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MEDIAL RINGS AND AN ASSOCIATED RADICAL

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1. INTRODUCTION AND PRELIMINARIES

This paper initiates the study in-depth associative rings with multiplicative semigroup satisfying the following identities:

1.1 $abcd = acbd$ (medial rings),
1.2 $abc = bac$ (left permutable rings),
1.3 $abc = acb$ (right permutable rings),
1.4 $abc = bac = acb$ (permutable rings).

We call these "the four identities." The terminology used here comes from the groupoid theory work of Ježek and Kepka [13, 14]. Semigroups satisfying these identities have been studied in detail (for example, see: [4, 5, 6, 11, 19, 20, 21, 22, 23, 25, 26]). The terminology used in semigroup theory is varied; e.g. "left (right) normal" and "right (left) commutative" for our "right (left) permutable," in Petrich [23, p. 121] and Nordahl [22].

Rings satisfying one of the four identities are special types of PI-rings; special enough so much more can be said than in the general PI situation. Immediate examples of rings satisfying one of the four identities are commutative rings, nilpotent rings of index $\leqq 4$ (for medial), $\leqq 3$ (for the other three identities), and direct sums of these rings, examples which in some sense are trivial and which did not motivate this study. Instead semigroup and groupoid theory were the motivations, especially through (1) semigroup rings formed from a commutative ring over a semigroup satisfying one of the four identities and (2) some general algebraic operation construction methods developed recently by the authors [1]. In Section 3 we discuss examples arising from these two procedures and several other classes of examples of rings satisfying one of the four identities. These examples will illustrate the complexity of the class of rings under investigation.

In some sense, most noncommutative rings are not medial (e.g., a medial ring with identity is commutative). Yet medial rings (and more generally, rings satisfying one of the four identities) are found as special subrings of every ring, and in a fashion that intimately ties them to the structure of the ring. This is shown in Section 4, through the development of a radical associated with medial rings. For any ring R, define $\mathcal{M}(R) = \{\Sigma x_n(a_nb_n - b_na_n)\, y_n \mid a_n, b_n, x_n, y_n \in R\}$. Then $\mathcal{M}(R)$ is an ideal of R which we call the *medial quasiradical of* R. Define $\bar{\mathcal{M}}(R)$ to be the sum of all ideals I of R such that $\mathscr{M}(I) = I$. Then $\bar{\mathscr{M}}$ defines a radical property and $\bar{\mathscr{M}}(R)$ is called the *medial radical of* R. An ideal of R which is itself a medial ring is called a *medial ideal.* We will prove that there are ideals $H \subseteq T \subseteq L$ of R which are permutable, right permutable, and medial, respectively, such that $\mathscr{M}(R) + L$ and $\mathscr{M}(R) + T$ are right essential in R and $\mathscr{M}(R) + H$ ideal essential in R. If R has D.C.C. on ideals, then $\mathscr{M}(R)$ can be replaced by $\bar{\mathscr{M}}(R)$. Examples are provided which illustrate and delimit the theory.

In Sections 5, 6 and 7 rings satisfying one of the four identities, especially mediality, are studied in depth. Prime, maximal and primitive ideals of medial rings are found to have many of the same properties they do in commutative rings; e.g., all prime ideals are completely prime. Chain conditions on ideals are shown to give some results in the medial case which require chain conditions on left (right) ideals for rings in general. As an example of these results: if R is medial with either A.C.C. or D.C.C. on ideals, then every nil one-sided ideal is nilpotent; in the D.C.C. case, every nonnilpotent ideal contains a nonzero idempotent. We give a series of decomposition theorems for rings satisfying the four identities.

The classification of subdirectly irreducible rings satisfying one of the four identities is taken up in Section 7. The heart of such a ring is either square zero or the ring is a field. In the former case, much can be said about the interrelations between the heart and the ring. As one moves from lesser to stricter conditions among the four identities, the results improve, finally yielding: if S is a permutable ring with D.C.C. on ideals, then S is a subdirect product of rings which are either commutative or nilpotent.

Throughout most of the paper it is possible to obtain a dual result of a given result by judiciously interchanging left and right. Since this is clear in the context of each result, we have often omitted stating these dual propositions.

In this paper ring will mean associative ring, not necessarily with identity (unity element). For any ring R, R^+ denotes the additive group of R, R^{opp} the opposite ring to R (i.e., the multiplication in R^{opp} is given by $a * b = b \cdot a$, where $b \cdot a$ is a product in R), and $N(R)$, (or often for convenience N) the set of nilpotent elements of R. For medial rings, N is an ideal and contains the commutator ideal $\langle R, R\rangle_R$. Recall $\langle R, R\rangle_R$ is the ideal of R generated by the set $[R, R] = \{[a, b] = ab - ba \mid a, b \in R\}$ For any nonempty subset S of R, $\langle S\rangle_R$ will be the ideal of R generated by S. If no ambiguity will result, we will use $\langle R, R\rangle$ and $\langle S\rangle$ for $\langle R, R\rangle_R$ and $\langle S\rangle_R$, respectively. Also $\langle S\rangle_l$ and $\langle S\rangle_r$ will denote, respectively, the left and right of R generated by S. The annihilator sets $l_R(S) = \{r \in R \mid rS = 0\}$, $r_R(S) = \{r \in R \mid Sr = 0\}$, and $\text{ann}_R(S) = \{r \in R \mid Sr = 0 = rS\}$, play a useful role in the theory developed herein. The socle of R, the sum of the minimal ideals of R, is denoted by $\text{Soc}(R)$. We use $R_{n \times n}$ for the full ring of n by n matrices over R.

A ring is *reduced* if it has no nonzero nilpotent elements and *entire* if it has no nonzero divisors of zero. A commutative entire ring is called a *domain.* An ideal which is itself a reduced ring is called a *reduced* ideal.

The symbol Z is reserved for the set of rational integers. Our notation for set inclusion is $A \subseteq B$ and for proper set inclusion, $A \subset B$.

2. BASIC PROPERTIES OF MEDIAL RINGS

Throughout this section R will denote a medial ring. A routine calculation shows: $R \cdot \langle R, R\rangle \cdot R = 0$; so $\langle R, R\rangle \subseteq N$. If S is a left (right) permutable ring, then $\langle S, S\rangle \cdot S = 0$, (respectively: $S \cdot \langle S, S\rangle = 0$).

The following useful and interesting identity was developed by Judith Covington [7].

Proposition 2.1. (Binomial Theorem for Medial Rings). *If $a, b \in R$, then for each integer $n \geqq 2$,*

$$(a + b)^n = \Sigma \binom{n-2}{k} \left[a^{n-k-1}b^k a + b^{n-k-1}a^k b + a^{n-k-1}b^{k+1} + \right.$$
$$\left. + b^{n-k-1}a^{k+1}\right],$$

where $k = 0, \ldots, n - 2$. (Here it is understood that if a^0 or b^0 appear, they are deleted.)

The proof is a straightforward induction on n; however, it differs somewhat from the proof for the Binomial Theorem for commutative rings.

Although it is known from other considerations that N is an ideal for rings satisfying $[x, y]^n = 0$ identically for some fixed n [9, Th. 54], the Binomial Theorem for Medial Rings yields directly that $N = N(R)$ is an ideal.

We shall give a plethora of examples showing medial rings need not be commutative or nilpotent of index $\leqq 4$, or direct sums of such rings. But in some ways the class of medial rings is delicately balanced away from the class of commutative rings, as the following shows.

Proposition 2.2. *If any of the following properties hold, then R is commutative:*

(i) *R contains a two-sided identity element;*

(ii) *R contains an element which is not a left zero divisor and an element which is not a right zero divisor;*

(iii) *R is a semiprime ring (e.g., R prime, reduced, simple, or von-Neumann regular).*

Proposition 2.3. *If I is a minimal ideal of R, then either $I^2 = 0$ or I is a field. Thus, if R is subdirectly irreducible, with heart H, then either $H^2 = 0$ or $R = H$ is a field.*

Proof. A minimal ideal of a ring is either square zero or is a simple ring [9, p. 135]. So $I^2 = 0$, or I is a simple medial ring, a field. If $H^2 \neq 0$, then H is a ring with nonzero identity. It is well known that if the heart of a ring contains a nonzero identity element, then the heart is the whole ring.

From this we immediately obtain a medial version of the Birkhoff Lemma [3].

If $R \neq 0$ has no nilpotent elements and is subdirectly irreducible, then R is a field.

Medial subdirectly irreducible rings will be considered in more detail in Section 7.

Proposition 2.4. *Let X be a one-sided ideal of a ring K. If $[K, K] \subseteq X$, then X is an ideal of K and $\langle K, K \rangle \subseteq X$. If K is medial and $N \subseteq X$, then X is an ideal of K and K/N is a commutative reduced ring.*

Proof. If $x \in X$, $r \in K$, then $rx - xr = y \in [K, K] \subseteq X$. If X is a right ideal, then $rx = xr + y \in X$, and X is a two-sided ideal of K. The left ideal case is similar. So $\langle K, K \rangle \subseteq X$. Now take K to be medial. Since $\langle K, K \rangle \subseteq N$, if $N \subseteq X$, then X/N is an ideal in the commutative reduced ring K/N.

If N is a summand of R, then $R = N \oplus C$, where C is a reduced ring (and hence commutative). As both the theory developed herein and the examples in Section 3 show, this is an exceptional case. However, it is possible to find a (right) ideal B which is a reduced ring such that $N + B$ is in some sense close to being R.

Theorem 2.5. *There exists a commutative right ideal B of R which is maximal among reduced right ideals of R, and $B \oplus N$ is an ideal of R which is essential as a right ideal of R.*

Proof. A standard Zorn's Lemma argument yields a right ideal B of R which is maximal among all reduced right ideals of R. Then B is commutative and by Proposition 2.4, $B \oplus N$ is an ideal of R.

Suppose X is a nonzero right ideal of R such that $X \cap (B + N) = 0$. Then $B \cap X = 0 = X \cap N$. Observe that $XB = 0$. Maximality of B yields $(B + X) \cap N \neq 0$. So there exist nonzero elements $x \in X$, $b \in B$ such that $0 = (b + x)^2 = b^2 + bx + x^2$, or $b^2 + bx = -x^2 \in B \cap X = 0$, contrary to x not nilpotent.

Note. If one replaces "right ideal" by ideal throughout the previous theorem and proof, a similar result is obtained.

Note that if X and Y are right ideals of a left permutable ring and I is an ideal of this ring such that $X \cap Y \subseteq I$, then $(XY)^2 \subseteq YXXY \subseteq Y \cap X \subseteq I$. In particular, if $X \cap Y = 0$, then $(XY)^2 = 0$.

Corollary 2.6. *If R is left permutable and B is as in Theorem 2.5, then $B \cap K \neq 0$ for each nonzero reduced right ideal K of R.*

Proof. Assume $B \cap K = 0$. Then $(BK)^2 = KBBK \subseteq B \cap K = 0$. If $b \in B$, $k \in K$ such that $(b + k)^2 = 0$, then $kb + k^2 = -b^2 \in B \cap K$. This forces $k = b = 0$, and $B + K$ is a reduced right ideal of R, contrary to maximality of B among such right ideals.

