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A GENERAL CONCEPT OF THE PSEUDOPROJECTIVE MODULE

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This paper can be viewed as a continuation of our investigations [9] and is devoted to some dual questions. Here we shall introduce a general type (in a certain sense) of the pseudoprojective module. For a characterization of such modules it is necessary to investigate some types of generalized hereditary and cohereditary preradicals. The basic properties of such preradicals are studied in Sec. 2. The main results are presented in Sec. 3 where we deal with various types of $\mathcal{a}$-pseudoprojective modules and with the modules pseudoprojective with respect to a pair of preradicals. We give several characterizations of these modules in the general case as well as in the case when the existence of projective covers must be assumed. In the last section some results concerning Morita equivalent rings and generalized pseudoprojective modules are presented without proofs.

1. PRELIMINARIES

In what follows R stands for an associative ring with identity. By the word "module" we shall always mean a unitary left R-module, unless specified otherwise. The category of all modules is denoted by R-mod. A module M is called *cofaithful* if every injective module is a homomorphic image of some direct copower $M^{(I)}$ of M. An exact sequence $0 \to K \to P \to M \to 0$ with P projective is said to be a *projective presentation* of M and is called a *projective cover* of M if K is small in P, i.e. if $K + L = P$, L a submodule of P, implies $L = P$. A ring R is said to be *left perfect* if every module possesses a projective cover. If $\mathcal{a}$ is any non-empty class of modules then $e(\mathcal{a})$ is the class of all epimorphic images of modules from $\mathcal{a}$ and $m(\mathcal{a})$ is the class of all modules which can be embedded in some module from $\mathcal{a}$.

A preradical r (for R-mod) is a subfunctor of the identity functor, i.e., r assigns to each module M its submodule $r(M)$ in such a way that every homomorphism $M \to N$ induces a homomorphism $r(M) \to r(N)$ by restriction. The identity functor denoted by id and the functor zer, zer $(M) = 0$ for each $M \in R$-mod, are preradicals. If r, s are preradicals and $r(M) \subseteq s(M)$ for every $M \in R$-mod then we shall write

$r \leqq s$. For a preradical r, a module M is said to be *r-torsion* (*r-torsionfree*) if $r(M) = M$ ($r(M) = 0$). We denote by $\mathscr{T}_r(\mathscr{F}_r)$ the class of all r-torsion (r-torsionfree) modules.

A preradical r is said to be

- *idempotent if* $r(M) \in \mathscr{T}_r$ for every $M \in R$-mod;
- a *radical* if $M/r(M) \in \mathscr{F}_r$ for every $M \in R$-mod;
- *hereditary* if $r(N) = N \cap r(M)$ for every $M \in R$-mod and every submodule N of M;
- *cohereditary* if $r(M/N) = (r(M) + N)/N$ for every $M \in R$-mod and every submodule N of M;
- *superhereditary* if it is hereditary and $\mathscr{T}_r$ is closed under direct products;
- *costable* if $r(M)$ is a direct summand in M for every projective module M.

For a preradical r and a module M let $\bar{r}(M) = \sum A$ and $\tilde{r}(M) = \bigcap B$, where A runs through all the r-torsion submodules of M and B runs through all the submodules of M with $M/B \in \mathscr{F}_r$. Then $\bar{r}$ is the largest idempotent preradical contained in r and $\tilde{r}$ is the least radical containing r. For every $M \in R$-mod put $ch(r)(M) = r(R)M$. Then $ch(r)$ is the largest cohereditary radical contained in r.

If $\{r_i \mid i \in I\}$ is a family of preradicals then the preradical $\sum_{i\in I} r_i$ is defined by the formula $(\sum_{i\in I} r_i)(M) = \sum_{i\in I} r_i(M)$ for each $M \in R$-mod. If $\mathscr{A}$ is a non-empty class of modules then the idempotent preradical $p_{\mathscr{A}}$ (radical $p^{\mathscr{A}}$) is defined for each $M \in R$-mod by $p_{\mathscr{A}}(M) = \sum \operatorname{Im} f$, $f \in \operatorname{Hom}_R(A, M)$, $A \in \mathscr{A}$ ($p^{\mathscr{A}}(M) = \bigcap \operatorname{Ker} f$, $f \in \operatorname{Hom}_R(M, A)$, $A \in \mathscr{A}$). For preradicals r, s we define a preradical $r \lhd s$ by $(r \lhd s)(M)/r(M) = s(M/r(M))$ for every $M \in R$-mod.

Recall the definitions of some special preradicals. For every module M, $\operatorname{Soc}(M)$ is the sum of all simple submodules of M (the socle), $Y(M) = \bigcap N$, where N runs through all the submodules of M such that M/N is cocyclic and small in the injective hull $E(M/N)$ of M/N.

If r is a preradical and N is a submodule of a module M then we define $C_r(N : M)$ by $C_r(N : M)/N = r(M/N)$. A submodule N of M is said to be $(r, 1)$-*codense* in M if there is an epimorphism $g : D \to M$ such that $g(r(g^{-1}(N))) = 0$, it is called $(r, 2)$-*codense* in M if $N \in \mathscr{F}_r$ and it is called $(r, 3)$-*codense* in M if $r(M) \cap N = 0$. These situations are denoted by $N \subseteq^{(r,i)} M$, $i = 1, 2, 3$.

If $\mathscr{A}$ is a non-empty class of modules then a module M is called $\mathscr{A}$-projective if it is projective with respect to all epimorphisms $B \to C$ with $B \in \mathscr{A}$. If r is a cohereditary radical then a module M is called *r-projective* if it is projective with respect to all epimorphisms $f : B \to C$ with $\operatorname{Ker} f \in \mathscr{F}_r$.

For further details we refer to [8].

Throughout the whole paper, unless specified otherwise, $\mathscr{A}$ is a non-empty class of modules and r is an idempotent cohereditary radical.

Definition 1. A preradical t is called *$(r, \mathscr{a})$-dcohereditary* if $t(B/A) = (t(B) + A)/A$ whenever $B \in \mathscr{a}$, $A \subseteq B$ and $r(A) = 0$. If $r = \text{zer}$ then we shall say that t is *$\mathscr{a}$-dcohereditary.*

Proposition 1. *The sum* $\sum_{i \in I} t_i$ *of $(r, \mathscr{a})$-dcohereditary preradicals t_i, $i \in I$, is $(r, \mathscr{a})$-dcohereditary.*

Proof. If $B \in \mathscr{a}$, $A \subseteq B$ and $r(A) = 0$ then $(\sum_{i \in I} t_i)(B/A) = \sum_{i \in I} t_i(B/A) = \sum_{i \in I} (t_i(B) + A)/A = ((\sum_{i \in I} t_i)(B) + A)/A$.

Proposition 2. *A preradical t is $\mathscr{a}$-dcohereditary iff for every $B \in \mathscr{a}$ and $t(B) \subseteq A \subseteq B$ we have $B/A \in \mathscr{F}_t$.*

Proof. If t is $\mathscr{a}$-dcohereditary then $t(B/A) = (t(B) + A)/A = 0$. If the condition is satisfied and $B \in \mathscr{a}$, $A \subseteq B$, then $t(B/(t(B) + A)) = 0$ and so $(t(B/A) + (t(B) + A)/A)/(t(B) + A)/A \subseteq t(B/A/(t(B) + A)/A) = 0$ gives $t(B/A) \subseteq (t(B) + A)/A \subseteq t(B/A)$.

Proposition 3. *A preradical t is $\mathscr{a}$-dcohereditary iff $B/t(B) \in \mathscr{F}_t$ for every $B \in \mathscr{a}$ and $B/A \in \mathscr{F}_t$ whenever $B \in e(\mathscr{a}) \cap \mathscr{F}_t$ and $A \subseteq B$.*

Proof follows immediately from Proposition 2.

Proposition 4. *Let t be an idempotent preradical. Then $t(B/t(B)) = 0$ for every $B \in m(\mathscr{a})$ iff for every exact sequence $0 \to A \to B \to C \to 0$ with $B \in m(\mathscr{a})$ and $A, C \in \mathscr{T}_t$ we have $B \in \mathscr{T}_t$.*

Proof. Let $0 \to A \rightarrowtail B \to C \to 0$ be an exact sequence with $B \in m(\mathscr{a})$ and $A, C \in \mathscr{T}_t$. Then $A = t(A) \subseteq t(B)$ and so $B/t(B)$ is a homomorphic image of $C \cong B/A$. Thus $B/t(B) \in \mathscr{T}_t \cap \mathscr{F}_t = 0$ and $B = t(B) \in \mathscr{T}_t$. Conversely, let $t(B/t(B)) = X/t(B)$, $X \subseteq B$, $B \in m(\mathscr{a})$. Then in the exact sequence $0 \to t(B) \to X \to X/t(B) \to 0$ we have $X \in \mathscr{T}_t$ by hypothesis so that $X \subseteq t(B)$ and $t(B/t(B)) = 0$.

Proposition 5. *Let t be a preradical. Then*

(i) *If t is $(r, m(\mathscr{a}))$-dcohereditary then for every $Q \in \mathscr{T}_t$ and every r-projective presentation $0 \to K \to P \to Q \to 0$ with $r(K) = 0$ we have $K + C_t(\mathrm{Ker} f : P) = P$ for each $f : P \to M$, $M \in \mathscr{a}$;*

(ii) *If t is idempotent and every $Q \in \mathscr{T}_t$ has an r-projective presentation $0 \to K \to P \to Q \to 0$ with $r(K) = 0$ such that $K + C_t(\mathrm{Ker} f : P) = P$ for each $f : P \to M$, $M \in \mathscr{a}$, then t is $(r, m(\mathscr{a}))$-dcohereditary.*

Proof. (i) Let $0 \to K \to P \to Q \to 0$ be an arbitrary r-projective presentation of $Q \in \mathscr{T}_t$ with $r(K) = 0$. If $f : P \to M$, $M \in \mathscr{a}$, then $P/\mathrm{Ker} f \in m(\mathscr{a})$ and $(K + \mathrm{Ker} f)/$

$/ \operatorname{Ker} f \cong K/K \cap \operatorname{Ker} f \in \mathscr{F}_r$, r being cohereditary. Now $P/(K + \operatorname{Ker} f)$ is t-torsion as a homomorphic image of $P/K \cong Q$, so that the hypothesis gives $P/(K + \operatorname{Ker} f) \cong$ $\cong P/\operatorname{Ker} f/(K + \operatorname{Ker} f)/\operatorname{Ker} f = (t(P/\operatorname{Ker} f) + (K + \operatorname{Ker} f)/\operatorname{Ker} f)/(K +$ $+ \operatorname{Ker} f)/\operatorname{Ker} f$ and consequently $C_t(\operatorname{Ker} f : P) + K = P$.

