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OSCILLATION THEOREMS FOR A CLASS OF LINEAR FOURTH ORDER DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

The present paper is a study of the oscillation of the differential equation

$$(\mathrm{R}) \qquad (L[y] =)\ y^{(4)} + P(t)\, y'' + R(t)\, y' + Q(t)\, y = 0$$

where $P(t)$, $R(t)$, $Q(t)$ are real-valued continuous functions on the interval $I = \langle a, \infty)$, $-\infty < a < \infty$.

We shall assume throughout that

$$(\mathrm{B}) \qquad P(t) \leqq 0\,, \quad R(t) \leqq 0\,, \quad R^2(t) \leqq 2\, P(t)\, Q(t)$$

for all $t \in I$ and $Q(t)$ not identically zero in any interval of I.

One can verify easily that the above assumptions are satisfied if $P(t) \leqq R(t) \leqq 0$, $2\, Q(t) \leqq R(t)$ for all $t \in I$.

This paper is the continuation of [6]. So we shall use the notations and results obtained earlier, without explaining them again here. There are proved some asymptotic properties of a solution $z(t)$ with $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$ and $z'''(t) \leqq 0$ on I. Oscillation theorems for equation (R) will be obtained by an application of the theory developed in [6]. For typical results on the subject we refer to the papers [1, 3, 4, 5, 6].

2. PRELIMINARIES

We begin by formulating preparatory results which are needed in proving theorems in section 3.

Theorem 1 [6]. *Suppose that* (B) *holds. Then* (R) *is oscillatory if and only if for every nonoscillatory solution $y(t)$ of* (R) *there holds either*

$$(1) \qquad y(t)\, y'(t) > 0\,, \quad y(t)\, y''(t) > 0\,, \quad y(t)\, y'''(t) > 0$$

on $\langle t_0, \infty)$ *for some* $t_0 \in I$, *or*

(1') $$y(t)\, y'(t) < 0$$

on I.

Suppose that every nonoscilatory solution $y(t)$ of (R) satisfies the conditions (1) or (1'). We will construct two linearly independent oscillatory solutions u and v of (R). The proof of it is similar to what was done in [1].

Throughout the remainder of this paper let z_0, z_1, z_2, z_3 denote solutions of (R) defined on I by the initial conditions

$$z_i^{(j)}(a) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \text{for} \quad i, j = 0, 1, 2, 3 .$$

For each natural number $n > a$ let $b_{0n}, b_{3n}, c_{2n}, c_{3n}$ be numbers satisfying

(2) $$b_{0n}^2 + b_{3n}^2 = c_{2n}^2 + c_{3n}^2 = 1 ,$$

(3) $$b_{0n}\, z_0(n) + b_{3n}\, z_3(n) = 0 ,$$

$$c_{2n}\, z_2(n) + c_{3n}\, z_3(n) = 0 .$$

Define $u_n(t)$ and $v_n(t)$ to be the solutions of (R) given by

$$u_n(t) = b_{0n}\, z_0(t) + b_{3n}\, z_3(t) , \quad v_n(t) = c_{2n}\, z_2(t) + c_{3n}\, z_3(t) .$$

By (2) there exists a sequence $\{n_k\}$ of natural numbers and numbers b_0, b_3, c_2, c_3 such that the sequences $\{b_{0n_k}\}$, $\{b_{3n_k}\}$, $\{c_{2n_k}\}$ and $\{c_{3n_k}\}$ converge to b_0, b_3, c_2 and c_3, respectively, where

$$b_0^2 + b_3^2 = c_2^2 + c_3^2 = 1 .$$

Let u and v be the solutions of (R) given by

$$u(t) = b_0\, z_0(t) + b_3\, z_3(t) , \quad v(t) = c_2\, z_2(t) + c_3\, z_3(t) .$$

Suppose u is nonoscillatory solution of (R). Since u is satisfying either (1) or (1') and $u'(a) = 0$, there exists a number $t_0 > a$ such that for all $t \geqq t_0$

$$\operatorname{sgn} u(t) = \operatorname{sgn} u^{(j)}(t) , \quad j = 1, 2, 3 .$$

Let τ be any number greater than t_0. Since $\{u_{n_k}(\tau)\}$, $\{u'_{n_k}(\tau)\}$, $\{u''_{n_k}(\tau)\}$ and $\{u'''_{n_k}(\tau)\}$ converge to $u(\tau)$, $u'(\tau)$, $u''(\tau)$ and $u'''(\tau)$, respectively, there exists a natural number n_0 such that

$$\operatorname{sgn} u_{n_k}(\tau) = \operatorname{sgn} u_{n_k}^{(j)}(\tau) , \quad j = 1, 2, 3$$

for all $n_k > n_0$. Hence by Lemma 2 [6]

$$\operatorname{sgn} u_{n_k}(t) = \operatorname{sgn} u_{n_k}^{(j)}(t) , \quad j = 1, 2, 3$$

for all $t > \tau$ and $n_k > n_0$. But this is a contradiction since $u_{n_k}(n_k) = 0$ for all natural numbers n_k. Therefore, u is oscillatory.