This corollary does not hold for arbitrary rings, as can be seen by taking

$$T = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}, \quad \text{and} \quad K = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot T,$$

where F is a field.

Proposition 2.7. *Let R be a right (left) permutable ring and X a left (right)* *ideal of R. Then* X^2 *and* RX (respectively, XR) *are ideals of* R *and are contained in* X.

Proof. $X^2R = XXR = XRX \subseteq X^2$, and $(RX)R = RRX \subseteq RX$.

We will make frequent use of the Peirce decomposition with respect to an idempotent in various guises. With this in mind, the following results, while elementary, are worth listing for future reference.

Proposition 2.8. *Let* I *be an ideal of* R, *and* e *a nonzero idempotent of* R. *Then*

(i) $I = eI \oplus \boldsymbol{r}_I(e)$, *as a direct sum of right ideals of* R, *with* $\boldsymbol{r}_I(e) = \boldsymbol{r}_I(eR)$ *an ideal of* R; *if* $e \in I$, *then* eI *is left permutable with left identity*;

(ii) *if* R *is right permutable, then* eR *is commutative with identity and* eI *is commutative*;

(iii) *if* R *is left permutable, then* eI *is a two-sided ideal of* R;

(iv) *if* R *is permutable, then* $R = eR \oplus \boldsymbol{r}_R(e)$ *as a direct sum of ideals of* R *and* eR *is a commutative ring with identity*;

(v) *if* e *is a right identity for* R, *then* $\boldsymbol{r}_I(e) = \langle R, R\rangle \cap I$;

(vi) *if* $e \in I$ *and* $I \cap \langle R, R\rangle = 0$, *then* e *is central in* R; *if furthermore,* e *is a left identity of the ring* I, *then* $R = I \oplus \boldsymbol{r}_R(I)$.

The proofs are straightforward, but medial or some version of permutable is used in each part. The obvious left-for-right or right-for-left interchanges give analogous results.

Definition 2.9. A nonempty subset X of a ring K is a *medial subset* of K if $abcd = acbd$ for each $a, b, c, d \in X$. Similarly, one can define *left permutable subset*, *right permutable subset*, and *permutable subset*.

Proposition 2.10. *If* X *is a medial (left permutable, right permutable, permutable) subset of a ring* K, *then the subring generated by* X *is a medial (left permutable, right permutable, permutable) ring.*

Proof. First show the desired properties for products of elements from X and then for sums of products.

3. EXAMPLES

In order to motivate and legitimatize the study of medial rings it behooves us to show via interesting examples that this class of rings is a rich one, indeed far richer than the class of rings which are direct sums of commutative or nilpotent rings. In this section we give several general procedures for constructing medial rings from other algebraic systems, including rings, and discuss some specific examples arising from these methods. At the end of the section we point out some places where not to look for medial rings.

If one begins with a medial (left permutable, right permutable, permutable) ring R, which is not commutative, then the ring of polynomials in any number of commuting indeterminants is a medial (left permutable, right permutable, permutable) ring which is not commutative. One can proceed similarly with formal power series rings or formal Laurent series rings over R.

The above assumes a medial, noncommutative ring as input; the following example class makes no such assumption and produces many interesting medial rings.

Example 3.1. Semigroup rings. Let S be a semigroup, written multiplicatively, and let R be a commutative ring. The semigroup ring $R[S]$ is medial (left permutable, right permutable, permutable) when S is medial (left permutable, right permutable, permutable). Semigroups with the desired properties abound (cf. [4, 5, 11, 20, 21, 22, 23, 26]).

Particularly far removed from both commutative rings and nilpotent rings is $R[S]$ where S is a rectangular band [6]. Moreover, these rings are medial, but neither left nor right permutable if the component sets defining the rectangular band S each have at least two elements.

An interesting class of left permutable semigroup rings arise from using a semigroup S defined via either

(i) for each $x \in S$, $xy = y$, for each $y \in S$; or

(ii) S has a two-sided zero z and given $x \in S$, either $xy = z$ for each $y \in S$, or $xy = y$ for each $y \in S$, and each occurs.

For these semigroups and any commutative ring R, $R[S]$ is a left permutable, right duo R-algebra [2]. The special case where $R = Z_2$ and S is a two element semigroup of type (i) is the smallest left permutable ring which is not permutable (and also, of course, the smallest noncommutative ring).

Note that a medial ring which is neither left nor right permutable can be obtained by using this four element semigroup ring, which for convenience we call V. Then $V \oplus V^{\mathrm{opp}}$ is a sixteen element ring with the desired properties. There are none of order lower than eight. It will be shown in Section 6 that the smallest permutable, noncommutative ring has order eight.

Because of the abundance of semigroups with the requisite properties, semigroup rings afford an excellent source of examples or counterexamples in the study of medial rings.

Example 3.2. Let W be a left R-module over a ring R, with R-homomorphisms $f\colon W \to R$ and $h\colon W \to W$, satisfying $h^2 = h$ and $fh = f$. Define $a * b = f(a)\, h(b)$ for each $a, b \in W$. Then $(W, +, *)$ is a ring and f is a ring homomorphism; if $f(W)$ is right permutable (commutative), then $(W, +, *)$ is medial (left permutable) [1, Proposition 4.5]. Some special cases are of particular interest.

3.2.1. Use as W the n by n matrices over a commutative ring R with identity, $n > 1$, and use $f(A) = \operatorname{trace}(A)$, h the identity map. This yields a left permutable ring $(W, +, *)$, which is not permutable. Every element in W with trace one is a left identity of $(W, +, *)$, the trace zero elements form a nilpotent ideal, and the left ideals are exactly the R-submodules of W. Narrowing down to the situation where R is a field, we have that the proper right ideals of $(W, +, *)$ are all contained in the set of trace zero elements, that set being the ideal N of nilpotent elements. So the proper ideals are exactly the R-submodules contained in N. For $n = 2$, $(W, +, *)$ is subdirectly irreducible.

If in the above, R is a right permutable ring which is not commutative, then $(W, +, *)$ is a medial ring which in general is neither left nor right permutable.

3.2.2. A variation on the theme of 3.2.1 is to use $h(a_{ij}) = (b_{ij})$, where $b_{ij} = 0$ if $i \neq j$ and $b_{ii} = a_{ii}$, the diagonalizer mapping. Other variations readily come to mind, e.g., $h(a_{ij}) = (c_{ij})$, where $c_{1j} = a_{1j}$ and $c_{ij} = 0$ otherwise.

3.2.3. Let W be a subdirect product of copies of a field R. For f use the j-th projection mapping on W and let $h = 1_W$. The proper right ideals of $(W, +, *)$ are exactly those subgroups of W^+ whose elements have a zero in the j-th component. Left ideals are exactly the subspaces of W. The set of all elements of W with zero j-th component is N.

3.2.4. Let W be a real or complex vector space with inner product $\langle \mid \rangle$, and let u_0 be a fixed vector in W. Use $f(v) = \langle v \mid u_0 \rangle$ and $h = 1_w$. Then $(W, +, *)$ is a left permutable algebra and if $\|u_0\| \leqq 1$, it is a normed algebra. If W is a Banach space, the algebra is a Banach algebra. Every nonzero idempotent is a left identity and there are uncountably many idempotents. The algebra can be decomposed as $W = W_1 \oplus \oplus W_2$, where W_1 is the subspace generated by u_0 and is a left ideal, and W_2 is the subspace perpendicular to u_0 and is a maximal ideal with $W_2 * W_2 = 0$.

This example suggests the study of normed medial algebras and more generally of medial topological algebras, which will be the subject of a subsequent paper by the authors.

Example 3.3. Let $(S, +, \cdot)$ be a ring with additive endomorphisms f and h satisfying

(1) for each $y \in f(S)$, $x \in h(S)$, $f(yx) = y f(x)$ and $h(yx) = y\, h(x)$, and

(2) $h^2 = h$, $fh = f$.

Define $a * b = f(a)\, h(b)$. Then $(S, +, *)$ is a ring. If $x, y \in f(S)$ implies $xy = yx$, then $(S, +, *)$ is left permutable. If $x, y \in f(S)$, $t \in h(S)$ implies $xyt = yxt$, then $(S, +, *)$ is medial. Less generally, if $(S, \cdot)$ is commutative, then $(S, *)$ is left permutable and if $(S, \cdot)$ is left permutable, then $(S, *)$ is medial. These results can be obtained directly by calculation or as corollaries of more general results in [1, Cor. 3.7, Cor. 4.2].

We next give a specific example of this class of rings.

3.3.1. Let R be a commutative ring with identity and define f and h on the full ring S of n by n matrices over R, $n > 1$, via: for any $A = (a_{ij}) \in S$, $f(A) = (b_{ij})$, where $b_{ij} = 0$ if $i \neq j$ and $b_{ii} = a_{ii}$; $h(A) = (c_{ij})$, where $c_{ij} = 0$ if $i > j$, and

$c_{ij} = a_{ij}$ otherwise. Then $(S, +, *)$ is a left permutable, but not right permutable ring. There are no left identity elements. Every element with all zeroes on the main diagonal is a left annihilator of the ring; if R is a field, these elements are the only nilpotents.

Since a medial group is commutative, group rings are of no interest to us in the context of Example 3.1; however, they do serve as useful examples of medial rings in the context of the following.

Example 3.4. Let R be a medial ring which is not commutative and let G be a commutative group. The group ring $R[G]$ is medial.

Example 3.5. Extending the line of thought in Examples 3.1 and 3.4, if R is a medial ring which is not commutative and S is a commutative semigroup, then $R[S]$ is medial and not commutative. Analogous results accrue when R is left permutable (right permutable, permutable).

Example 3.6. Let W be a free module over a commutative ring R with free basis $b_1, \ldots, b_n$. Define $b_i \,.\, b_j = \sum_k \lambda_{ijk} b_k$, where $\lambda_{ijk} \in R$. Then extend linearly to all of W. This yields an algebra over R (not necessarily associative). If the *structure constants* λ_{ijk} are chosen so that $(b_i \,.\, b_j) \,.\, b_l = b_i \,.\, (b_j \,.\, b_l)$, then the algebra W will be associative. If the structure constants are chosen so $b_1, \ldots, b_n$ is a medial set (left permutable, right permutable, permutable set), then the algebra W is medial (left permutable, right permutable, permutable).

3.6.1. Let R be a commutative ring and W a free R-module with a basis indexed by R, say $\mathscr{B} = \{b_\alpha \mid \alpha \in R\}$. Define $b_\alpha \,.\, b_\beta = ab_\beta$. This yields an associative, left permutable algebra which in general will not be right permutable. (If R is an integral domain, the algebra will not be right permutable.)

We next consider a class of rings which are only medial under very special circumstances.

Proposition 3.7. *Let R be the full ring of n by n matrices, $n \geqq 2$, over a ring K. Then*

(i) *R is medial if and only if $K^4 = 0$;*

(ii) *R is left (right) permutable if and only if $K^3 = 0$.*

Proof. The idea of the proof is conveyed fully in the case $n = 2$. If R is medial, then for each $a, b, c, d \in K$,

$$\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ abcd & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

so $abcd = 0$ and hence $K^4 = 0$. If $K^4 = 0$, then $R^4 = 0$ and R is medial.

If R is left permutable, then for each $a, b, c \in K$,

$$\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ abc & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So $K^3 = 0$. If $K^3 = 0$, then $R^3 = 0$.