(ii) Assume that t is idempotent and let $B \in m(\mathscr{A})$, $A \subseteq B$ and $r(A) = 0$. By hypothesis, the module $t(B/A) \in \mathscr{T}_t$ has an r-projective presentation $0 \to K \to P \to^g$ $\to^g t(B/A) \to 0$ with $r(K) = 0$ and so we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & P \arrow[r, "g"] \arrow[d, "h"'] & t(B/A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & C_t(A:B) \arrow[r, "\pi"] & t(B/A) \arrow[r] & 0
\end{tikzcd}$$

with exact rows and π the natural projection. Now h induces $\bar{h} : P/\operatorname{Ker} h \to C_t(A :$ $: B) \subseteq B$ in the natural way and hence $h(C_t(\operatorname{Ker} h : P)) = \bar{h}(C_t(\operatorname{Ker} h : P)/\operatorname{Ker} h) =$ $= \bar{h}(t(P/\operatorname{Ker} h)) \subseteq t(B)$ together with the hypothesis $P = K + C_t(\operatorname{Ker} h : P)$ yields $t(B/A) = g(P) = \pi h(C_t(\operatorname{Ker} h : P)) \subseteq \pi(t(B)) = (t(B) + A)/A \subseteq t(B/A)$.

Definition 2. For a preradical t and $A \in R$-mod put $dh_{\mathscr{A}}(t)(A) = \bigcap \{C_t(\operatorname{Ker} f : A) \mid$ $\mid f : A \to M;\ M \in \mathscr{A}\}$.

Definition 3. A preradical t is said to be *$\mathscr{A}$-dhereditary* if $(p^{\mathscr{A}} \lhd t)(A) = C_t(p^{\mathscr{A}}(A) :$ $: A) = dh_{\mathscr{A}}(t)(A)$ for every module A.

Proposition 6. *Let t be a preradical. Then*

(i) *$dh_{\mathscr{A}}(t)$ is a preradical;*

(ii) *$p^{\mathscr{A}} \lhd t \leqq dh_{\mathscr{A}}(t)$;*

(iii) *$dh_{\mathscr{A}}(t)(A) = t(A)$ for each $A \in m(\mathscr{A})$;*

(iv) *$dh_{\mathscr{A}}(dh_{\mathscr{A}}(t)) = dh_{\mathscr{A}}(t)$;*

(v) *$dh_{\mathscr{A}}(t) = p^{\mathscr{A}} \lhd dh_{\mathscr{A}}(t)$;*

(vi) *$dh_{\mathscr{A}}(t)$ is $\mathscr{A}$-dhereditary;*

(vii) *if s is an $\mathscr{A}$-dhereditary preradical with $t \leqq s$ then $dh_{\mathscr{A}}(t) \leqq p^{\mathscr{A}} \lhd s$.*

Proof. (i) Let $g : A \to B, f : B \to M$ be arbitrary homomorphisms, $M \in \mathscr{A}$. Denote $\operatorname{Ker} f = U$, $g^{-1}(U) = X$, $C_t(U : B) = V$, $C_t(X : A) = Y$. Since $g(X) \subseteq U$, g induces $\bar{g} : A/X \to B/U$ in the natural way and so $(g(Y) + U)/U = \bar{g}(Y/X) = \bar{g}(t(A/X)) \subseteq$ $\subseteq t(B/U) = V/U$ which proves that $g(C_t(g^{-1}(\operatorname{Ker} f) : A)) \subseteq C_t(\operatorname{Ker} f : B)$. Now $g(dh_{\mathscr{A}}(t)(A)) \subseteq g(\bigcap \{C_t(\operatorname{Ker} fg : A) \mid f : B \to M;\ \ M \in \mathscr{A}\}) \subseteq \bigcap \{g(C_t(g^{-1}(\operatorname{Ker} f) :$ $: A)) \mid f : B \to M;\ M \in \mathscr{A}\} \subseteq \bigcap \{C_t(\operatorname{Ker} f : B) \mid f : B \to M;\ M \in \mathscr{A}\} = dh_{\mathscr{A}}(t)(B)$.

(ii) For every module A we have $(p^{\mathscr{A}} \lhd t)(A) = C_t(p^{\mathscr{A}}(A) : A) = C_t((\bigcap \{\operatorname{Ker} f \mid f :$ $: A \to M;\ M \in \mathscr{A}\}) : A) \subseteq \bigcap \{C_t(\operatorname{Ker} f : A) \mid f : A \to M;\ M \in \mathscr{A}\} = dh_{\mathscr{A}}(t)(A)$.

(iii) If $A \in m(\mathscr{A})$ then there is a monomorphism $g : A \to M'$, $M' \in \mathscr{A}$, and so $t(A) \subseteq$ $\subseteq (p^{\mathscr{A}} \lhd t)(A) \subseteq dh_{\mathscr{A}}(t)(A) = \bigcap \{C_t(\operatorname{Ker} f : A) \mid f : A \to M;\ \ M \in \mathscr{A}\} \subseteq C_t(0 : A) =$ $= t(A)$.

(iv) By (iii), $dh_{\alpha}(t)(A/\operatorname{Ker} f) = t(A/\operatorname{Ker} f)$ for each $f : A \to M$, $M \in \alpha$, and so $dh_{\alpha}(dh_{\alpha}(t))(A) = \bigcap \{C_{dh_{\alpha}(t)}(\operatorname{Ker} f : A) \mid f : A \to M;\ M \in \alpha\} = \bigcap \{C_t(\operatorname{Ker} f : A) \mid f : A \to M;\ M \in \alpha\} = dh_{\alpha}(t)(A)$ for every $A \in R$-mod.

(v) By (ii) and (iv) we have $dh_{\alpha}(t) \leqq p^{\alpha} \lhd dh_{\alpha}(t) \leqq dh_{\alpha}(dh_{\alpha}(t)) = dh_{\alpha}(t)$.

(vi) By (v) and (iv) we have $p^{\alpha} \lhd dh_{\alpha}(t) = dh_{\alpha}(t) = dh_{\alpha}(dh_{\alpha}(t))$.

(vii) For each $A \in R$-mod we have $dh_{\alpha}(t)(A) = \bigcap \{C_t(\operatorname{Ker} f : A) \mid f : A \to M;\ M \in \alpha\} \subseteq \bigcap \{C_s(\operatorname{Ker} f : A) \mid f : A \to M; M \in \alpha\} = dh_{\alpha}(s)(A) = (p^{\alpha} \lhd s)(A)$, s being α-dhereditary.

Corollary 1. *$dh_{\alpha}(t)$ is the least α-dhereditary preradical s containing t and satisfying $p^{\alpha} \lhd s = s$.*

Proposition 7. *Every superhereditary preradical t is α-dhereditary.*

Proof. Let A be an arbitrary module and $a \in dh_{\alpha}(t)(A) = \bigcap \{C_t(\operatorname{Ker} f : A) \mid f : A \to M;\ M \in \alpha\}$. Then $a \in C_t(\operatorname{Ker} f : A)$, i.e. $a + \operatorname{Ker} f \in t(A/\operatorname{Ker} f)$ for every $f : A \to M$, $M \in \alpha$. Since t is superhereditary, there is an ideal I of R such that $t(B) = \{b \in B \mid Ib = 0\}$ for each $B \in R$-mod [8; I.2.E4]. Thus $Ia \subseteq \operatorname{Ker} f$ for each $f : A \to M$, $M \in \alpha$, i.e. $Ia \subseteq p^{\alpha}(A)$, so $a + p^{\alpha}(A) \in t(A/p^{\alpha}(A))$ and $a \in C_t(p^{\alpha}(A) : A)$. We have proved that $dh_{\alpha}(t)(A) \subseteq C_t(p^{\alpha}(A) : A)$ and Proposition 6(ii) completes the proof.

Proposition 8. *If $dh_{\alpha}(t)$ is a cohereditary radical then the preradical t is α-dcohereditary.*

Proof. Let $B \in \alpha$ and $t(B) \subseteq A \subseteq B$. Then $t(B) = dh_{\alpha}(t)(B)$ by Proposition 6(iii) so that Proposition 6(ii) and the hypothesis gives $t(B/A) \subseteq dh_{\alpha}(t)(B/A) = 0$, and it suffices to use Proposition 2.

Proposition 9. *If α is closed under direct products then every preradical t is α-dhereditary.*

Proof. For a module A let g be the composition of natural homomorphisms $A \to A/p^{\alpha}(A) \to \prod \{A/\operatorname{Ker} f \mid f : A \to M;\ M \in \alpha\} \to \prod \{M_f \mid f : A \to M;\ M_f = M \in \alpha\}$. Since by the hypothesis the last module lies in α and $\operatorname{Ker} g = p^{\alpha}(A)$, we have $dh_{\alpha}(t)(A) \subseteq C_t(\operatorname{Ker} g : A) = (p^{\alpha} \lhd t)(A)$ and Proposition 6(ii) completes the proof.

Definition 4. *For a preradical t and $A \in R$-mod put* $dch_{(r,\alpha)}(t)(A) = t(A) \cap (\bigcap \{g(C_t(\operatorname{Ker} f : P)) \mid f : P \to M;\ M \in \alpha\})$, where $0 \to K \to P \to^{g} A \to 0$ is an r-projective presentation of A with $r(K) = 0$. For $r = $ zer we shall write briefly $dch_{\alpha}(t)$.