Similarly, v is also oscillatory (we note that $v(a) = 0$).

Remark 1. An argument, similar to the one given to show that u and v are oscillatory, can be given to show that any nontrivial linear combination of u and v is oscillatory.

Further, we note that u and v are linearly independent since, otherwise, we would have $u = cz_3$, $c \neq 0$ and this would contradict the fact that u is oscillatory.

Lemma [2]. *Let $f(t)$ be a real valued function defined in $\langle t_0, \infty)$ for some real number $t_0 \geqq 0$. Suppose that $f(t) > 0$ and that $f'(t), f''(t)$ exist for $t \geqq t_0$. Suppose also that if $f'(t) \geqq 0$ eventually, then $\lim_{t\to\infty} f(t) = A < \infty$. Then*

$$\liminf_{t\to\infty} \left|t^\alpha f''(t) - \alpha t^{\alpha-1} f'(t)\right| = 0$$

for any $\alpha \leqq 2$.

Theorem 2. *Suppose that* (B) *holds and let*

$$\int_{\tau_0}^{\infty} t^{2+\alpha} Q(t)\,\mathrm{d}t = -\infty\,, \quad \tau_0 \geqq \max\{a, 0\}\,, \quad 0 \leqq \alpha < 1\,.$$

Then (R) *is nonoscillatory if and only if there exists a solution $y(t)$ of* (R) *and a number $t_0 \in I$ such that $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ for all $t \geqq t_0$.*

The proof is obtained similarly to that of Theorem 1.2 [5] using Theorem 6 [6] and Lemma 1.1 [5] and is omitted.

Theorem 3. *Suppose that* (B) *holds and let $Q(t) \leqq R(t)$ for all $t \in I$ and $\int_{t_0}^{\infty} s\,P(s)\,.$ $.\,\mathrm{d}s > -\infty$, $t_0 > \max\{a, 0\}$. Then there is not a solution $y(t)$ of* (R) *with $y(t) > 0$, $y'(t) > 0$ and $y''(t) < 0$ for $t \geqq t_0$.*

Proof. Suppose on the contrary that such a solution $y(t)$ exists. Pick $t_1 \geqq$ $\geqq \max\{t_0, 1\}$ such that $\int_{t_1}^{\infty} s\,P(s)\,\mathrm{d}s \geqq -1$. Multiply (R) by t and integrate by parts between t_1 and t, $t_1 < t$, to obtain

$$t\,y'''(t) - t_1\,y'''(t_1) - y''(t) + y''(t_1) + y''(t)\int_{t_1}^{t} s\,P(s)\,\mathrm{d}s - \tag{4}$$
$$- \int_{t_1}^{t} y'''(s)\int_{t_1}^{s} u\,P(u)\,\mathrm{d}u\,\mathrm{d}s + \int_{t_1}^{t} s\,R(s)\,y'(s)\,\mathrm{d}s + \int_{t_1}^{t} s\,Q(s)\,y(s)\,\mathrm{d}s = 0\,.$$

Since $-y''(t) \geqq y''(t)\int_{t_1}^{t} s\,P(s)\,\mathrm{d}s \geqq 0$ and $\int_{t_1}^{t} s\,R(s)\,y'(s)\,\mathrm{d}s \leqq 0$, (4) becomes

(5)
$$t\,y'''(t) - 2\,y''(t) + y''(t_1) - \int_{t_1}^{t} y'''(s)\int_{t_1}^{s} u\,P(u)\,\mathrm{d}u\,\mathrm{d}s \geqq t_1\,y'''(t_1) - \int_{t_1}^{t} s\,Q(s)\,y(s)\,\mathrm{d}s\,.$$

Note that $y'''(t) \leqq 0$ eventually is impossible with $y''(t) < 0$ and $y'(t) > 0$. Suppose that $y'''(t) \geqq 0$ for $t \geqq t_1$ (change t_1 if necessary). Then

$$- \int_{t_1}^{t} y'''(s) \int_{t_1}^{s} u\, P(u)\, du\, ds \leqq \int_{t_1}^{t} y'''(s)\, ds = y''(t) - y''(t_1)\,.$$

Therefore (5) becomes

$$t\, y'''(t) - y''(t) \geqq t_1\, y'''(t_1) - \int_{t_1}^{t} s\, Q(s)\, y(s)\, ds\,. \tag{6}$$

By Lemma $\liminf_{t\to\infty} (t\, y'''(t) - y''(t)) = 0$. But this contradicts the fact that the right hand side of (6) is positive and increasing. Theorem is proved for the case $y'''(t) \geqq 0$.