The right permutable case is similar.

Note that if $K^4 = 0$, but $K^3 \neq 0$, then R is medial but not left (right) permutable unless K is left (right) permutable.

Similar arguments hold for the lower triangular and upper triangular matrix rings. This shows matrix rings are not a convenient place to look for nonnilpotent medial rings.

It is worth noting that anti-commutative rings (those satisfying $ab = -ba$ identically) are always permutable.

Further examples of rings satisfying one or more of the identities 1.1–1.4 will be given in subsequent sections as the need arises to illustrate the theory. The richness of the class of medial rings should now be abundantly clear.

4. THE MEDIAL RADICAL

In section R will denote a ring *not* necessarily medial. Let $\mathscr{M}(R) = \langle R[R, R]\, R\rangle = \{\Sigma x_n(a_nb_n - b_na_n)\, y_n \mid a_n, b_n, x_n, y_n \in R\}$. Thus $\mathscr{M}$ is a function defined on the class of associative rings that assigns to every ring R a uniquely determined ideal $\mathscr{M}(R)$ of R such that: (i) $h(\mathscr{M}(R)) \subseteq \mathscr{M}(h(R))$ for every ring homomorphism h; (ii) $\mathscr{M}(R/\mathscr{M}(R)) = 0$ (i.e., $R/\mathscr{M}(R)$ is medial); (iii) $\mathscr{M}(X) \subseteq \mathscr{M}(R)$ for any subring X of R. Hence $\mathscr{M}$ is a complete quasiradical in the sense of Maranda-Michler [24, p. 54]. Consequently, we will call $\mathscr{M}(R)$ the *medial quasiradical* of R. Furthermore, a radical property, in the sense of Amitsur and Kurosh [24, p. 12] or [9, p. 3], can be defined via $\mathscr{M}$ by saying a ring X is a radical ring if $\mathscr{M}(X) = X$. We define the radical of R, $\bar{\mathscr{M}}(R)$, to be the sum of all radical ideals of R. The $\bar{\mathscr{M}}$-semisimple class of rings (i.e., R is semisimple if for any ideal $I \neq 0$, $\mathscr{M}(I) \neq I$) is the same as the class of rings which are semisimple with respect to the upper radical property determined by the class of medial rings [9, pp. 5–7]. In this section we will discuss the structure of a ring R in terms of $\mathscr{M}(R)$ and $\bar{\mathscr{M}}(R)$ and show that R is a right essential extension of $\mathscr{M}(R) + \mathscr{L}(R)$, where $\mathscr{L}(R) = l_R([R, R]\, R)$ is a medial ideal of R. Furthermore, $\mathscr{L}(R)$ is a right essential extension of an ideal T of R which is right permutable. Also T is a left essential extension of a permutable ideal of R. These relationships are indicated in Diagram 4.4. $\mathscr{L}(R)$ occupies a prominent position among medial ideals in that if K is a medial ideal of R and $K \cap \mathscr{L}(R) = 0$, then there exists an ideal Y of R such that $Y^4 = 0$ and Y is right essential in K. Finally, an iterative procedure is developed which, in the case of a ring with D.C.C. on ideals, allows us to "shrink" $\mathscr{M}(R)$ down to $\bar{\mathscr{M}}(R)$ and to expand $\mathscr{L}(R)$ to a larger medial ideal H of R such that $\bar{\mathscr{M}}(R) + H$ is right essential in R.

The following basic facts and definitions will be used throughout this section.

(i) Let X be a right R-submodule of a right R-module Y. We write $X \leqq'_R Y$ if, whenever $y \in Y$ and $y \notin X$, then there exists $s \in R$ such that $0 \neq ys \in X$. Observe that this condition implies that X is right essential in Y (i.e., every nonzero R-submodule of Y has nonzero intersection with X).

(ii) $\mathcal{M}(R) \subseteq \langle R, R \rangle$.

(iii) If R has a right identity, then $\mathcal{L}(R) = l_R([R, R])$ and $R[R, R] \subseteq \mathcal{M}(R)$. Thus, if R has an identity, then $\mathcal{M}(R) = \langle R, R \rangle$.

(iv) If I is a left, right, or two-sided ideal of R, then $\mathcal{M}(I)$ is a left, right, or two-sided ideal of R, respectively.

(v) Let I be an ideal of R. If $\mathcal{M}(I) = I$, then $I^2 = I$; hence $[\bar{\mathcal{M}}(R)]^2 = \bar{\mathcal{M}}(R)$.

(vi) $\mathcal{M}^1(R) = \mathcal{M}(R)$, $\mathcal{M}^2(R) = \mathcal{M}(\mathcal{M}(R)), \dots, \mathcal{M}^{n+1}(R) = \mathcal{M}^n(\mathcal{M}(R)), \dots$.

(vii) $\bar{\mathcal{M}}(R) \subseteq \mathcal{M}^n(R) \subseteq (R^4)^n$ for all positive integers n. Hence any nilpotent ring is $\bar{\mathcal{M}}$-semisimple.

(viii) If there exists a positive integer k such that $\mathcal{M}^{k+1}(R) = \mathcal{M}^k(R)$, then $\mathcal{M}^k(R) = \bar{\mathcal{M}}(R)$.

(ix) If X is a subring of R such that $\mathcal{M}(X) = X$, then $\mathcal{M}(\langle X \rangle) = \langle X \rangle$. Thus $\bar{\mathcal{M}}$ is a strong radical since it contains all one-sided $\bar{\mathcal{M}}$-ideals of R [16, p. 49]. In fact, $\bar{\mathcal{M}}(R)$ is the sum of all $\bar{\mathcal{M}}$-right (left) ideals of R.

(x) If I is a minimal ideal, either $\mathcal{M}(I) = I$, or $I^2 = 0$, or I is a field.

Theorem 4.1. *Let I be an ideal of R. Then:*

(i) *$\mathcal{L}(I) = l_I([I, I] I)$ is a medial ideal of R. Hence $\mathcal{L}(I) \langle \mathcal{L}(I), \mathcal{L}(I) \rangle_R \mathcal{L}(I) = 0$*

(ii) *$\mathcal{L}(I) \leqq'_I l_I(\mathcal{M}(I))$ and $[l_I(\mathcal{M}(I))] I \subseteq \mathcal{L}(I)$. Also, if X is a right ideal of I such that $X \cap \mathcal{M}(I) = 0$, then $X \subseteq \mathcal{L}(I)$.*

(iii) *If K is a right ideal of I and $K \leqq'_I \mathcal{L}(I)$, then $\mathcal{M}(I) + K \leqq'_I I$. In particular, $\mathcal{M}(I) = 0$ if and only if $I = \mathcal{L}(I)$.*

(iv) *If X is a right ideal of I, then $(X \cap \mathcal{M}(I)) + (X \cap \mathcal{L}(I)) \leqq'_I X$.*

Proof. (i) Clearly $\mathcal{L}(I) = \{x \in I \mid xabc = xbac \text{ for all } a, b, c \in I\}$ is a left ideal of R. Let $a, b, c \in I$, $x \in \mathcal{L}(I)$, and $y \in R$. Then $xy(ab - ba)c = x(ya)bc - x(yb)ac = x(by)ac - x(ay)bc = xabyc - xbayc = x(ab - ba)yc = 0$. Thus $\mathcal{L}(I)$ is a medial ideal of R.

(ii) Assume $x \in l_I(\mathcal{M}(I))$ but $x \notin \mathcal{L}(I)$. Then there exists $d \in [I, I] I$ such that $0 \neq xd$. Hence $0 = x(d(ab - ba)c) = xd(ab - ba)c$ for all $a, b, c \in I$. Thus $\mathcal{L}(I) \leqq'_I (\mathcal{M}(I))$. Similarly, $[l_I(\mathcal{M}(I))] I \subseteq \mathcal{L}(I)$ and $X \subseteq \mathcal{L}(I)$.

(iii) Assume $x \in I$, but $x \notin \mathcal{M}(I) + K$. If $x \in \mathcal{L}(I)$, then there exists $y \in I$ such that $0 \neq xy \in K$. If $x \notin \mathcal{L}(I)$, then there exists $a, b, c \in I$ such that $0 \neq x(ab - ba)c \in \mathcal{M}(I)$. Therefore, $\mathcal{M}(I) + K \leqq'_I I$.

(iv) See proof of part (iii).

Theorem 4.2. *Let I be an ideal of R and $S = \mathcal{L}(I)$. Then:*

(i) *$l_S([S, S]) = l_S(\langle S, S \rangle_R)$ and $l_I([I, I]) = l_I(\langle I, I \rangle_R)$ are ideals of R which are right permutable.*

(ii) $\boldsymbol{l}_I([I, I]) \leqq'_I \boldsymbol{l}_S([S, S]) \leqq'_S S$.

(iii) $\mathscr{M}(I) + \boldsymbol{l}_I([I, I]) \leqq'_I I$.

(iv) *Let* $H = \boldsymbol{l}_I([I, I])$. *If* $X \subseteq I$ *then* $\boldsymbol{l}_H(X)$ *is an ideal of* I.

(v) *Let* $P = \boldsymbol{l}_S([S, S]) + \langle S, S\rangle_R$. *Then* $P \leqq'_P S$ *and* P *is a right permutable ideal of* R.

Proof. (i) Clearly, $\boldsymbol{l}_S([S, S]) = \{x \in S \mid xab = xba \text{ for all } a, b \in S\}$ is a left ideal of R. Let $a, b \in S$, $x \in \boldsymbol{l}_S([S, S])$ and $y \in R$. Then $xy(ab - ba) = x(ya)\, b - x(yb)\, a = x(by)\, a - x(ay)\, b = xaby - xbay = x(ab - ba)\, y = 0$. Thus $\boldsymbol{l}_S([S, S])$ is an ideal of R. Similarly for $\boldsymbol{l}_I([I, I])$. The remainder of this part is obvious.

(ii) Clearly $\boldsymbol{l}_I([I, I]) \subseteq \boldsymbol{l}_S([S, S])$. If $x \in \boldsymbol{l}_S([S, S])$ such that $x \notin \boldsymbol{l}_I([I, I])$, then there exists $y \in [I, I]$ such that $0 \neq xy \in \boldsymbol{l}_S(R) \subseteq \boldsymbol{l}_S([S, S])$. Hence $\boldsymbol{l}_I([I, I]) \leqq'_I \boldsymbol{l}_S([S, S])$. Similarly, $\boldsymbol{l}_S([S, S]) \leqq'_S S$.

(iii) This part follows from part (ii) and Theorem 4.1 (iii).

(iv) Clearly $\boldsymbol{l}_H(X)$ is a left ideal of R. Let $a \in \boldsymbol{l}_H(X)$, $y \in I$, and $x \in X$. Since $a \in H$, then $ayx = axy = 0$. Thus $\boldsymbol{l}_H(X)$ is an ideal of I.