Proposition 10. *If t is a preradical then*

(i) *$dch_{(r,\mathscr{a})}(t)$ is a preradical. In particular, $dch_{(r,\mathscr{a})}(t)(A)$ does not depend on the particular choice of an r-projective presentation of A;*

(ii) *$dch_{(r,\mathscr{a})}(t) \leqq t$;*

(iii) *$dch_{(r,\mathscr{a})}(t)(A) = t(A)$ for every r-projective module A;*

(iv) *if u is an $(r, m(\mathscr{a}))$-dcohereditary preradical with $u \leqq t$ then $u \leqq dch_{(r,\mathscr{a})}(t)$.*

Proof. (i) If $h : A \to B$ is a homomorphism then we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] & P_1 \arrow[r, "g_1"] \arrow[d, "k"'] & A \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & K_2 \arrow[r] & P_2 \arrow[r, "g_2"] & B \arrow[r] & 0 \end{tikzcd}$$

the rows of which are r-projective presentations of the respective modules with $r(K_1) = r(K_2) = 0$. Using the induced maps for suitable factors (similarly as in the proof of 6(i)) one can easily check that $hg_1(C_t(k^{-1}(\mathrm{Ker} f) : P_1)) \subseteq g_2(C_t(\mathrm{Ker} f : P_2))$ for every $f : P_2 \to M$, $M \in \mathscr{a}$, and consequently $h(dch_{(r,\mathscr{a})}(t)(A)) \subseteq h(t(A) \cap \cap (\bigcap \{g_1(C_t(\mathrm{Ker} fk : P_1)) \mid f : P_2 \to M,\ M \in \mathscr{a}\})) \subseteq t(B) \cap (\bigcap \{hg_1(C_t(k^{-1}(\mathrm{Ker} f) : : P_1)) \mid f : P_2 \to M;\ M \in \mathscr{a}\}) \subseteq dch_{(r,\mathscr{a})}(t)(B)$. The special case follows by setting $h = 1_A$.

(ii) Obvious.

(iii) In Definition 4 we can take $P = A$ and $g = 1_A$. Then $t(A) \subseteq C_t(\mathrm{Ker} f : A)$ for each $f : A \to M$, $M \in \mathscr{a}$, yields $dch_{(r,\mathscr{a})}(t)(A) = t(A) \cap (\bigcap \{g(C_t(\mathrm{Ker} f : A)) \mid f : : A \to M;\ M \in \mathscr{a}\}) = t(A)$.

(iv) Let $A \in R$-mod be arbitrary and let $0 \to K \to P \to^g A \to 0$ be an r-projective presentation of A with $r(K) = 0$. Then $(K + \mathrm{Ker} f)/\mathrm{Ker} f \cong K/K \cap \mathrm{Ker} f \in \mathscr{F}_r$ for each $f : P \to M$, $M \in \mathscr{a}$, r being cohereditary. Now u is $(r, m(\mathscr{a}))$-dcohereditary, so that $P/\mathrm{Ker} f \in m(\mathscr{a})$ yields $C_u((K + \mathrm{Ker} f) : P)/(K + \mathrm{Ker} f) = u(P/(K + + \mathrm{Ker} f)) \cong u(P/\mathrm{Ker} f/(K + \mathrm{Ker} f)/\mathrm{Ker} f) = (u(P/\mathrm{Ker} f) + + (K + \mathrm{Ker} f)/\mathrm{Ker} f)/(K + \mathrm{Ker} f)/\mathrm{Ker} f \cong (C_u(\mathrm{Ker} f : P) + K)/(K + \mathrm{Ker} f)$. Consequently $u(A) = g(C_u(K : P)) \subseteq g(C_u((K + \mathrm{Ker} f) : P)) = g(C_u(\mathrm{Ker} f : P) + K) \subseteq \subseteq g(C_t(\mathrm{Ker} f : P))$ and hence $u(A) \subseteq t(A) \cap (\bigcap \{g(C_t(\mathrm{Ker} f : P)) \mid f : P \to M;\ M \in \in \mathscr{a}\}) = dch_{(r,\mathscr{a})}(t)(A)$.

Proposition 11. *If R is left perfect then $\overline{dch_{(r,\mathscr{a})}(t)}$ is the largest $(r, m(\mathscr{a}))$-dcohereditary idempotent preradical contained in t.*

Proof. If $u \leqq t$ is an arbitrary $(r, m(\mathscr{a}))$-dcohereditary idempotent preradical then $u \leqq \overline{dch_{(r,\mathscr{a})}(t)} \leqq t$ by Proposition 10(iv) and (ii) and it remains to show that $\overline{dch_{(r,\mathscr{a})}(t)}$ is $(r, m(\mathscr{a}))$-dcohereditary. Let Q be a $dch_{(r,\mathscr{a})}$-torsion module and let $0 \to K \to P \to^g Q \to 0$ be an r-projective cover of Q. Then necessarily $r(K) = 0$ and $Q = t(Q) \cap (\bigcap \{g(C_t(\mathrm{Ker}\, h : P)) \mid h : P \to M;\ M \in \mathscr{a}\})$ shows that $g(C_t(\mathrm{Ker}\, h : : P)) = Q$, hence $K + C_t(\mathrm{Ker}\, h : P) = P$ and consequently $C_t(\mathrm{Ker}\, h : P) = P$ for every $h : P \to M$, $M \in \mathscr{a}$, K being small in P.

Let $M \in \mathscr{A}$ and $f : P \to M$ be arbitrary. Then $0 \to \operatorname{Ker} f/r(\operatorname{Ker} f) \to P/r(\operatorname{Ker} f) \to^{v}$ $\to^{v} P/\operatorname{Ker} f \to 0$ is an r-projective presentation of $P/\operatorname{Ker} f$ with $r(\operatorname{Ker} f/r(\operatorname{Ker} f)) = 0$ and so $dch_{(r,\mathscr{A})}(t)\,(P/\operatorname{Ker} f) = t(P/\operatorname{Ker} f) \cap (\bigcap \{v(C_t(\operatorname{Ker} \bar{h} : P/r(\operatorname{Ker} f))) \mid \bar{h} :$ $: P/r(\operatorname{Ker} f) \to M;\ M \in \mathscr{A}\})$. However, denoting by h the composed map of $\bar{h}$ and the canonical projection $P \to P/r(\operatorname{Ker} f)$ we have $\operatorname{Ker} h/r(\operatorname{Ker} f) = \operatorname{Ker} \bar{h}$ and so $C_t(\operatorname{Ker} \bar{h} : P/r(\operatorname{Ker} f))/\operatorname{Ker} \bar{h} = t(P/r(\operatorname{Ker} f)/\operatorname{Ker} h/r(\operatorname{Ker} f)) \cong t(P/\operatorname{Ker} h) =$ $= C_t(\operatorname{Ker} h : P)/\operatorname{Ker} h \cong C_t(\operatorname{Ker} h : P)/r(\operatorname{Ker} f)/\operatorname{Ker} \bar{h}$, which together with the above part yields $v(C_t(\operatorname{Ker} \bar{h} : P/r(\operatorname{Ker} f))) = C_t(\operatorname{Ker} h : P)/\operatorname{Ker} f = P/\operatorname{Ker} f$. Hence $K + C_{\overline{dch_{(r,\mathscr{A})}(t)}}(\operatorname{Ker} f : P) = K + C_t(\operatorname{Ker} f : P) = P$ and $\overline{dch_{(r,\mathscr{A})}(t)}$ is $(r, m(\mathscr{A}))$-dcoherediary by Proposition 5.

3. GENERALIZED PSEUDOPROJECTIVE MODULES

Definition 5. A module Q is called $(r, \mathscr{A})$-*pseudoprojective* if $p_{\{Q\}}$ is $(r, m(\mathscr{A}))$-dcoherediatary, and is called $c(r, \mathscr{A})$-*pseudoprojective* if $\tilde{p}_{\{Q\}}$ is $(r, m(\mathscr{A}))$-dcoherediatary. For $r =$ zer we shall use the term $\mathscr{A}$-psuedoprojective and $c\mathscr{A}$-pseudodoprojective.

Lemma 1. *Let t be a preradical. Then the class $\mathscr{B}$ of all modules N having an r-projective presentation $0 \to K \to\to P \to N \to 0$ with $r(K) = 0$ and $K + C_t(\operatorname{Ker} f :$ $: P) = P$ for each $f : P \to M$, $M \in \mathscr{A}$, is closed under direct sums, homomorphic images and extensions of r-torsionfree modules.*

Proof. Let $N_i \in \mathscr{B}$, $i \in I$, and let $0 \to K_i \to P_i \to N_i \to 0$ be an r-projective presentation of N_i with $r(K_i) = 0$ and $K_i + C_t(\operatorname{Ker} f_i : P_i) = P_i$ for each $f_i : P_i \to M$, $M \in \mathscr{A}$. Then $0 \to \sum_{i\in I}^{\oplus} K_i \to \sum_{i\in I}^{\oplus} P_i \to \sum_{i\in I}^{\oplus} N_i \to 0$ is obviously an r-projective presentation of $\sum_{i\in I}^{\oplus} N_i$ with $r(\sum_{i\in I}^{\oplus} K_i) = 0$. If $\alpha_j : P_j \to \sum_{i\in I}^{\oplus} P_i$ is the canonical embedding then obviously $\sum_{i\in I}^{\oplus} \operatorname{Ker} f\alpha_i \subseteq \operatorname{Ker} f$ for every $f : \sum_{i\in I}^{\oplus} P_i \to M$, $M \in \mathscr{A}$, and by hypothesis $K_i + C_t(\operatorname{Ker} f\alpha_i : P_i) = P_i$ for each $i \in I$. Further, $\sum_{i\in I}^{\oplus} C_t(\operatorname{Ker} f\alpha_i : P_i) /$ $/ \sum_{i\in I}^{\oplus} \operatorname{Ker} f\alpha_i \cong \sum_{i\in I}^{\oplus} C_t(\operatorname{Ker} f\alpha_i : P_i)/\operatorname{Ker} f\alpha_i = \sum_{i\in I}^{\oplus} t(P_i/\operatorname{Ker} f\alpha_i) =$ $= t(\sum_{i\in I}^{\oplus}(P_i/\operatorname{Ker} f\alpha_i)) \cong t(\sum_{i\in I}^{\oplus} P_i/\sum_{i\in I}^{\oplus} \operatorname{Ker} f\alpha_i) = C_t(\sum_{i\in I}^{\oplus} \operatorname{Ker} f\alpha_i :$ $: \sum_{i\in I}^{\oplus} P_i)/\sum_{i\in I}^{\oplus} \operatorname{Ker} f\alpha_i$ gives $\sum_{i\in I}^{\oplus} K_i + C_t(\operatorname{Ker} f : \sum_{i\in I}^{\oplus} P_i) \supseteq \sum_{i\in I}^{\oplus} K_i + C_t(\sum_{i\in I}^{\oplus} \operatorname{Ker} f\alpha_i :$ $: \sum_{i\in I}^{\oplus} P_i) = \sum_{i\in I}^{\oplus} (K_i + C_t(\operatorname{Ker} f\alpha_i : P_i)) = \sum_{i\in I}^{\oplus} P_i$ and $\sum_{i\in I}^{\oplus} N_i \in \mathscr{B}$.

