^1,$$

whenever  $B$  is bounded in  $X^{1/2}$ .

Again, by [GT, Proposition 3.1], there exists a compact global attractor  $\tilde{\mathcal{A}}_0$  for  $S_0$  which is bounded in  $X^1$ .

Finally, the attractor  $\tilde{\mathcal{A}}_0$  can be naturally embedded into a compact set  $\mathcal{A}_0$  in  $X^{1/2} \times X$ . The set  $\mathcal{A}_0$  is defined by

$$\mathcal{A}_0 = \left\{ (\Phi, \Psi) \in X^{1/2} \times X; \Phi \in \tilde{\mathcal{A}}_0 \text{ and } \Psi = -\alpha\Phi - g(\|\Phi\|_{1/4}^2)A^{-1/2}\Phi \right\}.$$

Obviously,  $\mathcal{A}_0$  is bounded in  $X^1 \times X^{1/2}$ .

### 5. Upper semicontinuity of attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$ .

Recall that we are going to prove the property

$$\lim_{\varepsilon \rightarrow 0^+} \text{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

In Lemma 3.4, we have shown that there exists  $C_6 > 0$  such that

$$(5.1) \quad \begin{aligned} & \varepsilon^2 \|u_\varepsilon''(t)\|_{1/2}^2 + \|u_\varepsilon'(t)\|_1^2 + \|u_\varepsilon(t)\|_1^2 \leq C_6 \\ & \text{for each } \varepsilon \in (0, \varepsilon_0], t \in \mathbb{R} \text{ and any orbit} \\ & \{(u_\varepsilon(t), u_\varepsilon'(t)); t \in \mathbb{R}\} \subseteq \mathcal{A}_\varepsilon. \end{aligned}$$

Concerning the attractor  $\mathcal{A}_0$ , we have shown that there is  $C_7 > 0$  with the property

$$\|u_0'(t)\|_{1/2}^2 + \|u_0(t)\|_1^2 \leq C_7$$

for any orbit

$$\{(u_0(t), u_0'(t)); t \in \mathbb{R}\} \subseteq \mathcal{A}_0.$$

The idea of the proof is essentially the same as of [HR1]. Let us consider a sequence  $\varepsilon_n \rightarrow 0^+$  and an orbit

$$\{(u_n(t), u_n'(t)); t \in \mathbb{R}\} \subseteq \mathcal{A}_{\varepsilon_n}.$$

Since the set  $\bigcup_{t \in \mathbb{R}} \bigcup_{n \in \mathbb{N}} u_n(t)$  is bounded in  $X^1$  and

$$\|u_n'(t)\| \leq C_6 \quad \text{for each } n \in \mathbb{N} \text{ and } t \in \mathbb{R}.$$

By the Ascoli–Arzelà’s theorem we may thus extract a subsequence  $\{u_{n_1}\}$  of  $\{u_n\}$  which converges to  $\bar{u}$  in the space  $C(\langle -1, 1 \rangle; X^{1/2})$ . Again, there is a subsequence  $\{u_{n_2}\}$  which converges to  $\bar{u}$  in  $C(\langle -2, 2 \rangle; X^{1/2})$ . Thanks to the Cantor’s diagonalization process, there is a subsequence  $\{u_{n_k}\}$  of  $\{u_n\}$  such that  $u_{n_k} \rightarrow \bar{u}$  in  $C(J; X^{1/2})$  for any compact interval  $J \subseteq \mathbb{R}$ . Since

$$\sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}} \|u_n(t)\|_{1/2}^2 < +\infty,$$

then

$$\sup_{t \in \mathbb{R}} \|\bar{u}(t)\|_{1/2}^2 < +\infty.$$

On the one hand  $\frac{\partial u_{n_k}}{\partial t} \rightarrow \frac{\partial \bar{u}}{\partial t}$  in  $\mathcal{D}'(I; X^{1/2})$  (in the sense of distributions) for any bounded open interval  $I \subseteq \mathbb{R}$ .

On the other hand

$$\begin{aligned} u_{n_k}'(t) &= -A^{-1} \left\{ \varepsilon_{n_k}^2 \cdot u_{n_k}''(t) + \varepsilon_{n_k} \delta \cdot u_{n_k}'(t) \right\} - \alpha \cdot u_{n_k}(t) - \\ &\quad - g(\|u_{n_k}(t)\|_{1/4}^2) A^{-1/2} u_{n_k}(t). \end{aligned}$$

From (5.1) we observe that

$$\begin{aligned} \varepsilon_{n_k}^2 \|u_{n_k}''(t)\|_{1/2} &\longrightarrow 0 \quad \text{and} \quad \varepsilon_{n_k} |\delta| \cdot \|u_{n_k}'(t)\| \longrightarrow 0, \\ \text{as } \varepsilon_{n_k} &\longrightarrow 0^+. \end{aligned}$$

Therefore

$$\frac{\partial \bar{u}}{\partial t} = -\alpha \bar{u} - g(\|\bar{u}\|_{1/4}^2) A^{-1/2} \bar{u}.$$

Hence  $\bar{u}(t)$  is the solution of  $(1.1)_0$  which exists and is bounded on  $\mathbb{R}$ . Therefore

$$\{(\bar{u}(t), \bar{u}'(t)); t \in \mathbb{R}\} \subseteq \mathcal{A}_0.$$

Since  $(u_{n_k}(\cdot), u_{n_k}'(\cdot)) \longrightarrow (\bar{u}(\cdot), \bar{u}'(\cdot))$  in  $C(J; X^{1/2})$  for any compact interval  $J \in \mathbb{R}$  then we have

$$(u_{n_k}(0), u_{n_k}'(0)) \longrightarrow (\bar{u}(0), \bar{u}'(0)) \in \mathcal{A}_0 \quad \text{in } X^{1/2} \times X.$$

It means that

$$\lim_{\varepsilon \rightarrow 0^+} \text{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

Indeed, suppose to the contrary that there exists  $\varepsilon_n \longrightarrow 0^+$ ,  $\sigma > 0$  and a sequence  $(u_{n_0}, u_{n_0}') \in \mathcal{A}_{\varepsilon_n}$  such that

$$\text{dist}((u_{n_0}, u_{n_0}'), \mathcal{A}_0) \geq \sigma.$$

Obviously, there are orbits  $\{(u_{\varepsilon_n}(t), u_{\varepsilon_n}'(t)); t \in \mathbb{R}\} \subseteq \mathcal{A}_{\varepsilon_n}$ , for  $n \in \mathbb{N}$ , such that  $u_{\varepsilon_n}(0) = u_{n_0}$  and  $u_{\varepsilon_n}'(0) = u_{n_0}'$ . Then there exists a subsequence  $\varepsilon_{n_k}$  with the property

$$(u_{n_k}(0), u_{n_k}'(0)) \longrightarrow (\bar{u}(0), \bar{u}'(0)) \in \mathcal{A}_0,$$

a contradiction. Hence Theorem 1.1 is proved.  $\square$

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