(v) Clearly $P \leqq'_P S$. Let $x, x_1, x_2 \in \boldsymbol{l}_S([S, S])$ and $c, c_1, c_2 \in \langle S, S\rangle_R$. Then

$$\begin{aligned}(x + c)\,&[(x_1 + c_1)(x_2 + c_2) - (x_2 + c_2)(x_1 + c_1)] = \\ &= (x + c)\,[x_1x_2 + c_1x_2 + c_1c_2 - x_2x_1 - c_2x_1 - c_2c_1] = \\ &= (x + c)(x_1x_2 - x_2x_1) = c(x_1x_2 - x_2x_1)\,,\end{aligned}$$

since all x's and c's are elements of S and $x \in \boldsymbol{l}_S([S, S])$. Without loss of generality, assume $c = k(ab - ba)\, h$, where $a, b \in S$ and $k, h \in R$. Consider

$$\begin{aligned}kabh(x_1x_2 - x_2x_1) &= (ka)(bh)\, x_1x_2 - (ka)(bh)\, x_2x_1 \\ &= (ka)\, x_1(bh)\, x_2 - (ka)\, x_2(bh)\, x_1 \quad \text{(since } S \text{ is medial)} \\ &= (ka)\, x_1x_2(bh) - (ka)\, x_2x_1(bh) \quad \text{(since } x_1, x_2 \in \boldsymbol{l}_S([S, S])) \\ &= (ka)\,[x_1x_2 - x_2x_1]\,(bh) = 0 \quad \text{(since } ka \in S).\end{aligned}$$

Similarly, $kbah(x_1x_2 - x_2x_1) = 0$. Hence, $c(x_1x_2 - x_2x_1) = 0$. Therefore, P is a right permutable ideal of R.

From Theorem 4.2 we see that if I is medial, then I is "essentially" right permutable, and that I is right permutable if and only if $I = \boldsymbol{l}_I([I, I])$.

Theorem 4.3. *Let* I *be an ideal of* R, $S = \mathscr{L}(I)$, *and* T *any ideal of* R *which is right permutable and* $T \subseteq S$ (*e.g.*, $T = \boldsymbol{l}_I([I, I])$, *or* $\boldsymbol{l}_S([S, S])$). *Then*:

(i) $\boldsymbol{r}_T([T, T])$ *is a permutable ideal of* R.

(ii) *If* $x \in T$ *such that* $x \notin \boldsymbol{r}_T([T, T])$, *then there exists* $y \in [T, T]$ *such that* $0 \neq yx \in \boldsymbol{r}_T(S) \subseteq \boldsymbol{r}_T([T, T])$. *In particular*, $\boldsymbol{r}_T([T, T])$ *is left essential in* T.

(iii) *If* $T \leqq'_I S$, *then* $\mathscr{M}(I) + \boldsymbol{r}_T([T, T])$ *has nonzero intersection with every nonzero ideal of the ring* I.

(iv) *There exists a left ideal* Y *of* T (*of* R *contained in* T) *such that* $\langle T, T\rangle_R \cap Y = 0$, $\langle T, T\rangle_R \oplus Y \subseteq \boldsymbol{r}_T([T, T])$, *and* $\langle T, T\rangle_R \oplus Y$ *is a commutative ideal of* T

(and a left ideal of R) which is left essential in T. Furthermore, $\mathcal{M}(I) + (\langle T, T\rangle_R \oplus$ $\oplus\, Y)$ *has nonzero intersection with every nonzero ideal of R.*

Proof. (i) Clearly $r_T([T, T]) = \{x \in T \mid abx = bax$ for all $a, b \in T\}$ is a right ideal of R. Let $a, b \in T$, $x \in r_T([T, T])$, and $y \in R$. Then $(ab - ba)\, yx = a(by)\, x - bayx =$ $= a(yb)\, x - bayx$ (since $a \in S$) $= [(ay)\, b - b(ay)]\, x = 0$. Hence $yx \in r_T([T, T])$. Therefore, $r_T([T, T])$ is a permutable ideal of R.

(ii) Clear.

(iii) Let $x \in I$ and $x \notin \mathcal{M}(I) + r_T([T, T])$. If $x \notin S$, then there exists $a, b, c \in I$ such that $0 \neq x(ab - ba)\, c \in \mathcal{M}(I)$. So assume $x \in S$. Either $x \in T$ or there exists $k \in I$ such that $0 \neq xk \in T$. Then from part (ii), we have either $0 \neq yx \in r_T([T, T])$ or $0 \neq vxk \in r_T([T, T])$ for some y or $v \in I$.

(iv) By Zorn's Lemma there exists a left ideal of T (or R contained in T) which is maximal with respect to zero intersection with $\langle T, T\rangle_R$. Hence $\langle T, T\rangle_R \oplus Y$ is left essential in T. Since $T\langle T, T\rangle_R = 0$, $\langle T, T\rangle_R \oplus Y \subseteq r_T([T, T])$ and $\langle T, T\rangle_R \oplus$ $\oplus\, Y$ is a commutative ideal of T (a left ideal of R).

Example 4.5. Let

$$R = \begin{pmatrix} A & X \\ 0 & C \end{pmatrix},$$

where A and C are rings and X is an A, C – bimodule.

4.5.1. Let $A = X = C = F$, where F is a field. Then R is a subdirectly irreducible ring which has D.C.C. on ideals and is $\bar{\mathcal{M}}$-semisimple, but which is not medial. The components of Diagram 4.4 are as follows:

$$\mathcal{M}(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = \langle R, R\rangle\,, \quad \mathcal{M}^2(R) = 0 = \bar{\mathcal{M}}(R)\,,$$

$$\mathcal{L}(R) = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix} = l_R([R, R])\,; \quad \text{for} \quad T = \mathcal{L}(R)\,,$$

then

$$r_T([T, T]) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = \langle T, T\rangle_R\,.$$

We observe that in this case $\mathcal{M}(R) \subseteq \mathcal{L}(R)$ and that $\mathcal{L}(R)$ is right permutable. Also

$$r_R([R, R]) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$$

is a left permutable ideal of R.

4.5.2. Let

$$A = \begin{pmatrix} 2Z & 2Z \\ 0 & 2Z \end{pmatrix}, \quad X = \begin{pmatrix} 0 & Z_{128} \\ 0 & Z_{128} \end{pmatrix}, \quad C = 2Z\,.$$

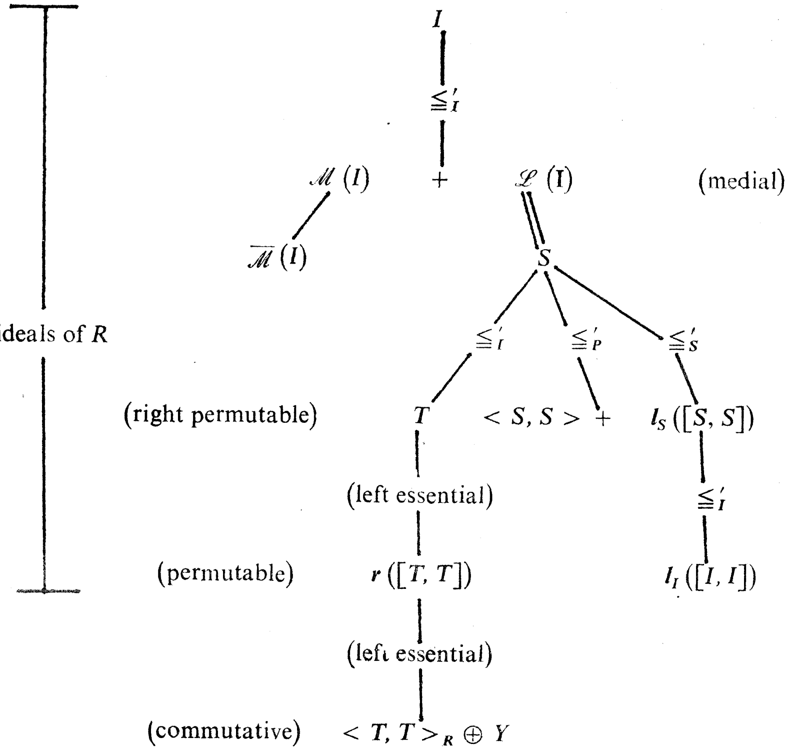

Diagram 4.4

Then R is a $\overline{\mathscr{M}}$-semisimple ring which is not medial $\mathscr{M}(R) \neq \langle R, R\rangle$, $(\mathscr{M}(R))^2 = 0 = \mathscr{M}^2(R)$, $\mathscr{M}(R) \nsubseteq \mathscr{L}(R)$, and $\mathscr{L}(R)$ is right permutable. Some of the components of Diagram 4.4 are:

$$\langle R, R\rangle = \begin{pmatrix} \begin{pmatrix} 0 & 4Z \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2Z_{128} \\ 0 & 2Z_{128} \end{pmatrix} \\ 0 & 0 \end{pmatrix},$$

$$\mathscr{M}(R) = \begin{pmatrix} \begin{pmatrix} 0 & 16Z \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 8Z_{128} \\ 0 & 8Z_{128} \end{pmatrix} \\ 0 & 0 \end{pmatrix},$$

$$\mathscr{L}(R) = \begin{pmatrix} \begin{pmatrix} 0 & 32Z \\ 0 & 32Z \end{pmatrix} & \begin{pmatrix} 0 & Z_{128} \\ 0 & Z_{128} \end{pmatrix} \\ 0 & 2Z \end{pmatrix} = l_S([S, S]),$$

$$l_R([R, R]) = \begin{pmatrix} \begin{pmatrix} 0 & 64Z \\ 0 & 64Z \end{pmatrix} & \begin{pmatrix} 0 & Z_{128} \\ 0 & Z_{128} \end{pmatrix} \\ 0 & 2Z \end{pmatrix},$$

for $T = \mathscr{L}(R)$

$$r_T([T,T]) = \begin{pmatrix} \begin{pmatrix} 0 & 32Z \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & Z_{128} \\ 0 & Z_{128} \end{pmatrix} \\ 0 & 64Z \end{pmatrix}.$$

4.5.3. Let $A = X = D$ and $C = F$, where D is a noncommutative division ring and F is a subfield of D (e.g., center of D). Then R has D.C.C. on ideals, and $0 \neq \bar{\mathscr{M}}(R) = \mathscr{M}(R) \not\subseteq \mathscr{L}(R)$. Some of the components of Diagram 4.4 are: $\bar{\mathscr{M}}(R) = \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix} = \langle R, R \rangle$, $\mathscr{L}(R) = \begin{pmatrix} 0 & D \\ 0 & F \end{pmatrix} = l_R([R,R])$. Note that $r_R([R,R]) = 0$, hence $\bar{\mathscr{M}}(R)$ is left essential in R and $\mathscr{L}(R)$ is right essential in R.

4.5.4. Let $A = F_{2\times 2}$, where F is a field, $X = 0$, and C be a medial ring. Then $R \approx A \oplus C$, $\bar{\mathscr{M}}(R) = A$ and $\mathscr{L}(R) = C$.

The next two results exhibit some connections between $\mathscr{L}(R)$ and arbitrary medial ideals.

Proposition 4.6. *If K is a medial ideal of R, then $K^2[R,R]K^2 = K^3[R,R]K = (K[R,R]K)^2 = (K\langle R,R\rangle)^4 = 0$. In particular, if $K \cap \mathscr{L}(R) = 0$, then $K \cap \mathscr{M}(R) \leqq'_R K$ and $(K \cap \mathscr{M}(R))^5 = 0$.*

Proof. Let $k_1, k_2, k_3, k_4 \in K$, and $a, b \in R$. Consider $k_1(k_2a)(bk_3)k_4 = k_1(bk_3)k_2(ak_4) = k_1(k_2b)k_3(ak_4) = k_1k_3(k_2ba)k_4 = k_1k_2bak_3k_4$. Hence, $k_1k_2(ab - ba)k_3k_4 = 0$. The remainder of the proof follows from the fact that K is a medial ideal and Theorem 4.1 (iv).