Now let $N \in \mathscr{B}$, $X \subseteq N$ and let $0 \to K \to P \to^{g} N \to 0$ be an r-projective presentation of N with $r(K) = 0$ and $K + C_t(\operatorname{Ker} f : P) = P$ for each $f : P \to M$, $M \in \mathscr{A}$. If we denote $L = g^{-1}(X)$ then it is easy to see that $0 \to L/r(L) \to P/r(L) \to N/X \to 0$ is an r-projective presentation of N/X with $r(L/r(L)) = 0$. Now let $\bar{f} : P/r(L) \to M$, $M \in \mathscr{A}$, be arbitrary and let f be $\bar{f}$ composed with the natural projection $P \to P/r(L)$. Then $C_t(\operatorname{Ker} \bar{f} : P/r(L))/\operatorname{Ker} \bar{f} = t(P/r(L)/\operatorname{Ker} f/r(L)) \cong t(P/\operatorname{Ker} f) =$

$= C_t(\mathrm{Ker}\, f : P)/\mathrm{Ker}\, f \cong C_t(\mathrm{Ker}\, f : P)/r(L)/\mathrm{Ker}\, \bar{f}$ gives $L/r(L) + C_t(\mathrm{Ker}\, \bar{f} : P/r(L)) =$ $= (L + C_t(\mathrm{Ker}\, f : P))/r(L) = P/r(L)$ and $N/X \in \mathscr{B}$.

Finally assume that r is idempotent and consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & N \arrow[r, "\pi"] \arrow[d] & L \arrow[d] & \\
0 \arrow[r] & P \arrow[r, "\iota"] \arrow[d, "l"] & P \oplus Q \arrow[r, "\pi"] \arrow[d, "g"] & Q \arrow[r] \arrow[d, "h"] \arrow[dl, dashed, "k"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $A, C \in \mathscr{B} \cap \mathscr{F}_r$ and the left and right columns are the corresponding r-projective presentations. Further, $pk = h$, g is induced by il and k, $N = \mathrm{Ker}\, g$ and obviously $pk\pi = pg$. Moreover, $\pi(r(N)) \subseteq r(L) = 0$, hence $r(N) \subseteq K$ and so $r(N) = 0$, r being idempotent. Now let $f : P \oplus Q \to M$, $M \in \mathcal{A}$, be arbitrary, and let $\alpha : Q \rightarrowtail P \oplus Q$ be the canonical embedding. Obviously, $C_t(\mathrm{Ker}\, f\iota : P) \subseteq C_t(\mathrm{Ker}\, f : P \oplus Q)$ and so $P = K + C_t(\mathrm{Ker}\, f\iota : P) \subseteq N + C_t(\mathrm{Ker}\, f : P \oplus Q)$. Now for $x \in L$ we have $pk(x) = h(x) = 0$ and so $k(x) = il(y)$ for some $y \in P$. Hence $g(x - y) = 0$ and thus $L \subseteq N + P$. But then $Q = L + C_t(\mathrm{Ker}\, f\alpha : Q) \subseteq N + C_t(\mathrm{Ker}\, f : P \oplus Q)$ as desired.

Theorem 1. *The following assertions are equivalent for a module Q:*

(i) *Q is $(r, \mathcal{A})$-pseudoprojective;*

(ii) *if $0 \to K \to P \to Q \to 0$ is any r-projective presentation of Q with $r(K) = 0$ then $K + C_{p\{Q\}}(\mathrm{Ker}\, f : P) = P$ for all $f : P \to M$, $M \in \mathcal{A}$;*

(iii) *there is an r-projective presentation $0 \to K \to P \to Q \to 0$ of Q with $r(K) = 0$ such that $K + C_{p\{Q\}}(\mathrm{Ker}\, f : P) = P$ for all $f : P \to M$, $M \in \mathcal{A}$;*

(iv) *for every epimorphisms $h : B \to A$, $p : A \to C$ with $B \in \mathcal{A}$ and $r(\mathrm{Ker}\, h) = 0$ and for every homomorphism $f : Q \to A$ with $pf \neq 0$ there are homomorphisms $k : Q \to Q/\mathrm{Ker}\, f$ and $g : Q \to B$ such that $phg \neq 0$ and $hg = \bar{f}k$ where $\bar{f} : Q/\mathrm{Ker}\, f \to A$ is induced by f in the natural way;*

(v) *for every epimorphisms $h : B \to A$, $p : A \to C$ with $B \in m(\mathcal{A})$ and $r(\mathrm{Ker}\, h) = 0$ and for every homomorphism $f : Q \to A$ with $pf \neq 0$ there is a homomorphism $g : Q \to B$ such that $phg \neq 0$.*

If $r =$ zer *then the above conditions are equivalent to:*

(vi) *for every exact sequence* $0 \to A \to B \to C \to 0$ *with* $B \in m(\mathscr{a})$ *and* $A, C \in \mathscr{T}_{p\{Q\}}$ *we have* $B \in \mathscr{T}_{p\{Q\}}$ *and for* $A \subseteq B$, $B \in e(m(\mathscr{a}))$ *with* $\mathrm{Hom}_R(Q, B) = 0$ *we have* $\mathrm{Hom}_R(Q, B/A) = 0$.

If the module Q *has an r-projective cover* $0 \to K \to P \to Q \to 0$ *then the conditions* (i)–(v) *are equivalent to the following ones*:

(vii) $p_{\{Q\}}(P/\mathrm{Ker}\, f) = P/\mathrm{Ker}\, f$ *for every* $f : P \to M$, $M \in \mathscr{a}$,

(viii) *for every* $f : P \to M$, $M \in \mathscr{a}$, *the factor-module* $P/\mathrm{Ker}\, f$ *is a homomorphic image of some direct copower of* Q;

(ix) $dh_{\mathscr{a}}(p_{\{Q\}})(P) = P$;

(x) $dh_{\mathscr{a}}(p_{\{Q\}}) = dh_{\mathscr{a}}(p_{\{P\}})$;

(xi) $p_{\{Q\}}(X) = p_{\{P\}}(X)$ *for each* $X \in m(\mathscr{a})$;

(xii) *for every* $X \in m(\mathscr{a})$ *with* $p_{\{P\}}(X) = X$ *we have* $p_{\{Q\}}(X) = X$.

If, moreover, $r = \mathrm{zer}$ *then the conditions* (i)–(xii) *are equivalent to*:

(xiii) *for every exact sequence* $0 \to A \to B \to C \to 0$ *with* $B \in m(\mathscr{a})$ *and* $A, C \in \mathscr{T}_{p\{Q\}}$ *we have* $B \in \mathscr{T}_{p\{Q\}}$ *and for each* $X \in e(m(\mathscr{a}))$, $\mathrm{Hom}_R(Q, X) = 0$ *iff* $\mathrm{Hom}_R(P, X) = 0$.

Proof. (i) $\Rightarrow$ (ii). By Proposition 5. (ii) $\Rightarrow$ (iii). Obvious. (iii) $\Rightarrow$ (iv). Let $h : B \to A$ and $p : A \to C$ be epimorphisms with $B \in \mathscr{a}$ and $r(\mathrm{Ker}\, h) = 0$ and let $f : Q \to A$ be a homomorphism with $pf \neq 0$. By (iii), Q has an r-projective presentation $0 \to K \to P \to^{q} Q \to 0$ with $r(K) = 0$ and $K + C_{p\{Q\}}(\mathrm{Ker}\, \alpha : P) = P$ for every $\alpha : P \to M$, $M \in \mathscr{a}$, and we can consider the following commutative diagram where $l = \bar{l}\sigma$ exists by the r-projectivity of P:

$$\begin{tikzcd}
 & & \mathrm{Ker}\, l \arrow[r] \arrow[d] & \mathrm{Ker}\, f \arrow[d] & \\
0 \arrow[r] & K \arrow[r] & P \arrow[r, "q"] \arrow[d, "\sigma"'] & Q \arrow[r] \arrow[d, "\pi"] & 0 \\
 & & P/\mathrm{Ker}\, l \arrow[r, "\bar{q}"] \arrow[d, "\bar{l}"'] & Q/\mathrm{Ker}\, f \arrow[d, "\bar{f}"] & \\
0 \arrow[r] & \mathrm{Ker}\, h \arrow[r] & B \arrow[r, "h"] & A \arrow[r, "p"] & C
\end{tikzcd}$$

Now if $pf\bar{q}t = 0$ for every $t : Q \to P/\mathrm{Ker}\, l$ then $0 = pf\bar{q}(p_{\{Q\}}(P/\mathrm{Ker}\, l)) = pf\bar{q}\sigma(C_{p\{Q\}}(\mathrm{Ker}\, l : P)) = pf\pi q(K + C_{p\{Q\}}(\mathrm{Ker}\, l : P)) = pf(Q) \neq 0$, which is a contradiction. Hence there is $t : Q \to P/\mathrm{Ker}\, l$ with $pf\bar{q}t \neq 0$ and we can set $k = \bar{q}t$ and $g = \bar{l}t$. Then $hg = h\bar{l}t = \bar{f}\bar{q}t = \bar{f}k$ and $phg = ph\bar{l}t = pf\bar{q}t \neq 0$ as desired.

(iv) $\Rightarrow$ (v). Without loss of generality we can suppose that $A = B/K$, $C = B/L$, $K \subseteq L$, $r(K) = 0$ and consider the following commutative diagram with $D \in \mathscr{a}$ and

$m : B \to D$ an embedding:

$$\begin{tikzcd}
& Q \arrow[d, "\pi"] \\
Q \arrow[r, "k"] \arrow[d, "g"] \arrow[dd, bend right, "\bar{g}"'] & Q/\mathrm{Ker}\, f \arrow[d, "\bar{f}"] \\
B \arrow[r, "h"] \arrow[d, "m"] & B/K \arrow[r, "p"] \arrow[d, "i"] & B/L \arrow[d, "j"] \\
D \arrow[r, "l"] & D/K \arrow[r, "q"] & D/L
\end{tikzcd}$$

The homomorphisms $k, \bar{g}$ with $l\bar{g} = i\bar{f}k$ and $ql\bar{g} \neq 0$ exist by the hypothesis since $qi\bar{f} = jp\bar{f} \neq 0$. The left bottom square is 1 pullback and so there is g with $mg = \bar{g}$ and $hg = \bar{f}k$. Henceforth, $jphg = qlmg = ql\bar{g} \neq 0$ and so $phg \neq 0$.