Suppose now that $y'''(t)$ has positive and negative values for arbitrary large t. Then there is a sequence of points $\{t_n\}$, $n \geqq 2$, $t_1 < t_2$, $\lim_{n\to\infty} t_n = \infty$, with the following properties: $t_i < t_{i+1}$, $i = 2, 3, \ldots$, $y'''(t_i) = 0$, $i = 2, 3, \ldots$, $\lim_{i\to\infty} y''(t_i) = 0$. The existence of such a sequence $\{t_n\}$ is clear since $y''(t) < 0$ and $\limsup_{t\to\infty} y''(t) = 0$.

Now, let

$$M = \int_{t_2}^{\infty} u\, P(u)\, du\,.$$

$M > -1$ by the choice of $t_2 > t_1$. Thus

$$- \int_{t_2}^{t} y'''(s) \int_{t_2}^{s} u\, P(u)\, du\, ds = \int_{t_2}^{t} y'''(s) \left(\int_{s}^{\infty} u\, P(u)\, du - M \right) ds =$$

$$= \int_{t_2}^{t} y'''(s) \int_{s}^{\infty} u\, P(u)\, du\, ds - M \int_{t_2}^{t} y'''(s)\, ds \leqq \int_{t_2}^{t} y'''(s) \int_{s}^{\infty} u\, P(u)\, du\, ds - y''(t_2)\,.$$

Substituting this into (5) (replacing t_1 by t_2) gives

$$t\, y'''(t) - 2\, y''(t) + \int_{t_2}^{t} y'''(s) \int_{s}^{\infty} u\, P(u)\, du\, ds \geqq - \int_{t_2}^{t} s\, Q(s)\, y(s)\, ds\,. \tag{7}$$

Denote

$$F(s) = \int_{s}^{\infty} u\, P(u)\, du\,.$$

Then

$$\int_{t_2}^{t} y'''(s)\, F(s)\, ds = y'''(t) \int_{t_2}^{t} F(s)\, ds - \int_{t_2}^{t} y^{(4)}(s) \int_{t_2}^{s} F(u)\, du\, ds = \tag{8}$$

$$= y'''(t) \int_{t_2}^{t} F(s)\, ds + \int_{t_2}^{t} P(s)\, y''(s) \int_{t_2}^{s} F(u)\, du\, ds + \int_{t_2}^{t} R(s)\, y'(s) \int_{t_2}^{s} F(u)\, du\, ds +$$

$$+ \int_{t_2}^{t} Q(s)\, y(s) \int_{t_2}^{s} F(u)\, du\, ds \leqq y'''(t) \int_{t_2}^{t} F(s)\, ds + \int_{t_2}^{t} R(s)\, y'(s) \int_{t_2}^{s} F(u)\, du\, ds +$$

$$+ \int_{t_2}^{t} Q(s)\, y(s) \int_{t_2}^{s} F(u)\, du\, ds \leqq$$

$$\leqq y'''(t) \int_{t_2}^{t} F(s)\, ds - \int_{t_2}^{t} (s - t_2)\, R(s)\, y'(s)\, ds - \int_{t_2}^{t} (s - t_2)\, Q(s)\, y(s)\, ds\,,$$

where the last inequality depends on the fact that $|F(u)| \leqq 1$. Applying Lemma 1.1 [3] to the solution $y(t)$, we obtain

$$-(s - t_2)\, R(s)\, y'(s) \leqq -R(s)\, y(s) \quad \text{for} \quad s > t_2$$

and hence

$$- \int_{t_2}^{t} (s - t_2)\, R(s)\, y'(s)\, ds \leqq - \int_{t_2}^{t} R(s)\, y(s)\, ds\,.$$

Sebstituting this into (8) yields

(9)

$$\int_{t_2}^{t} y'''(s)\, F(s)\, ds \leqq y'''(t) \int_{t_2}^{t} F(s)\, ds - \int_{t_2}^{t} R(s)\, y(s)\, ds - \int_{t_2}^{t} (s - t_2)\, Q(s)\, y(s)\, ds\,.$$