Corollary 4.7. *If $\mathscr{M}(R)$ contains no nonzero nilpotent ideal of R, then $\mathscr{L}(R) = l_R(\mathscr{M}(R))$. Hence $\mathscr{L}(R)$ contains every medial ideal of R.*

Proof. The proof follows from Proposition 4.6 and Theorem 4.1 (ii).

Proposition 4.8. *If H is a right permutable ideal of R, then $H^3[R,R] = (H \cap \langle R,R\rangle)^4 = H[R,R]H^2 = 0$. In particular, if $H \cap l_R([R,R]) = 0$, then $H^3 = 0$.*

Proof. Let $h_1, h_2, h_3 \in H$ and $a, b \in R$. Consider $h_1h_2(h_3a)b = h_1(h_3a)(h_2b) = h_1(h_2b)h_3a = h_1h_3h_2ba = h_1h_2h_3ba$. Therefore, $h_1h_2h_3(ab - ba) = 0$. The remainder of the proof follows from the fact that H is right permutable.

Proposition 4.9. *Let K be a medial ideal of R. Then:*

(i) *$N(K)$ is an ideal of R. In particular, $N(\mathscr{L}(R))$ is a nil ideal of R.*

(ii) *If $x \in N(R)$ and $y \in R$ such that $xy \in K$, then $xy \in N(K)$.*

(iii) *If X is any nil or nilpotent ideal of K, then $\langle X\rangle_R$ is a nil or nilpotent ideal of R, respectively.*

Proof. The proof is a straightforward application of the following observation. Let $k_1, k_2, \ldots, k_n \in K$ and $y_1, y_2, \ldots, y_n \in R$ where $n \geqq 4$. Consider $k_1(y_1k_2y_2)$.

$. k_3y_3k_4y_4 \dots k_ny_n = k_1k_3(y_1k_2y_2y_3)\, k_4y_4 \dots k_ny_4 = \dots = k_1k_3 \dots$
$\dots k_{n-1}(y_1k_2y_2y_3 \dots y_{n-1})\, k_ny_n.$

Theorem 4.10. *Let I be an ideal of R and $I_1 = \mathscr{M}(I) + \mathscr{L}(I)$, $I_2 = \mathscr{M}(I_1) + \mathscr{L}(I_1), \dots, I_{n+1} = \mathscr{M}(I_n) + \mathscr{L}(I_n)$ for $n = 0, 1, 2, \dots$, and $I = I_0$. Then:*

(i) $\mathscr{L}(I) \subseteq \mathscr{L}(I_1) \subseteq \dots \subseteq \mathscr{L}(I_n) \subseteq \dots$ *is an ascending chain of medial ideals of R.*

(ii) $I_{n+1} \leqq'_{I_n} I_n$, *hence* $I_n \leqq'_I I$.

(iii) $\mathscr{M}^2(I_n) \subseteq \mathscr{M}(I_{n+1}) \subseteq \mathscr{M}^2(I_n) + [\mathscr{M}(I_n)]\,[\mathscr{L}(I_n)]$.

(iv) $I_{n+1} = \mathscr{M}^{n+1}(I) + \mathscr{L}(I_n)$. *In particular, if $\mathscr{M}^{n+1}(I) = 0$, then $\mathscr{L}(I_n) \leqq'_I I$.*

Proof. (i) From Theorem 4.1 $\mathscr{L}(I_n)$ is a medial ideal of R. Let $x \in \mathscr{L}(I_n) \subseteq I_{n+1}$. Since $I_{n+1} \subseteq I_n$, then $x(ab - ba)\, c = 0$ for all $a, b, c \in I_{n+1}$. Hence $x \in \mathscr{L}(I_{n+1})$.

(ii) Assume $x \in I_n$, but $x \notin I_{n+1}$. Then there exists $a, b, c \in I_n$ such that $0 \neq x(ab - ba)\, c \in \mathscr{M}(I_n) \subseteq I_{n+1}$. Hence $I_{n+1} \leqq'_{I_n} I_n$.

(iii) Let $v \in \mathscr{M}(I_{n+1})$. Without loss of generality, assume $v = (y_1 + a_1)\,[(y_2 + a_2) \cdot (y_3 + a_3) - (y_3 + a_3)(y_2 + a_2)]\,(y_4 + a_4)$ where $y_i \in \mathscr{M}(I_n)$ and $a_i \in \mathscr{L}(I_n)$ for $i = 1, 2, 3, 4$. Since $[\mathscr{L}(I_n)]\,[\mathscr{M}(I_n)] = 0$ (Theorem 4.1 (ii)) and $a_1 \in \mathscr{L}(I_n)$, then

$$\begin{aligned} v &= y_1[y_2y_3 + y_2a_3 + a_2a_3 - y_3y_2 - y_3a_2 - a_3a_2]\,(y_4 + a_4) = \\ &= y_1[y_2y_3 - y_3y_2]\, y_4 + \\ &+ y_1[y_2y_3 + y_2a_3 + a_2a_3 - y_3y_2 - y_3a_2 - a_3a_2]\, a_4 \in \\ &\in \mathscr{M}^2(I_n) + [\mathscr{M}(I_n)]\,[\mathscr{L}(I_n)]\,. \end{aligned}$$

Hence $\mathscr{M}^2(I_n) \subseteq \mathscr{M}(I_{n+1}) \subseteq \mathscr{M}^2(I_n) + [\mathscr{M}(I_n)]\,[\mathscr{L}(I_n)]$.

(iv) Clearly, $I_1 = \mathscr{M}^1(I) + \mathscr{L}(I_0)$. By part (iii), $\mathscr{M}^2(I) \subseteq \mathscr{M}(I_1) \subseteq \mathscr{M}^2(I) + [\mathscr{M}(I)]\,[\mathscr{L}(I)]$. But $[\mathscr{M}(I)]\,[\mathscr{L}(I)] \subseteq \mathscr{L}(I) \subseteq \mathscr{L}(I_1)$, by part (i). Hence $I_2 = \mathscr{M}^2(I) + \mathscr{L}(I_1)$. Again, by part (iii), $\mathscr{M}^2(I_1) \subseteq \mathscr{M}(I_2) \subseteq \mathscr{M}^2(I_1) + [\mathscr{M}(I_1)] \cdot [\mathscr{L}(I_1)]$. Clearly, $\mathscr{M}^3(I) \subseteq \mathscr{M}^2(I_1) \subseteq \mathscr{M}(\mathscr{M}^2(I) + [\mathscr{M}(I)]\, \mathscr{L}(I)])$. If we take $v_1 \in \mathscr{M}(\mathscr{M}^2(I) + [\mathscr{M}(I)]\,[\mathscr{L}(I)])$ and calculate in a manner similar to that for v in part (iii), we have $\mathscr{M}^2(I_1) \subseteq \mathscr{M}(\mathscr{M}^2(I) + [\mathscr{M}(I)]\,[\mathscr{L}(I)]) \subseteq \mathscr{M}^3(I) + [\mathscr{M}(I)]\,[\mathscr{L}(I)]$. Thus $\mathscr{M}^3(I) \subseteq \mathscr{M}(I_2) \subseteq \mathscr{M}^3(I) + \mathscr{L}(I_1)$. Hence $\mathscr{M}^3(I) + \mathscr{L}(I_2) \subseteq \mathscr{M}(I_2) + \mathscr{L}(I_2) = I_3 \subseteq \mathscr{M}^3(I) + \mathscr{L}(I_1) + \mathscr{L}(I_2) = \mathscr{M}^3(I) + \mathscr{L}(I_2)$. Therefore, $I_3 = \mathscr{M}^3(I) + \mathscr{L}(I_2)$. Consequently, iterating on this process yields $I_{n+1} = \mathscr{M}^{n+1}(I) + \mathscr{L}(I_n)$ for $n = 0, 1, 2, \dots$.

We note that Theorem 4.10 (iv) shows that a nilpotent ideal is "essentially" medial. This can also be easily verified directly. From Example 4.5.1 we observe that

$$\boldsymbol{r}_R([R, R]) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$$

is a left permutable ideal. Now $R = \boldsymbol{l}_R([R, R]) + \boldsymbol{r}_R([R, R])$. Hence a sum of medial ideals is not necessarily medial. However, the next result shows that such a sum is "essentially" medial.

Corollary 4.11. *Let $K = \sum_{i=1}^{n} K_i$, where the K_i are medial ideals of R. Then there exists a medial ideal W of R such that $W \leqq'_K K$.*

Proof. Let $V = \Sigma(K_i \cap \mathcal{M}(K))$. By Proposition 4.6, V is a nilpotent ideal of R. Let $P = V + \mathcal{L}(K)$. Let $0 \neq \Sigma k_i \in K$, where $k_i \in K_i$. If $\Sigma k_i \notin \mathcal{L}(K)$, then there exists $a, b, c \in K$ such that $0 \neq (\Sigma k_i)(ab - ba) c \in V$. Thus $P \leqq'_K K$. Let $x \in \mathcal{M}(P)$. Without loss of generality, assume $x = (v_1 + a_1)[v_2 + a_2)(v_3 + a_3) - (v_3 + a_3) \cdot (v_2 + a_2)](v_4 + a_4)$, where $v_j \in V$ and $a_j \in \mathcal{L}(K)$ for $j = 1, 2, 3, 4$. Since $[\mathcal{L}(K)][(\mathcal{M}(K)] = 0$ (Theorem 4.1 (ii)) and $a_1 \in \mathcal{L}(K)$, then $x = v_1[v_2v_3 + v_2a_3 + a_2a_3 - v_3v_2 - v_3a_2 - a_3a_2](v_4 + a_4) = v_1[v_2v_3 - v_3v_2] v_4 + v_1[v_2v_3 + v_2a_3 + v_2a_3 + a_2a_3 - v_3v_2 - v_3a_2 - a_3a_2] a_4 \in \mathcal{M}(V) + [\mathcal{M}(K)][\mathcal{L}(K)]$. Hence $\mathcal{M}(P)$ is a nilpotent ideal of R. Thus there exists a positive integer n such that $\mathcal{M}^n(P) = 0$. By Theorem 4.10 (iv), there exists $W \leqq'_K K$.