(v) $\Rightarrow$ (ii). Let $0 \to K \to P \to Q \to 0$ be an r-projective presentation of Q with $r(K) = 0$ and suppose that for some $k : P \to M$, $M \in \mathscr{a}$, we have $K + C_{p_{\{Q\}}}(\mathrm{Ker}\, k : P) \subsetneqq P$. Then for the natural epimorphisms

$$\begin{tikzcd}
& P/K \cong Q \arrow[d, "f"] \\
P/\mathrm{Ker}\, k \arrow[r, "h"] & P/(K + \mathrm{Ker}\, k) \arrow[r, "p"] & P/(K + C_{p_{\{Q\}}}(\mathrm{Ker}\, k : P))
\end{tikzcd}$$

we have $pf \neq 0$, $P/\mathrm{Ker}\, k \in m(\mathscr{a})$ and $r(\mathrm{Ker}\, h) = r((K + \mathrm{Ker}\, k)/\mathrm{Ker}\, k) \cong r(K/K \cap \mathrm{Ker}\, k) = 0$, r being cohereditary. Hence by (v) there is a homomorphism $g : P/K \to P/\mathrm{Ker}\, k$ such that $phg \neq 0$. However, $\mathrm{Im}\, g \subseteq p_{\{Q\}}(P/\mathrm{Ker}\, k) = C_{p_{\{Q\}}}(\mathrm{Ker}\, k : P)/\mathrm{Ker}\, k$ and so $ph(\mathrm{Im}\, g) = 0$, which is a contradiction.

(iii) $\Rightarrow$ (i). Let $\mathscr{B}$ be the class of all modules N having an r-projective presentation $0 \to K \to P \to N \to 0$ with $r(K) = 0$ and $K + C_{p_{\{Q\}}}(\mathrm{Ker} f : P) = P$ for each $f : P \to M$, $M \in \mathscr{a}$. Since $Q \in \mathscr{B}$ by the hypothesis, Lemma 1 gives $\mathscr{T}_{p_{\{Q\}}} \subseteq \mathscr{B}$ and it suffices to use Proposition 5.

(i) $\Leftrightarrow$ (vi). It follows easily from Propositions 3 and 4.

(ii) $\Rightarrow$ (vii), (vii) $\Rightarrow$ (iii), (vii) $\Rightarrow$ (viii), (xi) $\Rightarrow$ (xii). Obvious.

(viii) $\Rightarrow$ (ix). We have $dh_{\mathscr{a}}(p_{\{Q\}})(P) = \bigcap\{C_{p_{\{Q\}}}(\mathrm{Ker} f : P) \mid f : P \to M;\ M \in \mathscr{a}\} = P$ by hypothesis.

(ix) $\Rightarrow$ (x). Obviously $p_{\{Q\}} \leqq p_{\{P\}}$ and $p_{\{P\}} \leqq dh_{\mathscr{a}}(p_{\{Q\}})$ by hypothesis. Hence Proposition 6(iv) gives $dh_{\mathscr{a}}(p_{\{P\}}) \leqq dh_{\mathscr{a}}(dh_{\mathscr{a}}(p_{\{Q\}})) = dh_{\mathscr{a}}(p_{\{Q\}}) \leqq dh_{\mathscr{a}}(p_{\{P\}})$.

(x) $\Rightarrow$ (xi). By Proposition 6(iii).

(xii) $\Rightarrow$ (vii). If $f : P \to M$, $M \in \mathscr{a}$, then $P/\mathrm{Ker} f \in m(\mathscr{a})$ and so $p_{\{P\}}(P/\mathrm{Ker} f) = P/\mathrm{Ker} f = p_{\{Q\}}(P/\mathrm{Ker} f)$.

(vi), (xi) $\Rightarrow$ (xiii). The first part of (xiii) follows immediately from (vi). Since $p_{\{Q\}} \leqq p_{\{P\}}$, $\mathrm{Hom}_R(P, X) = 0$ implies $\mathrm{Hom}_R(Q, X) = 0$. Let $\mathrm{Hom}_R(Q, X) = 0$ and

$X = Y/Z$ for some $Y \in m(\alpha)$. If $p : Y \to X$ is the canonical projection and $f : P \to X$ is an arbitrary homomorphism then there is $g : P \to Y$ with $pg = f$, P being in this case projective. Thus $\operatorname{Im} g \subseteq p_{\{P\}}(Y) = p_{\{Q\}}(Y) \subseteq Z$ by (xi) and the hypothesis and so $f(P) = pg(P) = 0$.

(xiii) $\Rightarrow$ (vii). Let $f : P \to M$, $M \in \alpha$, be arbitrary. Then $P/\operatorname{Ker} f \in m(\alpha)$ so that the hypothesis and Proposition 4 give $X = P/\operatorname{Ker} f/p_{\{Q\}}(P/\operatorname{Ker} f) \in \mathscr{F}_{p\{Q\}}$. But then $X \in \mathscr{F}_{p\{P\}}$ by the hypothesis and consequently $X = 0$.

Theorem 2. *The following assertions are equivalent for a module Q:*

(i) *Q is c α-pseudoprojective;*

(ii) *if $0 \to K \to P \to Q \to 0$ is any projective presentation of Q then $K + C_{\tilde{p}\{Q\}}(\operatorname{Ker} f : P) = P$ for all $f : P \to M$, $M \in \alpha$;*

(iii) *there is a projective presentation $0 \to K \to P \to Q \to 0$ of Q such that $K + C_{\tilde{p}\{Q\}}(\operatorname{Ker} f : P) = P$ for all $f : P \to M$, $M \in \alpha$;*

(iv) *if $A \subseteq B$, $B \in e(m(\alpha))$ and $\operatorname{Hom}_R(Q, B) = 0$ then $\operatorname{Hom}_R(Q, B/A) = 0$.*

If the module Q has a projective cover $0 \to K \to P \to Q \to 0$ then the above conditions are equivalent to the following ones:

(v) *$dh_\alpha(\tilde{p}_{\{Q\}})(P) = P$;*

(vi) *$dh_\alpha(\tilde{p}_{\{Q\}}) = dh_\alpha(p_{\{P\}})$;*

(vii) *$\tilde{p}_{\{Q\}}(X) = p_{\{P\}}(X)$ for every $X \in m(\alpha)$;*

(viii) *if $X \in m(\alpha)$ is such that $p_{\{P\}}(X) = X$ then $\operatorname{Hom}_R(Q, X/Z) \neq 0$ for each $Z \subsetneqq X$;*

(ix) *if $0 \neq Y \in e(m(\alpha))$ is a homomorphic image of P then $\operatorname{Hom}_R(Q, Y) \neq 0$;*

(x) *$\tilde{p}_{\{Q\}}(X) = p_{\{P\}}(X)$ for every $X \in e(m(\alpha))$;*

(xi) *for every $X \in e(m(\alpha))$ we have $\operatorname{Hom}_R(Q, X) = 0$ iff $\operatorname{Hom}_R(P, X) = 0$.*

Proof. (i) $\Rightarrow$ (ii). By Proposition 5.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). By hypothesis and Lemma 1 we have $\mathscr{T}_{\tilde{p}\{Q\}} \subseteq \mathscr{B}$, where $\mathscr{B}$ is the class of all modules N having a projective presentation $0 \to K \to P \to N \to 0$ with $K + C_{\tilde{p}\{Q\}}(\operatorname{Ker} f : P) = P$ for each $f : P \to M$, $M \in \alpha$. Now it suffices to use Proposition 5.

(i) $\Leftrightarrow$ (iv). By Proposition 3.

(ii) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (vi). We have $p_{\{P\}} \leqq dh_\alpha(\tilde{p}_{\{Q\}})$ by hypothesis. The obvious inequality $p_{\{Q\}} \leqq p_{\{P\}}$ yields $\tilde{p}_{\{Q\}} \leqq p_{\{P\}}$, $p_{\{P\}}$ being a radical. Now Proposition 6(iv) gives $dh_\alpha(p_{\{P\}}) \leqq dh_\alpha(dh_\alpha(\tilde{p}_{\{Q\}})) = dh_\alpha(\tilde{p}_{\{Q\}}) \leqq dh_\alpha(p_{\{P\}})$.

(vi) $\Rightarrow$ (vii). By Proposition 6(iii).

(vii) ⇒ (viii). If $X \in m(\mathscr{a})$, $p_{\{P\}}(X) = X$ and $Z \subsetneqq X$ then $\tilde{p}_{\{Q\}}(X) = X$ by hypothesis and so $\tilde{p}_{\{Q\}}(X/Z) = X/Z$ yields $p_{\{Q\}}(X/Z) \neq 0$. Thus $\mathrm{Hom}_R(Q, X/Z) \neq 0$.
(viii) ⇒ (ix). Let $Y = X/Z$, $X \in m(\mathscr{a})$, and let $g : P \to Y$ be an epimorphism. There is a homomorphism $f : P \to X$ which composed with $X \to X/Z$ gives g. Now $p_{\{P\}}(\mathrm{Im}\, f) = \mathrm{Im}\, f$ and so by hypothesis $\mathrm{Hom}_R(Q, \mathrm{Im}\, f/Z \cap \mathrm{Im}\, f) \neq 0$. However, it is easy to see that $X = Z + \mathrm{Im}\, f$ and so $Y \cong \mathrm{Im}\, f/Z \cap \mathrm{Im}\, f$.
(ix) ⇒ (x). Let $X = Y/Z$, $Y \in m(\mathscr{a})$, be such that $\tilde{p}_{\{Q\}}(Y) \subsetneqq p_{\{P\}}(Y)$. Then there is $f : P \to Y$ with $\mathrm{Im}\, f \nsubseteq \tilde{p}_{\{Q\}}(Y)$ and so, by (ix), there is $0 \neq g : Q \to \mathrm{Im}\, f/\mathrm{Im}\, f \cap$ $\cap\, \tilde{p}_{\{Q\}}(Y)$. But $\mathrm{Im}\, f/\mathrm{Im}\, f \cap \tilde{p}_{\{Q\}}(Y) \cong (\tilde{p}_{\{Q\}}(Y) + \mathrm{Im}\, f)/\tilde{p}_{\{Q\}}(Y) \in \mathscr{F}_{p\{Q\}}$, which is a contradiction.
(x) ⇒ (xi). Immediate since $\mathrm{Hom}_R(A, B) = 0$ means $p_{\{A\}}(B) = 0$.
(xi) ⇒ (v). If $f : P \to M$, $M \in \mathscr{a}$, is arbitrary then $P/\mathrm{Ker}\, f \in m(\mathscr{a})$ and for $\tilde{p}_{\{Q\}}(P/\mathrm{Ker}\, f) = X/\mathrm{Ker}\, f$ we have $\tilde{p}_{\{Q\}}(P/X) = 0 = \mathrm{Hom}_R(Q, P/X)$. Thus $\mathrm{Hom}_R(P, P/X) = 0$ since $P/X \in e(m(\mathscr{a}))$ and consequently $X = P$.
(v) ⇒ (iii). Obvious.