It follows from (7) and (9) that

$$t\, y'''(t) - 2\, y''(t) + y'''(t) \int_{t_2}^{t} F(s)\, ds - \int_{t_2}^{t} R(s)\, y(s)\, ds - \int_{t_2}^{t} (s - t_2)\, Q(s)\, y(s)\, ds \geqq$$

$$\geqq - \int_{t_2}^{t} s\, Q(s)\, y(s)\, ds\,.$$

Combining the last three terms gives

$$t\, y'''(t) - 2\, y''(t) + y'''(t) \int_{t_2}^{t} F(s)\, ds \geqq \int_{t_2}^{t} [R(s) - t_2\, Q(s)]\, y(s)\, ds\,. \tag{10}$$

Replacing t by t_i in (10) where $\{t_i\}$ is the sequence defined above yields

$$-2\, y''(t_i) \geqq \int_{t_2}^{t_i} [R(s) - t_2\, Q(s)]\, y(s)\, ds\,. \tag{11}$$

The right hand side of (11) is positive and increasing in t_i while the left hand side of (11) converges to zero as $i \to \infty$. This contradiction proves the theorem.

Theorem 4. *Suppose that $P(t) \leqq R(t) \leqq 0$, $2\, Q(t) \leqq R(t)$ for all $t \in I$ and let*

$$\int_{t_0}^{\infty} s\, P(s)\, ds > -\infty\,, \quad t_0 > \max\{a, 0\}\,.$$

Then there is not a solution $y(t)$ of (R) *with $y(t) > 0$, $y'(t) > 0$ and $y''(t) < 0$ for $t \geqq t_0$.*

The proof follows along the lines of the proof of the previous theorem since the assumptions $P(t) \leqq R(t) \leqq 0$, $2\,Q(t) \leqq R(t)$ for $t \in I$ imply the assumptions (B). We remark that in this case we should take a sequence of points $\{t_n\}$ such that $n \geqq 2$, $t_2 > 2$, $\lim_{n\to\infty} t_n = \infty$.

Theorem 5. *Suppose that* (B) *holds and let*

$$\int_{t_0}^{\infty} s^{2+\alpha} Q(s)\,\mathrm{d}s = -\infty\,, \quad \int_{t_0}^{\infty} s^{2+\alpha} R(s)\,\mathrm{d}s > -\infty\,,$$

$t_0 \geqq \max\{a, 0\}$, $0 \leqq \alpha < 1$. *Then for every solution $y(t)$ of* (R) *such that $y(t)\,y'(t) \leqq \leqq 0$, $y(t)\,y''(t) \geqq 0$ and $y(t)\,y'''(t) \leqq 0$ for $t \geqq t_0$ there holds*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0\,.$$

Proof. Suppose that $y(t) > 0$ for $t \geqq t_0$. Then by the above conditions it follows that $y'(t) \leqq 0$, $y''(t) \geqq 0$ and $y'''(t) \leqq 0$ for $t \geqq t_0$. From this it follows easily that $\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0$.

Pick $t_1 \geqq t_0$ such that $\int_{t_1}^{\infty} s^{2+\alpha} R(s)\,\mathrm{d}s \geqq -1$. Multiplying (R) by $t^{2+\alpha}$, $0 \leqq \alpha < 1$, integrating from t_1 to t, we obtain

$$\begin{aligned}(12)\quad & [y'''(s)\,s^{2+\alpha}]_{t_1}^{t} - [(2+\alpha)\,s^{1+\alpha}\,y''(s)]_{t_1}^{t} + [(2+\alpha)(1+\alpha)\,s^{\alpha}\,y'(s)]_{t_1}^{t} - \\ & - [(2+\alpha)(1+\alpha)\,\alpha s^{\alpha-1}\,y(s)]_{t_1}^{t} + (2+\alpha)(1+\alpha)\,\alpha(\alpha-1)\int_{t_1}^{t} s^{\alpha-2}\,y(s)\,\mathrm{d}s + \\ & + \int_{t_1}^{t} s^{2+\alpha} P(s)\,y''(s)\,\mathrm{d}s + \int_{t_1}^{t} s^{2+\alpha} R(s)\,y'(s)\,\mathrm{d}s + \int_{t_1}^{t} s^{2+\alpha} Q(s)\,y(s)\,\mathrm{d}s = 0\,.\end{aligned}$$