Corollary 4.12. *Let R be a ring with* D.C.C. *on ideals and $R_1 = \mathcal{M}(R) + \mathcal{L}(R)$ $R_2 = \mathcal{M}(R_1) + \mathcal{L}(R_1), \ldots, R_{n+1} = \mathcal{M}(R_n) + \mathcal{L}(R_n)$ for $n = 0, 1, 2, \ldots$ and $R = R_0$. Then:*

(i) *There exists an integer n such that $\bar{\mathcal{M}}(R) + \mathcal{L}(R_n) \leqq'_R R$.*

(ii) *Let $S = \mathcal{L}(R_n)$ and $H = l_S([S, S])$. Then H is an ideal of R which is right permutable, $H \leqq'_S S$, and $\bar{\mathcal{M}}(R) + H \leqq'_R R$.*

(iii) *If $\bar{\mathcal{M}}(R) = 0$ and I is an ideal of R, then there exists a medial ideal K of R such that $K \leqq'_I I$.*

(iv) *If $\mathcal{L}(R_n) = 0$ and I is an ideal of R, then there exists an ideal Y of R such that $\bar{\mathcal{M}}(I) + Y \leqq'_I I$ and $Y^4 = 0$.*

Proof. (i) From Theorem 4.10 there exists n such that $R_n = R_{n+1} = \ldots$. Hence $\mathcal{M}^{n+1}(R) = \bar{\mathcal{M}}(R)$. Thus $\bar{\mathcal{M}}(R) + \mathcal{L}(R_n) \leqq'_R R$.

(ii) This part follows from Theorem 4.2.

(iii) This is a consequence of Theorem 4.10 and the fact that $\bar{\mathcal{M}}(I) \subseteq \bar{\mathcal{M}}(R)$.

(iv) From Theorems 4.1 and 4.10 there exists an integer n such that $I_n = I_{n+1} + \ldots$ and $I_{n+1} = \bar{\mathcal{M}}(I) + \mathcal{L}(I_n) \leqq'_I I$. Let $Y = \mathcal{L}(I_n) \cap \mathcal{M}(R)$. By Proposition 4.6, $Y^4 = 0$. By Theorem 4.10 (i), $\mathcal{L}(R) = 0$. From Theorem 4.1 (iv), $Y \leqq'_T \mathcal{L}(I_n)$. Now Theorem 4.1 (iii) and Theorem 4.10 yield $\bar{\mathcal{M}}(I) + Y \leqq'_I I$.

Corollary 4.13. *Let R be a semiprime ring. Then:*

(i) *$\mathcal{L}(R)$ is a reduced commutative ideal of R which contains all medial ideals of R.*

(ii) *$\mathcal{L}(R) = l_R([R, R]) = l_R(\mathcal{M}^n(R))$, and $\mathcal{M}^n(R) \oplus \mathcal{L}(R) \leqq'_{P_n} R$ for $n = 1, 2, \ldots,$ where $P_n = \mathcal{M}^n(R) \oplus \mathcal{L}(R)$.*

(iii) *If R has* D.C.C. *on ideals, then $\mathcal{L}(R)$ (as a ring) has* D.C.C. *on annihilators, $\mathcal{L}(R)$ contains all $\bar{\mathcal{M}}$-semisimple ideals of R, and $\bar{\mathcal{M}}(R) \oplus \mathcal{L}(R) \leqq'_P$ where $P = \bar{\mathcal{M}}(R) \oplus \mathcal{L}(R)$.*

Proof. (i) By Theorem 4.1 (i), $\langle \mathcal{L}(R), \mathcal{L}(R) \rangle_R$ is a nilpotent ideal of R, hence $\mathcal{L}(R)$ is commutative. From Corollary 4.7 and Proposition 4.9, $\mathcal{L}(R)$ is reduced and contains all medial ideals of R.

(ii) From Proposition 4.6, $\mathscr{L}(R) \cap [R, R] = 0$, hence $\mathscr{L}(R) = l_R([R, R])$ by Theorem 4.2 (ii). From part (i) and Theorem 4.10, $\mathscr{L}(R) = \mathscr{L}(R_n)$ for all n. Since $l_R(\mathscr{M}(R)) \cap \mathscr{M}(R) = 0$, Theorem 4.1 shows that $\mathscr{L}(R) = l_R(\mathscr{M}(R))$. By Theorem 4.10, $\mathscr{M}^n(R) \oplus \mathscr{L}(R) \leqq'_R R$ for all n. Now let X be a nonzero right ideal of R such that $0 = X \cap \mathscr{M}^n(R)$ for some $n > 1$. Assume $0 \neq y \in X \cap \mathscr{M}(R)$. There exists $x \in \mathscr{M}^n(R)$ and $a \in \mathscr{L}(R)$ such that either $y = x + a$ or there exists $d \in R$ such that $0 \neq yd = x + a$. Hence $a \in \mathscr{L}(R) \cap \mathscr{M}(R) = 0$. Thus $0 \neq x \in X \cap \mathscr{M}^n(R)$. Contradiction! Hence $X \subseteq \mathscr{L}(R)$. Therefore, $l_R(\mathscr{M}^n(R)) \subseteq \mathscr{L}(R)$. Consequently $l_R(\mathscr{M}^n(R)) = \mathscr{L}(R)$. Hence $P \leqq'_P R$.

(iii) From part (ii) and Theorem 4.2 (iv), $\mathscr{L}(R)$ has D.C.C. on annihilators. There exists n such that $\mathscr{M}^n(R) = \bar{\mathscr{M}}(R)$. From part (ii), $\mathscr{L}(R) = l_R(\bar{\mathscr{M}}(R))$, so $\mathscr{L}(R)$ contains all $\bar{\mathscr{M}}$-semisimple ideals of R; and $\bar{\mathscr{M}}(R) \oplus \mathscr{L}(R) \leqq'_R R$.

Proposition 4.14. (i) *Let X be an ideal of R such that $\mathscr{M}(X) = X$. Then $\mathscr{M}(X_{n\times n}) = X_{n\times n}$ for any positive integer n. Consequently, $(\bar{\mathscr{M}}(R))_{n\times n} \subseteq \bar{\mathscr{M}}(R_{n\times n})$.*

(ii) *R is medial (right permutable) if and only if the left ideal*

$$\begin{bmatrix} R & 0 \\ R & 0 \end{bmatrix}$$

of $R_{2\times 2}$ is medial (right permutable).

Proof. (i) The result is a consequence of the following calculation which generalizes to the $n \times n$ case. Let $a_j, b_j, x_j, y_j \in X$ for $j = 1, 2, 3, 4$. Then

$$\begin{bmatrix} x_1(a_1b_1 - b_1a_1)\,y_1 & x_2(a_2b_2 - b_2a_2)\,y_2 \\ x_3(a_3b_3 - b_3a_3)\,y_3 & x_4(a_4b_4 - b_4a_4)\,y_4 \end{bmatrix} =$$

$$= \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} - \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \right) \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 \\ x_3 & x_4 \end{bmatrix} \left(\begin{bmatrix} a_3 & 0 \\ 0 & a_4 \end{bmatrix} \begin{bmatrix} b_3 & 0 \\ 0 & b_4 \end{bmatrix} - \begin{bmatrix} b_3 & 0 \\ 0 & b_4 \end{bmatrix} \begin{bmatrix} a_3 & 0 \\ 0 & a_4 \end{bmatrix} \right) \begin{bmatrix} y_3 & 0 \\ 0 & y_4 \end{bmatrix}.$$

(ii) The part is also a straightforward calculation.

We note that in Proposition 4.14 (ii) the size of the matrix ring could be $n \times n$ and any left ideal with R's in a column and zeros elsewhere could be used in place of

$$\begin{bmatrix} R & 0 \\ R & 0 \end{bmatrix}.$$

This leads to the surprising fact that a $\bar{\mathscr{M}}$-radical ring can be a direct sum of medial left ideals. Let

$$R = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$$

where F is a field. Then R is a simple $\bar{\mathscr{M}}$-radical ring. However,

$$R = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$$

is a direct sum of right permutable left ideals of R. Also observe that $\bar{\mathscr{M}}(F)_{2\times 2} = 0 + \bar{\mathscr{M}}(F_{2\times 2}) = F_{2\times 2}$. More generally, if R is any semiprime ring, Proposition 3.7 shows that $R_{n\times n}$ $(n > 1)$ contains no medial ideals. If R also has D.C.C. on ideals, then by Corollary 4.13, $R_{n\times n}$ is "essentially" a $\bar{\mathscr{M}}$-radical ring.

Proposition 4.15. *Let* $R[x]$ *and* $R[[x]]$ *denote the ring of polynomials and the ring of formal power series over a commuting indeterminate* x, *respectively.* Then $(\mathscr{M}(R))[x] = \mathscr{M}(R[x])$ and $(\mathscr{M}(R))[[x]] = \mathscr{M}(R[[x]])$. Hence $(\bar{\mathscr{M}}(R))[x] \subseteq \bar{\mathscr{M}}(R[x])$ and $(\bar{\mathscr{M}}(R))[[x]] \subseteq \bar{\mathscr{M}}(R[[x]])$.

Proof. A straightforward calculation shows that $\mathscr{M}(R[x]) \subseteq (\mathscr{M}(R))[x]$. Let $a, b, c, d \in R$ and consider $c(ab - ba)\,dx^n \in (\mathscr{M}(R))[x]$. Then $c(ab - ba)\,dx^n = c(a(bx^n) - (bx^n)\,a)\,d \in \mathscr{M}(R[x])$. It follows that $(\mathscr{M}(R))[x] = \mathscr{M}(R[x])$. Similarly for $(\mathscr{M}(R))[[x]] = \mathscr{M}(R[[x]])$.

5. PROPERTIES OF MAXIMAL AND PRIME IDEALS

Throughout this section R will denote a medial ring. If M is a maximal ideal of R, then R/M is simple and commutative. If $(R/M)^2 = 0$, then $R^2 \subseteq M$. Otherwise, R/M is a field and M is prime.

Proposition 5.1. *The following are equivalent for an ideal* P *of* R:

(i) P *is a completely prime ideal.*
(ii) R/P *is an entire ring.*
(iii) P *is a prime ideal.*

Proof. The first two implications are immediate. For (iii) $\Rightarrow$ (i), consider any $x, y \in R$ such that $xy \in P$. Then $xRyR \subseteq xyR \subseteq P$. If $yR \subseteq P$, then $(Ry)^2 = R(yR)\,y \subseteq P$. So either $xR \subseteq P$ or $Ry \subseteq P$. For any $r, s \in R$, $m, n \in Z$, $(rx + nx).(sy + my) = rsxy + mrxy + nxsy + nmxy \in P$; so $\langle x\rangle_l \langle y\rangle_l \subseteq P$ and hence $\langle x\rangle_l \subseteq P$ or $\langle y\rangle_l \subseteq P$.

Similarly, we have:

Proposition 5.2. *The following are equivalent for an ideal* S *of* R:

(i) *If* $x^2 \in S$, *then* $x \in S$ (S *is completely semiprime*).
(ii) S *is a semiprime ideal.*
(iii) R/S *is reduced.*

We denote the prime radical of R by $\beta(R)$.

Proposition 5.3. *The nil radical and prime radical of R coincide.*

Proof. N is the nil radical of R and by Proposition 5.1, $N \subseteq \beta(R)$. However, N is also semiprime, so $\beta(R) \subseteq N$.

Note that N is contained in the Jacobson radical of R, $J(R)$. Since there are commutative domains that are Jacobson radical rings, it is possible for $N \neq J(R)$, for R medial. For rings is general, D.C.C. on ideals does not guarantee the Jacobson radical and nil radical will coincide, but for medial rings it does as will be shown in the next section (Theorem 6.5).

Proposition 5.4. *A proper primitive ideal of R is both a maximal and a prime ideal.*

Proof. If D is a proper primitive ideal of R, then R/D is primitive (hence prime) and commutative. So R/D is a field [12, p. 7], and D is a maximal ideal and a prime ideal.