Definition 6. Let s be a preradical. A module Q is said to be (r, s, i)-*pseudoprojective*, $i = 1, 2, 3$, if for every diagram

$$\begin{tikzcd}
 & Q \arrow[d, "\pi"] & \\
Q \arrow[r, dashed, "k"] \arrow[d, dashed, "g"'] & Q/\mathrm{Ker}\, f \arrow[d, "\bar{f}"] & \\
B \arrow[r, "h"] & A \arrow[r, "p"] & C
\end{tikzcd}$$

where h, p are epimorphisms, $f = \bar{f}\pi$, $r(\mathrm{Ker}\, h) = 0$, $h^{-1}(\mathrm{Im}\, f) \subseteq^{(s,i)} B$ and $pf \neq 0$, there are homomorphisms $k : Q \to Q/\mathrm{Ker}\, f$ and $g : Q \to B$ such that $phg \neq 0$ and $hg = \bar{f}k$.

Lemma 2. *Let s be a preradical. A module Q is (r, s, i)-pseudoprojective iff it is $(r, \mathscr{a}_i)$-pseudoprojective where $\mathscr{a}_i = \{M \mid M \subseteq^{(s,i)} M\}$, $i = 1, 2, 3$.*

Proof. Necessity easily follows from the definitions and the simple fact that $B \subseteq^{(s,i)} B$ implies $h^{-1}(\mathrm{Im}\, f) \subseteq^{(s,i)} B$.

As concerns sufficiency, the diagram

$$\begin{tikzcd}
 & Q \arrow[d, "\pi"] & \\
 & Q/\mathrm{Ker}\, f \arrow[d, "\bar{f}"] & \\
B \arrow[r, "h"] & A \arrow[r, "p"] & C
\end{tikzcd}$$

with epimorphisms h, p, $r(\mathrm{Ker}\, h) = 0$, $h^{-1}(\mathrm{Im}\, f) \subseteq^{(s,i)} B$ and $p\bar{f} \neq 0$ induces the commutative diagram

$$\begin{tikzcd}
& Q \arrow[d, "\pi"] & \\
& Q/\mathrm{Ker}\, f \arrow[d, "\bar{f}"] & \\
h^{-1}(\mathrm{Im}\, f) \arrow[r, "h"] \arrow[d] & \mathrm{Im}\, f \arrow[r, "p"] \arrow[d] & \mathrm{Im}\, pf \arrow[d] \\
B \arrow[r, "h"] & A \arrow[r, "p"] & C
\end{tikzcd}$$

in the natural way. Now $h^{-1}(\mathrm{Im}\, f) \subseteq^{(s,i)} B$ yields $h^{-1}(\mathrm{Im}\, f) \subseteq^{(s,i)} h^{-1}(\mathrm{Im}\, f)$ and hence by hypothesis there are homomorphisms $k : Q \to Q/\mathrm{Ker}\, f$ and $g : Q \to$ $\to h^{-1}(\mathrm{Im}\, f)$ with $phg \neq 0$ and $\bar{f}k = hg$, from which the assertion easily follows.

Proposition 12. *Let s be a preradical and Q a module. Then*
(i) *Q is $(r, s, 1)$-pseudoprojective iff it is $(r, ch(s), 2)$-pseudoprojective;*
(ii) *Q is $(r, s, 2)$-pseudoprojective iff it is $(r, s, 3)$-pseudoprojective.*

Proof. It is easy to see that $M \subseteq^{(s,1)} M$ iff $M \in \mathscr{F}_{ch(s)}$ and $M \subseteq^{(s,2)} M$ iff $M \in \mathscr{F}_s$ iff $M \subseteq^{(s,3)} M$. Now it suffices to use Lemma 2.

Proposition 13. *If s is a cohereditary radical then a module Q is $(r, s, 2)$-pseudoprojective iff $Q/s(Q)$ is so.*

Proof is a simple consequence of Lemma 2.

Theorem 3. *If s is a preradical then the following assertions are equivalent for a module Q:*
(i) *Q is $(r, s, 2)$-pseudoprojective;*
(ii) *if $0 \to K \to P \to Q \to 0$ is any r-projective presentation of Q with $r(K) = 0$ then $K + (\tilde{s} \lhd p_{\{Q\}})(P) = P$;*
(iii) *there is an r-projective presentation $0 \to K \to P \to Q \to 0$ of Q with $r(K) = 0$ such that $K + (\tilde{s} \lhd p_{\{Q\}})(P) = P$;*
(iv) *$P_{\{Q\}}$ is $(r, \mathscr{F}_s)$-dcohereditary;*
(v) *for every epimorphisms $h : B \to A$, $p : A \to C$ with $B \in \mathscr{F}_s$ and $r(\mathrm{Ker}\, h) = 0$ and for every homomorphism $f : Q \to A$ with $pf \neq 0$ there is a homomorphism $g : Q \to B$ such that $phg \neq 0$.*

If $r =$ zer then the above conditions are equivalent to:
(vi) *for every exact sequence $0 \to A \to B \to C \to 0$ with $B \in \mathscr{F}_s$ and $A, C \in \mathscr{T}_{p\{Q\}}$*

we have $B \in \mathscr{T}_{p\{Q\}}$ *and for* $A \subseteq B$, $B \in e(\mathscr{F}_s)$, *with* $\mathrm{Hom}_R(Q, B) = 0$ *we have* $\mathrm{Hom}_R(Q, B/A) = 0$.

If the module Q *has an r-projective cover* $0 \to K \to P \to Q \to 0$ *then the conditions* (i)–(v) *are equivalent to the following ones*:

(vii) $(\tilde{s} \lhd p_{\{Q\}})(P) = P$;

(viii) $P/\tilde{s}(P)$ *is a homomorphic image of some direct copower of* Q;

(ix) $\tilde{s} \lhd p_{\{Q\}} = \tilde{s} \lhd p_{\{P\}}$;

(x) $p_{\{Q\}}(X) = p_{\{P\}}(X)$ *for each* $X \in \mathscr{F}_s$;

(xi) *for every* $X \in \mathscr{F}_s$ *with* $p_{\{P\}}(X) = X$ *we have* $p_{\{Q\}}(X) = X$.

If, moreover, $r =$ zer *then the above conditions are equivalent to*:

(xii) *for every exact sequence* $0 \to A \to B \to C \to 0$ *with* $B \in \mathscr{F}_s$ *and* $A, C \in \mathscr{T}_{p\{Q\}}$ *we have* $B \in \mathscr{T}_{p\{Q\}}$ *and for each* $X \in e(\mathscr{F}_s)$, $\mathrm{Hom}_R(Q, X) = 0$ *iff* $\mathrm{Hom}_R(P, X) = 0$.

Proof follows immediately from Theorem 1 and from the simple fact that $dh_{\mathscr{F}_s}(t) = \tilde{s} \lhd t$ for every preradicals s, t.

Theorem 4. *If* s *is a preradical then the following assertions are equivalent for a module* Q:

(i) Q *is c* $\mathscr{F}_s$*-pseudoprojective*;

(ii) *if* $0 \to K \to P \to Q \to 0$ *is any projective presentation of* Q *then* $K + (\tilde{s} \lhd \bar{p}_{\{Q\}})(P) = P$;

(iii) *there is a projective presentation* $0 \to K \to P \to Q \to 0$ *of* Q *such that* $K + (\tilde{s} \lhd \bar{p}_{\{Q\}})(P) = P$;

(iv) *if* $A \subseteq B$, $B \in e(\mathscr{F}_s)$ *and* $\mathrm{Hom}_R(Q, B) = 0$ *then* $\mathrm{Hom}_R(Q, B/A) = 0$.

If the module Q *has a projective cover* $0 \to K \to P \to Q \to 0$ *then the above conditions are equivalent to the following ones*:

(v) $(\tilde{s} \lhd \bar{p}_{\{Q\}})(P) = P$;

(vi) $\tilde{s} \lhd \bar{p}_{\{Q\}} = \tilde{s} \lhd p_{\{P\}}$;

(vii) $\bar{p}_{\{Q\}}(X) = p_{\{P\}}(X)$ *for every* $X \in \mathscr{F}_s$;

(viii) *if* $0 \neq Y \in e(\mathscr{F}_s)$ *is a homomorphic image of* P *then* $\mathrm{Hom}_R(Q, Y) \neq 0$;

(ix) *if* $X \in \mathscr{F}_s$ *is such that* $p_{\{P\}}(X) = X$ *then* $\mathrm{Hom}_R(Q, X/Z) \neq 0$ *for each* $Z \subsetneqq X$;

(x) *for every* $X \in e(\mathscr{F}_s)$ *we have* $\mathrm{Hom}_R(Q, X) = 0$ *iff* $\mathrm{Hom}_R(P, X) = 0$.

Proof follows easily from Theorem 2.

Proposition 14. *If* s *is a preradical then a* (zer, s, 2)*-pseudoprojective module* Q *is c* $\mathscr{F}_s$*-pseudoprojective. The converse holds provided that one of the following conditions is satisfied*:

(1) *there is a projective presentation* $0 \to K \to P \to Q \to 0$ *of* Q *such that* $p_{\{Q\}}(P/\tilde{s}(P))$ *is a direct summand of* $P/\tilde{s}(P)$;

(2) $\tilde{s} \lhd p_{\{Q\}}$ *is costable.*

Proof. In view of $\tilde{s} \lhd p_{\{Q\}} \leqq \tilde{s} \lhd \tilde{p}_{\{Q\}}$ it suffices to use Theorem 3(iii) and Theorem 4(iii).

