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OSCILLATION OF n-TH ORDER LINEAR AND NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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This paper is a continuation of the paper [3]. Likewise as in [3] we consider the equation

(1) $$u^{(n)}(t) + p(t)\,|u(\tau(t))|^{\alpha}\operatorname{sign} u(\tau(t)) = 0\,,$$

where

(i) $0 \leqq p(t) \in C_{[t_0,\infty)}$, $p(t)$ is not identically zero in any neighborhood $O(\infty)$,
(ii) $\tau(t) \in C_{[t_0,\infty)}$, $\tau(t) \leqq t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$,
(iii) $n \geqq 2$, $\alpha \geqq 1$.

Without mentioning them again, we shall assume the validity of conditions (i), (ii) and (iii) throughout the paper.

Suppose that there exist solutions of the equation (1) on an interval of the form $[b, \infty)$, where $b \geqq t_0$. In the sequel we shall use the term "solution" only to denote a solution which exists on $[b, \infty)$ where $b \geqq t_0$. Moreover, we shall exclude from our considerations solutions of (1) with the property that $u(t) \equiv 0$ for $t \geqq T_1 \geqq t_0$.

A solution $u(t)$ of (1) is called *oscillatory* for $t \geqq t_0$ if there exists an infinite sequence of points $\{s_i\}_{i=1}^{\infty}$ such that $u(s_i) = 0$ and $s_i \to \infty$ for $i \to \infty$. A solution $u(t)$ of (1) is called *nonoscillatory* if there exists a number $T_2 \geqq t_0$ such that $u(t) \neq 0$ for $t \geqq T_2$. A nonoscillatory solution is said to be *strongly monotone*, if it tends monotonically to zero together with its first $n - 1$ derivatives as $t \to \infty$.

In this paper we extend the main result obtained in [3], which gives sufficient conditions for all solutions of (1) to be oscillatory in the case n is even and for every solution of (1) to be either oscillatory or strongly monotone when n is odd.

Let us denote $\gamma(t) = \sup\{s \geqq t_0 \mid \tau(s) \leqq t\}$ for $t \geqq t_0$. It is clear that $t \leqq \gamma(t)$ and $\tau(\gamma(t)) = t$.

In the paper [3] (if we take into consideration the remark on p. 494) the following three assertions are proved.

Theorem 1. *Let $\alpha \geqq 1$ and let the equation (1) have an unbounded nonoscillatory* *solution. Then*

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^{\infty} p(x)\,dx = 0\,, \quad \textit{if} \quad \alpha > 1$$

and

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^{\infty} p(x)\,dx \leqq (n-1)!\,, \quad \textit{if} \quad \alpha = 1\,.$$

Theorem 2. *Let $\alpha > 0$ and*

$$\limsup_{t\to\infty} t^{n-1} \int_{t}^{\infty} p(x)\,dx = \infty\,.$$

Then, for n even, all nonoscillatory solutions of (1) are unbounded, while, for n odd, every nonoscillatory solution of (1) is either unbounded or strongly monotone.

Theorem 3. *Let $\alpha > 1$. If*

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^{\infty} p(x)\,dx > 0$$

and

$$\limsup_{t\to\infty} t^{n-1} \int_{t}^{\infty} p(x)\,dx = \infty\,,$$

then every solution of (1) is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.

It is clear that in the same way as we obtained (in [3]) Theorem 3 from Theorems 1 (case $\alpha > 1$) and 2, we obtain the following result from Theorems 1 (case $\alpha = 1$) and 2.

Theorem 4. *Let $\alpha = 1$. If*

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^{\infty} p(x)\,dx > (n-1)!$$

and

$$\limsup_{t\to\infty} t^{n-1} \int_{t}^{\infty} p(x)\,dx = \infty\,,$$

then every solution of (1) is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.

The study of the paper [3] shows that all considerations remain valid also in the case $\tau(t) \equiv t$. Then the above theorems are valid for the ordinary differential equation

$$u^{(n)}(t) + p(t)\,|u(t)|^{\alpha}\,\text{sign}\,u(t) = 0\,. \tag{2}$$

Because $\gamma(t) \equiv t$ if $\tau(t) \equiv t$, so according to Theorems 3 and 4 we can formulate the following theorem.

Theorem 5. *Let* $\alpha \geqq 1$. *If*

$$\limsup_{t\to\infty} t^{n-1} \int_t^\infty p(x)\,dx = \infty\,,$$

then every solution of (2) *is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.*

Now we may state an assertion which will be useful for a comparison of assumptions of the theorems.

Lemma 1. *Let* $f(t)$ *be a continuous and nonnegative function defined on a neighborhood* $O(\infty)$. *Let k be a natural number. If*

$$\limsup_{t\to\infty} t^k \int_t^\infty f(x)\,dx = \infty\,, \tag{3}$$

then

$$\int^\infty t^k f(t)\,dt = \infty\,. \tag{4}$$

Proof. Suppose that the condition (3) holds. Then, by Theorem 5, every solution of the equation

$$v^{(k+1)}(t) + f(t)\,|v(t)|^\alpha \operatorname{sign} v(t) = 0\,, \quad \alpha > 1\,,$$

is oscillatory if k is odd, and is either oscillatory or strongly monotone if k is even. This together with Theorem 2 in [1] (a necessary and sufficient condition for oscillation of our equation) implies that the condition (4) holds and the lemma is proved.

It is evident that the result of Theorem 2 has a considerable influence on the results of Theorems 3, 4 and 5. Therefore, we present Theorem 2.1 in [2], more precisely its corollary 2.1, which has (for $\tau(t) \leqq t$ and also for $\tau(t) \equiv t$) the following wordig.

Theorem 6. *Let* $\alpha > 0$. *The equation* (1) *has a nonoscillatory solution* $u(t)$ *such that* $\lim_{t\to\infty} u(t) = a \neq 0$ *if and only if*

$$\int^\infty t^{n-1} p(t)\,dt < \infty\,.$$

Now, Theorems 1 and 6 yield the following results.

Theorem 7. *Let* $\alpha > 1$. *If*

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^\infty p(x)\,dx > 0 \tag{5}$$

and

$$\int^\infty t^{n-1} p(t)\,dt = \infty\,, \tag{6}$$

then every solution of (1) *is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.*

Proof. On the contrary, let $u(t)$ be a nonoscillatory solution of (1). Moreover, let $u(t)$ be not strongly monotone if n is odd. Then, by (5) and Theorem 1, $u(t)$ must be bounded. On the other hand, by (6) and Theorem 6, $u(t)$ must be unbounded if n is even, and $u(t)$ must be either unbounded or strongly monotone if n is odd. This is a contradiction and the theorem is proved.

The proofs of the following two theorems are the same as that of Theorem 7 and thus are omitted.

Theorem 8. *Let* $\alpha = 1$. *If*

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^{\infty} p(x)\,\mathrm{d}x > (n-1)!$$

and

$$\int^{\infty} t^{n-1}\, p(t)\,\mathrm{d}t = \infty\,,$$

then every solution of (1) *is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.*

Theorem 9. *Let* $\alpha = 1$. *If*

$$\limsup_{t\to\infty} t^{n-1} \int_{t}^{\infty} p(x)\,\mathrm{d}x > (n-1)!$$

and

$$\int^{\infty} t^{n-1}\, p(t)\,\mathrm{d}t = \infty\,,$$

then every solution of (2) *is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.*

Remark. From Lemma 1 we know that Theorems 7, 8 and 9 are better than Theorems 3, 4 and 5.

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