An examination of the proof of Krull's theorem on prime ideals in Kaplansky [15, pp. 1–2], shows this proof works, *mutatis mutandis*, for left permutable or right permutable rings (with no assumption of identity element). With the modification of using a product of three elements instead of two in equation (1) of that proof, the process can be carried through for medial rings.

Theorem 5.5. *Let S be a multiplicatively closed set in R and let I be an ideal of R maximal with respect to the exclusion of S. Then I is a prime ideal.*

6. CHAIN CONDITIONS

In this section some of the major results for rings with chain conditions on left (right) ideals are obtained for medial rings under the weaker hypothesis of a chain condition on two-sided ideals. Also, various decompositions are given for medial (one-sided permutable, permutable) rings with chain conditions. These decompositions have no exact counterpart for nonmedial rings.

Theorem 6.1. *If R is a medial ring with* D.C.C. *on ideals, then every nil one-sided ideal is nilpotent, and every non-nilpotent ideal contains a nonzero idempotent element.*

Proof. For some $k \geqq 1$, $N^k = N^{k+1}$. Let $T = N^k$ and suppose $T \neq (0)$. Then $T^3 = T \neq (0)$. Select an ideal B minimal among those ideals S for which $TST \neq (0)$. There exists $b \in B$ such that $TbT \neq 0$ and hence $TbT = B$. Write $b = \sum_j t_j b t'_j$, where $t_j, t'_j \in T$. Then for each $t \in T$, $bt = \sum_j t_j b t'_j t = \sum_j t_j t'_j bt = (\sum_j t_j t'_j)\, bt$. Let $\sum_j t_j t'_j = u$ Then $bt = ubt = u^n bt$ for each $n \geqq 1$. Select n so that $u^n = 0$ to obtain $bt = 0$, which yields $TbT = 0$, a contradiction. Thus N is nilpotent. Each nil ode-sided ideal of R is contained in N and hence is nilpotent. If I is a nonnilpotent ideal in R, then either $I \cap N = 0$ and I contains a minimal ideal of R which is a field, or $I \cap N \neq 0$, and I maps homomorphically onto a nonzero ideal $\bar{I}$ of R/N. Since R/N is a finite direct sum of fields, so is $\bar{I}$. The unity element in $\bar{I}$ pulls back to a non-nilpotent element $b \in I$ such that $b^2 - b$ is nilpotent, a condition which guarantees the existence of a nonzero idempotent in I [10, p. 22].

Note that N nilpotent can be obtained by the ostensibly weaker hypothesis of D.C.C. on nil ideals, but the existence of the nonzero idempotent cannot.

As a consequence of Theorem 6.1 and Proposition 4.9 we have: if K is a ring with D.C.C. on ideals and I is a medial ideal of K, then $N(I)$ is a nilpotent ideal of K.

For right (left) permutable rings more can be said.

Corollary 6.2. *If R is a right permutable ring with* D.C.C. *on ideals, then every non-nil, one-sided ideal contains a nonzero idempotent.*

Proof. If L is a non-nil left ideal of R, then RL is a non-nil ideal of R which is contained in L. Thus, using Theorem 6.1, we have every non-nil left ideal of R contains a nonzero idempotent, i.e., R is an I-ring. However, every non-nil right ideal of an I-ring contains a nonzero idempotent [12, p. 210].

Theorem 6.3. *If R is a medial ring with* A.C.C. *on nilpotent ideals, then every nil one-sided ideal is nilpotent.*

Proof. Select a maximal nilpotent ideal I. If $b \in N$, with $b^k = 0$, then $I + NbN$ will be a nilpotent ideal containing I. For if $I^n = 0$, use the Binomial Theorem for Medial Rings with $m = \max\{2k+1, 2n+1\}$, to obtain $(I + NbN)^m = (0)$. So $I + NbN = I$, or $N^3 \subseteq I$. Thus N and consequently all nil one-sided ideals of R are nilpotent.

Example 6.4. Let K be the real algebra defined via the basis $\{x, y\}$ for $\mathbb{R}^2$ by $x^2 = x$, $yx = y$, $xy = y^2 = 0$. This yields a ring in which every proper, nonzero right ideal is a one dimensional subspace and which has neither D.C.C. nor A.C.C. on left ideals. (See Divinsky [9, Ex. 5, p. 37].) This ring is right permutable. Then the ring $R = K \oplus K^{\text{opp}}$ is a medial ring with D.C.C. and A.C.C. on ideals, but with neither chain condition on left or on right ideals.

Theorem 6.5. *If R is a medial ring with* D.C.C. *on ideals, then*

(i) *Every prime ideal P of R is a maximal ideal of R and R/P is a field.*

(ii) *The number of prime ideals is finite.*

(iii) $J(R) = N$.

(iv) *If $R^2 = R$, then the sets of maximal and of prime ideals coincide.*

Proof. R/P is a commutative domain with D.C.C., so it is a field and P is maximal. Because of the latter, *mutatis mutandis* the standard commutative ring proof can be used to obtain that there are finitely many prime ideals. From $J(R)/N \subseteq J(R/N) = 0$, we obtain $J(R) = N$.

From (ii) and Proposition 5.4 we have:

Corollary 6.6. *The structure space of a medial ring with* D.C.C. *is finite.*

Main Decomposition Theorem 6.7. *Let R be a medial ring with* D.C.C. *on ideals. Then $R = F \oplus A$, as a direct sum of ideals, where*:

(i) *F is either zero of a finite direct sum of fields.*

(ii) *$N \subseteq A$ and* $(\operatorname{Soc} A)^2 = 0$.

(iii) *If $N \neq A$, then $A = l_A(e) \oplus A \cdot e$, as a direct sum of left ideals of R, where e*

is a nonzero idempotent in A, $l_A(e) = l_A(Ae)$ is an ideal of R, and Ae is a right permutable ring with right identity and with D.C.C. *on ideals.*

(iv) *$Ae = r_{Ae}(e) + eAe$, as a direct sum of right ideals of Ae, where $r_{Ae}(e) = r_{Ae}(eAe)$ is an ideal of Ae, and eAe is a commutative ring with identity which has* D.C.C. *on ideals.*

(v) *If $N \neq A$, then there exists a nonzero integer k such that $k \cdot l_A(e)$ and $k \cdot r_A(e)$ are contained in N. If e has infinite additive order, then $k = 1$.*

Proof. For any minimal ideal I of R with $I^2 \neq 0$, I is a field with identity i. Then $R = l_R(i) \oplus I$, where $l_R(i) = l_R(I) = \mathrm{ann}_R(I)$. Repeat this process in $l_R(I)$ and successive annihilators to obtain: $R = I_1 \oplus \ldots \oplus I_n \oplus A$, where each minimal ideal I_j is a field and A is an ideal of R with $N \subseteq A$ and $(\mathrm{Soc}\, A)^2 = 0$.

If $A \neq N$, then A/N is a finite direct sum of fields. Let $u + N$ be the identity in A/N. This yields a nonzero idempotent of the form $e = u \cdot q(u)$, where $q(u) = \Sigma a_j u^j$ is a polynomial with integer coefficients [10, p. 22]. Then $A = l_A(e) \oplus Ae$ is a direct sum of left ideals of A (and hence of R), with $l_A(e) = l_A(Ae)$ an ideal of R, Ae a right permutable ring with right identity and $Ae \approx A/l_A(e)$ satisfying the D.C.C. on ideals. Apply a similar, but right-sided, decomposition to Ae to obtain: $Ae = r_{Ae}(e) + eAe$, as a direct sum of right ideals of Ae, with $r_{Ae}(e) = r_{Ae}(eAe)$ an ideal of Ae, and eAe a commutative ring with identity and D.C.C. on ideals.

Since $u + N$ is the identity in A/N, then for each $x \in A$, $xu^n = x + c_n$, where $c_n \in N$; for each natural number n. So $xe = (\Sigma a_j + c$, where $c \in N$ and c depends on x. Let $\Sigma a_j = k$. Note $k \neq 0$, for otherwise, $e = e^2 = 0e + c \in N$. For each $x \in l_A(e)$, $0 = kx + c$, or $kx = -c \in N$. Proceed similarly to get right annihilator results.

Using $x = e$, we have $e = e^2 = ke + c$, or $(1 - k)\, e = c$. So $(1 - k)^n e = 0$ for some n. If e has infinite additive order, then $k = 1$.

In commentary on this theorem and its proof, note that $l_A(e)$ may be zero; but if $l_A(e) \neq 0$, then $N \cap l_A(e) \neq 0$, for otherwise $l_A(e)$ would contain a minimal ideal on R which is not square zero. If $N \cap Ae = 0$, i.e., if $N = l_A(e)$, then Ae is a finite direct sum of fields. Also, if A^+ is torsion-free, then $l_A(e) \subseteq N$. So if $N \cap Ae = 0$ and A^+ is torsion-free, then $l_A(e) = N$. Note that every right ideal of $l_A(e)$ is a right ideal of A.

Theorem 6.8. *If R is a right permutable ring with* D.C.C. *on ideals, then $R = F \oplus L \oplus S$, as a direct sum of ideals, where:*

(i) *F is either zero or is a finite direct sum of fields.*

(ii) *$L \subseteq N$, hence L is nilpotent.*

(iii) *S is either zero or S has a right identity $\hat{e}$.*

(iv) *If $S \neq (0)$, then either S is a commutative ring with identity, or $S = r(\hat{e}) \oplus \hat{e}S\hat{e}$, as a direct sum of right ideals of S, with $\hat{e}S\hat{e}$ being commutative with identity and having* D.C.C. *on ideals, and $N(r_S(\hat{e})) \neq (0)$.*

Proof. Proceed as in the Main Decomposition Theorem. For each $a \in A$, $r \in R$,

$(ae)r = (ar)e \in Ae$. So $A = l_A(e) \oplus Ae$ as a direct sum of ideals of A (and of R), If $l_A(e) \nsubseteq N$, then $l_A(e)$ will contain a nonzero idempotent e_1 and using $T = l_A(e)$, $T = l_T(e_1) \oplus Te_1$, as a direct sum of ideals of R. Repeat this procedure until a left annihilator contained in N is reached, which must occur because of the minimal conditions. Each of the other summands obtained in the process will each have a right identity. So $A = L \oplus S$, as a direct sum of ideals, where $L \subseteq N$ and S has a right identity $\hat{e}$. Either $\hat{e}$ is also a left identity for S, and S is a commutative ring with identity, or we can decompose S as: $S = r_S(\hat{e}) \oplus \hat{e}S$, as a direct sum of right ideals of S with the resulting properties listed above.

In the following we use the notation of the Main Decomposition Theorem.

Corollary 6.9. *Let R be a ring which is either*

(i) *Medial with* D.C.C. *on right ideals, or*

(ii) *Right permutable with* D.C.C. *on ideals.*

If $l_A(e) \subseteq N$, then $l_A(e)$ is nilpotent with torsion additive group. Hence, if A^+ is torsion-free, then $l_A(e) = 0$ and $R = F \oplus Ae$, as a direct sum of ideals.

Proof. Either (i) or (ii) implies $l_A(e)$ has D.C.C. on ideals and N is nilpotent. A nilpotent ring with D.C.C. on ideals has torsion additive group [17, p. 40]. If A^+ is torsion-free, then $l_A(e) \subseteq N$.