In order to prove the converse let us suppose that the projective presentation of Q from (1) fulfils $K + (\tilde{s} \lhd p_{\{Q\}})(P) \neq P$. Note that if (2) holds then any projective presentation of Q has the property from (1). Denoting $X = (\tilde{s} \lhd p_{\{Q\}})(P)$ we have $P/\tilde{s}(P) = X/\tilde{s}(P) \oplus Y$, $Y \cong P/X$ and since $\tilde{s}(P) \subseteq X$, $Y \in e(\mathscr{F}_s)$. Further, $\mathrm{Hom}_R(Q, P/(K + X)) \neq 0$ and Theorem 4(iv) yields $\mathrm{Hom}_R(Q, Y) \neq 0$. If $i : Y \to P/\tilde{s}(P)$ and $p : P/\tilde{s}(P) \to Y$ are the canonical embedding and projection, respectively, then for $0 \neq f : Q \to Y$ we have $0 \neq if : Q \to P/\tilde{s}(P)$ and $p(if) = f \neq 0$. On the other hand, $p_{\{Q\}}(P/\tilde{s}(P)) = X/\tilde{s}(P)$ means that for each $g : Q \to P/\tilde{s}(P)$ we have $\mathrm{Im}\, g \subseteq X/\tilde{s}(P)$ and so $pg = 0$. This contradiction shows that $K + (\tilde{s} \lhd p_{\{Q\}})(P) = P$ and Theorem 3(iii) can be applied.

Proposition 15. *Every direct sum of* (r, α)*-pseudoprojective modules is* (r, α)*-pseudoprojective.*

Proof. If $p_{\{Q_i\}}$, $i \in I$, are $(r, m(\alpha))$-dcohereditary then $p_{\{\sum\limits_{i\in I}\oplus Q_i\}} = \sum\limits_{i \in I} p_{\{Q_i\}}$ is $(r, m(\alpha))$-dcohereditary by Proposition 1.

Proposition 16. *If* A, B *are modules with* $p_{\{A\}}(B) = B$ *then* $A \oplus B$ *is* (r, α)*-pseudoprojective iff* A *is so.*

Proof. From $p_{\{A\}}(B) = B$ we get $p_{\{B\}} \leqq p_{\{A\}}$ and hence $p_{\{A\oplus B\}} = p_{\{A\}} + p_{\{B\}} = p_{\{A\}}$.

Proposition 17. *If every cocyclic factor-module of* Q *is* α*-pseudoprojective then* Q *is so.*

Proof. Let $h : B \to A$, $p : A \to C$ be epimorphisms with $B \in m(\alpha)$ and let $f : Q \to A$ be an arbitrary homomorphism with $pf \neq 0$. Obviously, there is $k : C \to D$ with D cocyclic ano $kpf \neq 0$ and we can treat the following diagram

$$
\begin{tikzcd}
 & & Q \arrow[d, "\sigma"] \arrow[dll, dashed, "g"'] & & \\
 & & Q/\mathrm{Ker}\, kpf \arrow[d, "\bar{f}"] \arrow[dll, dashed, "\bar{g}"'] & & \\
B \arrow[r, "h"] & A \arrow[r, "\pi"] & A/\mathrm{Ker}\, kp \arrow[r, "\bar{p}"] & C/\mathrm{Ker}\, k \arrow[r, "\bar{k}"] & D
\end{tikzcd}
$$

where σ, π are natural projections and $\bar{f}$, $\bar{p}$, $\bar{k}$ are naturally induced by f, p, k, respectively. Now $Q/\mathrm{Ker}\, kpf \cong \mathrm{Im}\, kpf \subseteq D$ is cocyclic and so by hypothesis and Theorem

1(v) there is $\bar{g} : Q/\mathrm{Ker}\, kpf \to B$ with $\bar{k}\bar{p}\pi h\bar{g} \neq 0$. Setting $g = \bar{g}\sigma$ we have $phg \neq 0$ since $kphg = \bar{k}\bar{p}\pi h\bar{g}\sigma \neq 0$ and it suffices to use Theorem 1(v).

Proposition 18. *A simple module S is $(r, \mathscr{a})$-pseudoprojective iff it is $(r, \mathscr{a})$-projective.*

Proof. Only necessity requires verification. Let $h : B \to A$, $B \in \mathscr{a}$, $r(\mathrm{Ker}\, h) = 0$ be an epimorphism, $p = 1_A$ and let $0 \neq f : S \to A$ be arbitrary. By Theorem 1(iv) there are $k : S \to S$ and $g : S \to B$ with $fk = hg \neq 0$. So k is obviously an isomorphism and $hgk^{-1} = f$.

Proposition 19. *If $\mathscr{a}$ is closed under direct products and Q is an $\mathscr{a}$-pseudoprojective cofaithful module such that $(p^{a} \lhd Y)(Q) = Q$ then $dh_{\mathscr{a}}(p_{\{Q\}}) = p^{a} \lhd p_{\{Q\}} = p^{a} \lhd Y$.*

Proof. From $(p^{a} \lhd Y)(Q) = Q$ it follows that $p_{\{Q\}} \leqq p^{a} \lhd Y$ and [8; I.4.E6] gives $p^{a} \lhd p_{\{Q\}} \leqq p^{a} \lhd (p^{a} \lhd Y) = (p^{a} \lhd p^{a}) \lhd Y = p^{a} \lhd Y$, p^{a} being a radical. To complete the proof it remains to show that $Y(X) \subseteq p_{\{Q\}}(X)$ for each $X \in m(\mathscr{a})$, since then by Proposition 9 (and Definition 3) we have $p^{a} \lhd Y = dh_{\mathscr{a}}(Y) \leqq dh_{\mathscr{a}}(p_{\{Q\}}) =$
$= p^{a} \lhd p_{\{Q\}}$.

Let $X \in m(\mathscr{a})$ be arbitrary and assume that for $N = X/p_{\{Q\}}(X)$ we have $Y(N) \neq 0$. Since $Y(N) = \bigcap \{K \subseteq N \mid N/K$ cocyclic and small in $E(N/K)\}$ and $\bigcap \{L \subseteq N \mid N/L$ cocyclic$\} = 0$, there is a cocyclic factor-module C of N which is not small in $E(C)$. Thus $E(C)$ contains a proper submodule D with $C + D = E(C)$. Now Q is $\mathscr{a}$-pseudoprojective, $N \in e(m(\mathscr{a}))$ and so $p_{\{Q\}}(C/C \cap D) = 0$ by Theorem 1(vi) and Proposition 3. On the other hand, $p_{\{Q\}}(E(C)) = E(C)$, Q being cofaithful, and so $C/C \cap D \cong E(C)/D \in \mathscr{T}_{p_{\{Q\}}}$. This contradiction shows that $Y(X/p_{\{Q\}}(X)) = 0$ and consequently $Y(X) \subseteq p_{\{Q\}}(X)$.

Corollary 2. *If s is a preradical and Q is a (zer, s, 2)-pseudoprojective cofaithful module such that $(\bar{s} \lhd Y)(Q) = Q$ then $\bar{s} \lhd p_{\{Q\}} = \bar{s} \lhd Y$.*

Proof. The class $\mathscr{F}_s$ is closed under direct products and $dh_{\mathscr{F}_s}(t) = \bar{s} \lhd t$ for every preradicals s, t.

Proposition 20. *If R is a left perfect ring, $\mathscr{a}$ a class of modules closed under homomorphic images and direct products and Q is a cofaithful module then the following assertions are equivalent*:

(i) $p^{a} \lhd p_{\{Q\}} = p^{a} \lhd Y$;

(ii) $\mathscr{T}_{p^{a} \lhd p_{\{Q\}}} = \mathscr{T}_{p^{a} \lhd Y}$;

(iii) *Q is $\mathscr{a}$-pseudoprojective and $(p^{a} \lhd Y)(Q) = Q$.*

Proof. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iii). The radical p^{a} is cohereditary since $\mathscr{a}$ is cohereditary. Further, R is left perfect so that by [8; I.10.E1] the radical Y is cohereditary and so $p^{a} \lhd Y$ is cohere-

ditary by [8; Proposition I.4.8]. Consequently, if $0 \to K \to P \to Q \to 0$ is a projective cover of Q then $Q \in \mathscr{T}_{p^{\alpha} \lhd p_{\{Q\}}}$ yields $(p^{\alpha} \lhd Y)(Q) = Q$ by hypothesis and so $(p^{\alpha} \lhd Y)(P) = P$. Now if $f : P \to M$, $M \in \alpha$, is arbitrary then (ii), Proposition 9 and Proposition 6(iii) yield $P/\mathrm{Ker}\, f = (p^{\alpha} \lhd Y)(P/\mathrm{Ker}\, f) = (p^{\alpha} \lhd p_{\{Q\}})(P/\mathrm{Ker}\, f) = dh_{\alpha}(p_{\{Q\}})(P/\mathrm{Ker}\, f) = p_{\{Q\}}(P/\mathrm{Ker}\, f)$ and it suffices to use Theorem 1(vii).
(iii) $\Rightarrow$ (i). By Proposition 19.

Corollary 3. *If R is left perfect and s is a cohereditary radical then the following conditions are equivalent for a cofaithful module Q:*
(i) $s \lhd p_{\{Q\}} = s \lhd Y$;
(ii) $\mathscr{T}_{s \lhd p_{\{Q\}}} = \mathscr{T}_{s \lhd Y}$;
(iii) *Q is* (zer, s, 2)*-pseudoprojective and* $(s \lhd Y)(Q) = Q$.

Definition 7. A module Q is said to be *strongly (r, α)-pseudoprojective* if there is an r-projective presentation $0 \to K \to P \to Q \to 0$ of Q with $r(K) = 0$ such that $dh_{\alpha}(p_{\{Q\}}) = dh_{\alpha}(p_{\{P\}})$.

Proposition 21. *The class of all strongly (r, α)-pseudoprojective modules is closed under arbitrary direct sums.*

Proof. Let Q_i, $i \in I$, be strongly (r, α)-pseudoprojective modules. Then there is an r-projective presentation $0 \to K_i \to P_i \to Q_i \to 0$ of Q_i with $r(K_i) = 0$ such that $dh_{\alpha}(p_{\{Q_i\}}) = dh_{\alpha}(p_{\{P_i\}})$ for each $i \in I$. Then for every $X \in m(\alpha)$ we have $p_{\{Q_i\}}(X) = dh_{\alpha}(p_{\{Q_i\}})(X) = dh_{\alpha}(p_{\{P_i\}})(X) = p_{\{P_i\}}(X)$ by Proposition 6(iii). Denoting $Q = \sum_{i \in I}^{\oplus} Q_i$, $P = \sum_{i \in I}^{\oplus} P_i$, $K = \sum_{i \in I}^{\oplus} K_i$, $0 \to K \to P \to Q \to 0$ is an r-projective presentation of Q with $r(K) = 0$. Now for each $X \in m(\alpha)$ we have $p_{\{Q\}}(X) = \sum_{i \in I} p_{\{Q_i\}}(X) = \sum_{i \in I} p_{\{P_i\}}(X) = p_{\{P\}}(X)$, which easily yields the equality $dh_{\alpha}(p_{\{Q\}}) = dh_{\alpha}(p_{\{P\}})$.