Since

$$\int_{t_1}^{t} s^{2+\alpha} R(s)\,y'(s)\,\mathrm{d}s = y'(t)\int_{t_1}^{t} s^{2+\alpha} R(s)\,\mathrm{d}s - \int_{t_1}^{t} y''(s)\int_{t_1}^{s} u^{2+\alpha} R(u)\,\mathrm{d}u\,\mathrm{d}s$$

and

$$y'(t)\int_{t_1}^{t} s^{2+\alpha} R(s)\,\mathrm{d}s \leqq -y'(t)\,,$$

$$-\int_{t_1}^{t} y''(s)\int_{t_1}^{s} u^{2+\alpha} R(u)\,\mathrm{d}u\,\mathrm{d}s \leqq y'(t) - y'(t_1)\,,$$

it follows that

$$(13)\qquad \int_{t_1}^{t} s^{2+\alpha} R(s)\,y'(s)\,\mathrm{d}s \leqq -y'(t_1)\,.$$

From the above inequalities (12) and (13) we obtain

$$t^{2+\alpha}\, y'''(t) \geqq K - \int_{t_1}^{t} s^{2+\alpha}\, Q(s)\, y(s)\, \mathrm{d}s\,, \tag{14}$$

where K is a constant.

Now suppose that $\lim_{t\to\infty} y(t) = B > 0$. Since $y(t)$ has a finite limit and $0 \leqq \alpha < 1$ from (14) it follows that

$$t^{2+\alpha}\, y'''(t) \geqq K - B \int_{t_1}^{t} s^{2+\alpha}\, Q(s)\, \mathrm{d}s\,.$$

Hence it follos that $y'''(t) > 0$ for sufficiently large t. But this is a contradiction and the proof is complete.

3. OSCILLATION THEOREM

Now, oscillation theorem for equation (R) will be obtained by using preceding results.

Theorem 6. *Suppose that*

$$\int_{\tau_0}^{\infty} t^{2+\alpha}\, Q(t)\, \mathrm{d}t = -\infty\,, \quad \tau_0 > \max\{a, 0\} \quad \textit{for some} \quad 0 \leqq \alpha < 1 \tag{15}$$

and let (B) *holds and* $\int_{\tau_0}^{\infty} t\, P(t)\, \mathrm{d}t > -\infty$, $Q(t) \leqq R(t)$ *for all* $t \geqq \tau_0$, *or* (15) *holds and* $\int_{\tau_0}^{\infty} t\, P(t)\, \mathrm{d}t > -\infty, P(t) \leqq R(t) \leqq 0$, $2\, Q(t) \leqq R(t)$ *for all* $t \in I$.

Then (R) *is oscillatory and there exists a fundamental system of solutions of* (R) *such that two solutions of this system are oscillatory, other solutions of this system are nonoscillatory and one of them tends monotonically to* ∞ *as* $t \to \infty$ *and the other of them tends to zero if* $\int_{\tau_0}^{\infty} s^{2+\alpha}\, R(s)\, \mathrm{d}s > -\infty$.

Proof. It follows from Theorems 2, 3, 4 and Theorem 1 that (R) is oscillatory. Then (R) has oscillatory solutions

$$u(t) = b_0\, z_0(t) + b_3\, z_3(t)\,, \quad v(t) = c_2\, z_2(t) + c_3\, z_3(t)$$

whose construction has already shown in the previous section. It follows from Theorem 2 [6] that there exists assolution z with the properties $z > 0$, $z' < 0$, $z'' > 0$ and $z''' \leqq 0$ for $t \in I$. By Theorem 5 $\lim_{t\to\infty} z(t) = 0$.

Note that z_3 has no zero to the right of a by Lemma 2 [6] and $\lim_{t\to\infty} z_3(t) = \infty$.

The solutions $z(t)$, $u(t)$, $v(t)$ and $z_3(t)$ form the fundamental system of (R). In fact, their Wronskian $W[z(t), u(t), v(t), z_3(t)]_{t=a} = -b_0 c_2\, z'(a) \neq 0$, since $z'(a) < 0$ and $b_0 \neq 0$, otherwise it would be $u(t) = b_3\, z_3(t)$, which would contradict the fact that

$u(t)$ is oscillatory and $z_3(t)$ has no zeros to the right of a. By the same argument $c_2 \neq 0$. The proof of Theorem is complete.

Remark. Theorem 6 is a generalization of Theorem 1.7 [5]. If $R(t) \equiv 0$, $P(t) \equiv 0$ for $t \in I$ we obtain wellnown results for equation $y^{(4)} + Q(t)\,y = 0$ [1, 3].

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