Corollary 6.10. *If R is left permutable with* D.C.C. *on ideals, then Ae is a commutative ring with identity.*

Corollary 6.11. *If R is permutable with* D.C.C. *on ideals, then $R = F \oplus L \oplus S$, where F and L are as in Theorem* 6.8, *and S is a commutative ring with identity and $S \cap N \neq 0$ with* $(\mathrm{Soc}\ S)^2 = 0$. *If also, R has a right identity, then $L = 0$ and R is commutative.*

There are many permutable, noncommutative rings with D.C.C. on ideals. The following example is the smallest such ring.

Example 6.12. Let R be the set of all matrices of the form

$$\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix},$$

where $a, b, c \in Z_2$, with the usual matrix operations. Then $R^3 = 0$, but R is noncommutative and has eight elements. Note: Rings of orders n, $1 \leqq n \leqq 7$, are all commutative, except for some one-sided permutable ones of order four.

In the special situation where R is a medial ring with right identity e, and hence R is right permutable, any $x \in I$, I an ideal of R, can be written as $x = ex + c$, where $c \in I \cap \langle R, R\rangle$. Also, eI is an ideal of the commutative unital ring $eR \approx R/\langle R, R\rangle$. Every one-sided ideal of R containing $\langle R, R\rangle$ is a two-sided ideal. In this setting we obtain a result which is both an extension of Cohen's Theorem [15, Th. 8] and a variation on the theme of Hopkin's Theorem [9, Th. 15].

Theorem 6.13. *Let R be a medial ring with right identity e such that either* (1) *R has* D.C.C. *on ideals, or* (2) *every prime ideal of R is finitely generated. If $\langle R, R\rangle$ is finitely generated as an ideal of R, then every right ideal of R containing $\langle R, R\rangle$ is finitely generated. If every ideal of R contained in $\langle R, R\rangle$ is finitely generated, then R has* A.C.C. *on right ideals.*

Proof. If (1), then eR has D.C.C. on ideals and hence has A.C.C. on ideals. If (2), then every prime ideal of eR is finitely generated, which by Cohen's Theorem yields eR has A.C.C. on ideals. Any ideal I of R which contains $\langle R, R\rangle$ yields $y_1, \ldots, y_n \in I$ so that $ey_1, \ldots, ey_n$ generates eI as an ideal of eR. If $g_1, \ldots, g_m$ generate $\langle R, R\rangle$ as an ideal of R, then for any $x = ex + c \in I$, with $c \in \langle R, R\rangle$, we have $r_1, \ldots, r_n$, $s_1, \ldots, s_m \in R$ such that $x = \Sigma ey_i er_i + \Sigma g_j s_j = \Sigma ey_i r_i + \Sigma g_j s_j$. (Note: $R\cdot\langle R, R\rangle = 0$, so the $g_1, \ldots, g_m$ generate $\langle R, R\rangle$ as a right ideal of R.)

Now assume every ideal of R contained in $\langle R, R\rangle$ is finitely generated (as an ideal of R). Then for any right ideal X of R, both $X \cap \langle R, R\rangle$ and $X + \langle R, R\rangle$ are ideals of R and are finitely generated as right R-modules. Let $y_1, \ldots, y_m \in X \cap \langle R, R\rangle$ be a set of generators for $X \cap \langle R, R\rangle$. Since $X/(X \cap \langle R, R\rangle)$ is an R-homomorphic image of $X + \langle R, R\rangle$, it is finitely generated by some set $x_1 + X \cap \langle R, R\rangle, \ldots, x_n + {} + X \cap \langle R, R\rangle$, where each $x_j \in X$. Then $y_1, \ldots, y_m, x_1, \ldots, x_n$ generate X as a right ideal of R.

The condition that each ideal of R contained in $\langle R, R\rangle$ be finitely generated as an ideal of R, could be replaced by $\langle R, R\rangle$ has A.C.C. on ideals, which is equivalent to the additive group of $\langle R, R\rangle$ having A.C.C. on subgroups. Each of these conditions is stronger than that of $\langle R, R\rangle$ being finitely generated as an ideal of R, as the following example shows.

Example 6.14. Let R be the matrix ring

$$\begin{bmatrix} 0 & Q \\ 0 & Q \end{bmatrix},$$

where Q is the rational number field. Then R is right permutable with right identities,

$$\langle R, R\rangle = \begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix},$$

R has A.C.C. and D.C.C. on right ideals, but neither A.C.C. nor D.C.C. on left ideals the only proper nonzero ideal of R is $\langle R, R\rangle$, and $\langle R, R\rangle$ has neither A.C.C. nor D.C.C. on ideals.

7. SUBDIRECTLY IRREDUCIBLE RINGS

In this section we begin the classification of medial, subdirectly irreducible rings. The highly satisfactory classification of commutative, subdirectly irreducible rings by McCoy [18] and Divinsky [8] serves as a model. Nilpotent, subdirectly irreducible rings have been considered by Kruse and Price [17, pp. 7–9]. An analysis herein which reduces matters to commutative or nilpotent rings will be considered satisfactory.

We will use H to denote the heart of a subdirectly irreducible ring. Recall (Proposition 2.3); either $H^2 = 0$ or the ring is a field.

Theorem 7.1. *Let R be a subdirectly irreducible medial ring, with $H^2 = 0$ and $\langle R, R\rangle \neq 0$. If $HR = RH = 0$, then H^+ is isomorphic to a cyclic group of prime order. If one of HR and RH is not zero, it must be H and the other must be zero.*

Proof. If $HR = RH = 0$, then every subgroup of H^+ is an ideal of R. This forces $H^+ \approx C(p)$. If $HR \neq 0$, then HR is an ideal of R and must be H. Then $R \cdot H = R \cdot (HR) \subseteq R \cdot \langle R, R\rangle \cdot R = 0$.

Corollary 7.2. *If R is a subdirectly irreducible medial ring with $H^2 = 0$, $\langle R, R\rangle \neq 0$, and R^+ torsion free, then one of HR or RH is nonzero.*

Theorem 7.3. *Let R be a subdirectly irreducible medial ring, with $H^2 = 0$, $\langle R, R\rangle \neq 0$, and $HR \neq 0$. Then*

(i) *There exists $h_0 \in H$ such that $h_0R = H$ and $r_0 \in R$, $r_0 \notin N$, such that $h_0r_0 = h_0$.*

(ii) *If S is a nonzero subgroup of H^+, then $SR = H$.*

(iii) *$N \subseteq \boldsymbol{r}_R(H) = \mathrm{ann}_R(H) \subset R$, (so $N \neq R$).*

(iv) *H is a minimal right ideal of R.*

(v) *If d is a left zero divisor in R (i.e., there is a nonzero $d' \in R$ such that $dd' = 0$), the either $Hd = 0$ or $\boldsymbol{r}_R(d) \subseteq \mathrm{ann}_R(H)$.*

Proof. Since $HR = H$, there exists $h_0 \in H$ such that $h_0R \neq 0$. But h_0R is an ideal of R, so $h_0R = H$. Then there exists $r_0 \in R$ such that $h_0r_0 = h_0$. So $h_0 = h_0r_0 = h_0r_0^n$, for all n, and consequently $r_0 \notin N$. For a nonzero subgroup S of H^+, if $SR = 0$, then $S = H$; while if $SR \neq 0$, then SR is a nonzero ideal of R and must be H.

If $h \in H$ and $h \cdot N \neq 0$, then $h \cdot N = H$. So there exists x with $x^n = 0$, such that $hx = h$. Then $h = hx = hx^n = 0$. So $h \cdot N = 0$ for all $h \in H$. Since $R \cdot H = 0$, we have $N \subseteq \boldsymbol{r}_R(H) = \mathrm{ann}_R(H)$.

Any nonzero right ideal T of R which is contained in H is also an ideal of R. And finally, if $Hd \neq 0$, then either $HdR = 0$ and $Hd = H$, leading to $0 = HdR = HR = H$, or HdR is a nonzero ideal of R and $H = HdR$. Then for any $h \in H$, there exist $h_i \in H$, $r_i \in R$ such that $h = \Sigma h_i dr$, or $hd' = \Sigma h_i dr_i d' = \Sigma h_i r_i dd' = 0$. Thus $\boldsymbol{r}_R(d) \subseteq \boldsymbol{r}_R(H)$.

Theorem 7.4. *Let R be right permutable and subdirectly irreducible with $H^2 = 0$ and $\langle R, R\rangle \neq 0$. If R contains a nonzero idempotent e, then*

(i) *e is a right identity, $R = Re$, and $\boldsymbol{l}_R(e) = 0$.*

(ii) *$H \cdot R \neq 0$ and $RH = 0$.*

(iii) $R = \langle R, R\rangle \oplus eR$, *as a direct sum of right ideals of* R, *with* eR *a commutative ring with identity*, $H \subseteq \langle R, R\rangle$.

(iv) *Any left ideal of R contained in eR is square zero and no nonzero ideal of R is contained in eR.*

Proof. Any left ideal of a right permutable, subdirectly irreducible ring is either square zero or contains the heart. Since Re is not square zero, then $H \subseteq Re$, which forces $l_R(e)$ to be zero and $R = Re$. Then rest of the conclusion follows immediately from Theorem 7.1 and Proposition 2.8.

Corollary 7.5. *If R si a right permutable, subdirectly irreducible ring with* D.C.C. *on ideals, then either*

(i) *R is commutative,*

(ii) *R is nilpotent, or*

(iii) $\langle R, R\rangle \neq 0$, *e is a right identity element in* R, *and* $R = \langle R, R\rangle \oplus eR$, *as a direct sum of right ideals, where eR is a commutative ring with identity and with* D.C.C. *on ideals.* $(H \subseteq r_R(e) = \langle R, R\rangle.)$

Proof. If R is noncommutative and non-nilpotent, then Theorem 7.4 yields the right identity e, and $r_R(e) \neq 0$, because otherwise $R = Re$ is commutative.

Corollary 7.6. *A permutable, subdirectly irreducible ring S with* D.C.C. *on ideals is either commutative or nilpotent. Every permutable ring with* D.C.C. *on ideals is a subdirect product of rings which are commutative or nilpotent.*

Proof. If S is not commutative nor nilpotent, then $S = r_S(e) \oplus eS$, but this is a direct sum ideals in the permutable case, so $S = \langle S, S\rangle = r_S(e)$, or $S = eS$, which are nilpotent or commutative, respectively. Invoking Birhoff's Theorem [3] completes the proof.

Added in Proof. It has recently come to our attention that S. Pellegrini Manara has considered medial near-rings (c.f., *Medial near-rings in which each element is a power of itself*, Riv. Mat. Univ. Parma (4) *11* (1985), 223–228 (MR 88a: 16070); *On regular medial near-rings*, Boll. Un. Mat. Ital. D (6) *4* (1985), 131–136 (MR 88a: 16071); *On medial near-rings*, *Near-rings and Near-fields* (Tübingen, 1985), 199–209, North-Holland Math. Stud., 137, North-Holland, Amsterdam–New York, 1987 (MR 88c: 16051)). In the paper *On medial near-rings*, she has a section on medial rings in which her main result (Theorem 9) is contained in our Propositions 2.1 and 5.1. However, our methods are independent of hers.

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