Proposition 22. *Every strongly (r, α)-pseudoprojective module Q is (r, α)-pseudoprojective. The converse holds provided Q has an r-projective cover* $0 \to K \to P \to Q \to 0$.

Proof. For every $f : P \to M$, $M \in \alpha$, we have $K + C_{p_{\{Q\}}}(\mathrm{Ker}\, f : P) \supseteq K + dh_{\alpha}(p_{\{Q\}})(P) = K + dh_{\alpha}(p_{\{P\}})(P) = P$ and it suffices to use Theorem 1(iii), while the converse follows from Theorem 1(x).

Definition 8. A module Q is said to be an *α-generator* if $dh_{\alpha}(p_{\{Q\}}) = \mathrm{id}$. If s is a preradical then Q is called an s-generator if it is an $\mathscr{F}_s$-generator.

Proposition 23. *The following assertions are equivalent for a module Q:*
(i) *Q is an α-generator*;

(ii) *Q is strongly α-pseudoprojective and every simple module from $e(m(\alpha))$ is a homomorphic image of Q;*

(iii) *Q is α-pseudoprojective and every simple module from $e(m(\alpha))$ is a homomorphic image of Q.*

Moreover, if Q has a projective cover $0 \to K \to P \to Q \to 0$ then the above conditions are equivalent to:

(iv) *Q is α-pseudoprojective and P is an α-generator.*

Proof. (i) $\Rightarrow$ (ii). If $0 \to K \to P \to Q \to 0$ is any projective presentation of Q then (i) and the inequality $p_{\{Q\}} \leqq p_{\{P\}}$ give $dh_\alpha(p_{\{P\}}) = dh_\alpha(p_{\{Q\}}) = \mathrm{id}$. Let $S = Y/X$, $Y \in m(\alpha)$, be a simple module. Then $Y = dh_\alpha(p_{\{Q\}})(Y) = p_{\{Q\}}(Y)$ by Proposition 6(iii). Consequently, there is a homomorphism $f : Q \to Y$ with $\mathrm{Im}\, f \nsubseteq X$ and so the composed map $g : Q \to Y \to S$ is an epimorphism, S being simple.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). It is easy to see that Q is an α-generator iff each module from $m(\alpha)$ is $p_{\{Q\}}$-torsion. So, assume that $p_{\{Q\}}(X) \neq X$ for some $X \in m(\alpha)$. Then there is a submodule $p_{\{Q\}}(X) \subseteq Y \subseteq X$ such that X/Y is cocyclic with a simple submodule S. By Theorem 1(vi) the factor-module X/Y, and consequently S, are $p_{\{Q\}}$-torsionfree which contradicts the hypothesis since obviously $S \in e(m(\alpha))$.

(i) $\Rightarrow$ (iv). It has been shown in part (i) $\Rightarrow$ (ii).

(iv) $\Rightarrow$ (i). By hypothesis and Theorem 1(x) we have $dh_\alpha(p_{\{Q\}}) = dh_\alpha(p_{\{P\}}) = \mathrm{id}$.

Corollary 4. *If s is a preradical then the following assertions are equivalent for a module Q:*

(i) *Q is an s-generator;*

(ii) *Q is strongly $\mathscr{F}_s$-pseudoprojective and every simple module from $e(\mathscr{F}_s)$ is a homomorphic image of Q;*

(iii) *Q is* (zer, s, 2)*-pseudoprojective and every simple module from $e(\mathscr{F}_s)$ is a homomorphic image of Q.*

Moreover, if Q has a projective cover $0 \to K \to P \to Q \to 0$ then the above conditions are equivalent to:

(iv) *Q is* (zer, s, 2)*-pseudoprojective and P is an s-generator.*

Proposition 24. *Let $Q = \sum_{S \in \mathscr{S}}^{\oplus} S$, where $\mathscr{S}$ is the representative set of simple modules. Then the following conditions are equivalent:*

(i) *Q is α-pseudoprojective;*

(ii) Soc *is $m(\alpha)$-dcohereditary;*

(iii) *Q is an α-generator;*

(iv) *every module is α-projective;*

(v) *every simple module is α-projective;*

(vi) *Q is α-projective.*

Moreover, if $m(\alpha)$ is closed under injective hulls then these conditions are equivalent to:

(vii) *every $X \in m(\alpha)$ is injective.*

Proof. (i) $\Leftrightarrow$ (ii). Obvious since Soc $= p_{\{Q\}}$.
(i) $\Rightarrow$ (iii). By Proposition 23.
(iii) $\Rightarrow$ (iv). Q is an α-generator means that each module from $m(\alpha)$ is $p_{\{Q\}}$-torsion, i.e. completely reducible. Thus each exact seqeunce $0 \to A \to B \to C \to 0$ with $B \in \alpha$ splits and (iv) immediately follows.
(iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (i). Obvious.
(iii) $\Rightarrow$ (vii). If $X \in m(\alpha)$ then $E(X) \in m(\alpha)$ is completely reducible by (iii) and so $E(X) = \mathrm{Soc}\,(E(X)) = \mathrm{Soc}\, X = X$.
(vii) $\Rightarrow$ (iii). If $X \in m(\alpha)$ then every submodule Y of X is a direct summand of X, Y being injective. Hence X is completely reducible.

Corollary 5. *Let $Q = \sum\limits_{S \in \mathscr{S}}^{\oplus} S$, where $\mathscr{S}$ is the representative set of simple modules. If s is a preradical then the following conditions are equivalent:*

(i) *Q is* (zer, s, 2)*-pseudoprojective;*

(ii) Soc *is $\mathscr{F}_s$-dcohereditary;*

(iii) *Q is an s-generator;*

(iv) *every module is $\mathscr{F}_s$-projective;*

(v) *every simple module is $\mathscr{F}_s$-projective;*

(vi) *Q is $\mathscr{F}_s$-projective.*

Moreover, if s is hereditary then these conditions are equivalent to:

(vii) *every s-torsionfree module is injective.*

4. PSEUDOPROJECTIVITY AND MORITA EQUIVALENCE

Let $F : R\text{-mod} \to S\text{-mod}$, $G : S\text{-mod} \to R\text{-mod}$ be the functors which represent the Morita equivalence of the rings R, S, and let $f : FG \to 1_{S\text{-mod}}$, $g : GF \to 1_{R\text{-mod}}$ be the corresponding natural isomorphisms. If r is a preradical for R-mod and $N \in S$-mod then we define $_S r(N) = f_N(F(r(G(N))))$. The well-known properties of Morita equivalence immediately imply that $_S r$ is a preradical for S-mod.

Proposition 25. *Let R and S be Morita equivalent via $F : R\text{-mod} \to S\text{-mod}$, $G : S\text{-mod} \to R\text{-mod}$. Then an R-module Q is (r, α)*-pseudoprojective iff $F(Q)$ is

$({}_S r, F(\alpha))$-pseudoprojective. In particular, $E({}_R R)$ is (r, α)-pseudoprojective iff $E({}_S S)$ is $({}_S r, F(\alpha))$-pseudoprojective.

Corollary 6. *Let R and S be Morita equivalent rings via $F : R\text{-mod} \to S\text{-mod}$, $G : S\text{-mod} \to R\text{-mod}$ and let s be a preradical for R-mod. Then an R-module Q is $(r, s, 2)$-pseudoprojective iff $F(Q)$ is $({}_S r, {}_S s, 2)$-pseudoprojective. In particular, $E({}_R R)$ is $(r, s, 2)$-pseudoprojective (pseudoprojective) iff $E({}_S S)$ is $({}_S r, {}_S s, 2)$-pseudoprojective (pseudoprojective).*

References

[1] *J. A. Beachy:* A generalization of injectivity, Pacif. J. Math. *41* (1972), 313—327.

[2] *L. Bican:* QF-3' modules and rings, Comment. Math. Univ. Carolinae *14* (1973), 295—303.

[3] *L. Bican:* Preradicals and injectivity, Pacif. J. Math. *56* (1975), 367—372.

[4] *L. Bican:* Corational extensions and pseudo-projective modules, Acta Math. Acad. Sci. Hungar. *28* (1976), 5—11.

[5] *L. Bican:* General concept of injectivity, Universale Algebren und Theorie der Radikale, Akademie Verlag, Berlin, 1976, 37—40.

[6] *L. Bican, P. Jambor, T. Kepka, P. Němec:* Hereditary and cohereditary preradicals, Czech. Math. J. *26* (1976), 192—206.

[7] *L. Bican, P. Jambor, T. Kepka, P. Němec:* Pseudoprojective modules, Math. Slovaca *29* (1979), 106—115.

[8] *L. Bican, T. Kepka P. Němec:* Rings, modules and preradicals, Lecture notes in pure and applied mathematics, Vol. 75, Marcel Dekker Inc., New York and Basel (1982).

[9] *L. Bican, J. Jirásko:* Generalized pseudoinjectivity (to appear).

[10] *J. P. Jans:* Torsion asociated with duality, Tôhoku Math. J. *24* (1972), 449—452.

[11] *J. Jirásko:* Pseudohereditary and pseudocohereditary preradicals, Comment. Math. Univ. Carolinae *20* (1979), 317—327.

[12] *J. Jirásko:* Preradicals and generalizations of QF-3' modules, Comment. Math. Univ. Carolinae *23* (1982), 25—40.

[13] *J. Jirásko:* Preradicals and generalizations of QF-3' modules II, Comment. Math. Univ. Carolinae *23* (1982), 269—284.

[14] *H. Jirásková, J. Jirásko:* Generalized projectivity, Czech. Math. J. *28* (1978), 632—646.

[15] *A. I. Kašu:* When a radical associated with module is torsion (Russian), Mat. Zametki *16* (1974), 41—48.

[16] *Y. Kurata, H. Katayama:* On a generalization of QF-3' rings, Osaka J. Math. *13* (1976), 407—418.

[17] *F. F. Mbuntum, K. Varadarajan:* Half-exact preradicals, Comm. Algebra *5* (1977), 555—590.

[18] *K. M. Rangaswamy:* Codivisible modules, Comm. Algebra *2* (1974), 475—489.

[19] *G. M. Zuckerman:* On pseudoinjective modules and self-pseudoinjective rings (Russian), Mat. Zametki *7* (1970), 369—380.

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