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ON THE DE MORGAN FORMULAE AND THE ANTITONY OF COMPLEMENTS IN LATTICES

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1. Introduction. It is well-known that in a uniquely complemented lattice L satisfying the De Morgan formulae the mapping $x \rightarrow x'$ ($x, x' \in L$; x' is the unique complement of x) is *antitone*, i.e. $x \leqq y$ implies $x' \geqq y'$, and it is not difficult to show that also the converse statement is true. In this paper we begin with showing that certain weaker forms of the De Morgan conditions are still equivalent to the original ones (Theorem 1). By making use also of this result we give a necessary condition for a uniquely complemented lattice to be non-modular (Theorem 2). In Section 3 we extend in two essentially different ways the sense of the De Morgan formulae and the antitony of complements to complemented lattices in which the complementation is not unique, and investigate the interrelations of these extended properties. Finally, in Section 4, we discuss these generalized conditions in modular lattices.

For the notation and terminology not explained here we refer to [1].

The author expresses his gratitude to Prof. J. JAKUBÍK who called the author's attention to an inaccuracy in the original formulation of the conditions UMM and UMJ.

2. On uniquely complemented lattices. We begin with proving the following theorem:

Theorem 1. *Let L be a uniquely complemented lattice. Then the following assertions are equivalent:*

(i) $(x \cap y)' = x' \cup y'$ *for each* $x, y \in L$.
(ii) $(x \cap y)' = x' \cup y'$ *for each comparable pair* $x, y \in L$.
(iii) $(x \cap y)' = x' \cup y'$ *for each incomparable pair* $x, y \in L$.
(iv) $(x \cup y)' = x' \cap y'$ *for each* $x, y \in L$.
(v) $(x \cup y)' = x' \cap y'$ *for each comparable pair* $x, y \in L$.
(vi) $(x \cup y)' = x' \cap y'$ *for each incomparable pair* $x, y \in I$.
(vii) $x \leqq y$ *implies* $x' \geqq y'$ *for each* $x, y \in L$.

Proof. By the lattice theoretical duality principle it suffices to show that (i), (ii), (iii) and (vii) are pairwise equivalent.

First of all, the implications (i) ⇒ (ii) ⇒ (vii) and (i) ⇒ (iii) are obvious.

We show that (iii) implies (vii). Assume to this end that (vii) is not satisfied in L. Then there exist elements a, b in L such that $a < b$ and a' is neither greater than nor equal to b'. Suppose $a' < b'$. Then

$$b \cap a' \leqq b \cap b' = o \quad \text{and} \quad b \cup a' \geqq a \cup a' = i\,,$$

i.e., a' would be a complement of b, too, in contradiction to the fact that L is uniquely complemented. The only remaining possibility is that

$$a' \parallel b'\,. \tag{1}$$

It follows that (iii) cannot be satisfied in L because it would imply[1])

$$(a' \cap b')' = a'' \cup b'' = a \cup b = b\,,$$

i.e. $a' \cap b' = b'$, in contradiction to (1). Thus we have proved that (iii) implies (vii).

Finally we show that (vii) implies (i). Suppose (vii). Then the inequalities $x \cap y \leqq$ $\leqq x$ and $x \cap y \leqq y$ imply

$$(x \cap y)' \geqq x' \cup y'\,. \tag{2}$$

Similarly, $x' \cup y' \geqq x'$ and $x' \cup y' \geqq y'$ imply

$$(x' \cup y')' \leqq x'' \cap y'' = x \cap y\,. \tag{3}$$

By (vii) again, (2) and (3) imply

$$x \cap y = (x \cap y)'' \leqq (x' \cup y')' \leqq x \cap y$$

whence $(x \cap y)' = x' \cup y'$. This means that (vii) implies (i), indeed, and the theorem is proved.

Using this result, we prove

Theorem 2. *Let L be a non-modular, uniquely complemented lattice. Then there exist elements a, b in L such that $a < b$ but $a' \parallel b'$.*

Proof. It is known that every uniquely complemented lattice in which the De Morgan formulae hold is distributive (see, e.g., [1], p. 122). It follows, by Theorem 1, that the complementation in the lattice in question cannot be antitone. Consequently, the assertion of Theorem 2 follows by the same argument used above for proving the implication (iii) ⇒ (vii).

[1]) x'' is an abbreviated notation for $(x')'$.

3. The generalizations of the De Morgan formulae and the antitony of complements. There are two essentially different ways offering themselves for extending the sense of the De Morgan conditions and the antitony of complements to lattices in which the complementation is not unique. These two possibilities are to require that one of the conditions (i)–(vii) in Section 2 be satisfied either for *arbitrary* complements of the elements occuring in this condition or only for *suitably chosen* ones. Let us examine conditions of these types.

Proposition 1. *Let L be a complemented lattice in which $x \leqq y$ implies $x' \geqq y'$ for each $x, y \in L$ and for any complements x', y' of the elements x, y, respectively. Then L is uniquely complemented.*

Proof. Suppose that there exists an element a in L which has two different complements, u and v. Without loss of generality we may suppose that u is not greater than v. Then we reach a contradiction by setting $x = y = a$, $x' = u$ and $y' = v$. Thus Proposition 1 is verified.

In accordance with this result we introduce the following definition, furnishing an effective generalization of the antitony of complements:

Definition 1. A complemented lattice L is said to *satisfy the antitony of complements universally* if the following condition UA is fulfilled:

UA. For any elements $x, y \in L$ and for any complements x', y' of the elements x, y, respectively, $x < y$ implies $x' \geqq y'$.

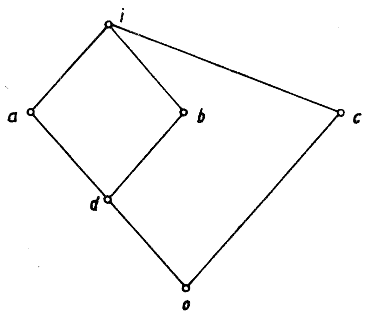

Diagram 1

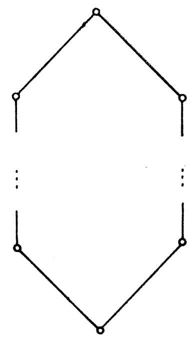

Diagram 2

Example. The lattice given in Diagram 1 satisfies UA without being uniquely complemented.

If we want to get a generalization of the De Morgan formulae, then we may not confine ourselves to exclude only the pairs x, y with $x = y$. This is shown by[2])

[2]) This proposition and its proof are due to Prof. J. Jakubík.

Proposition 2. *Let L be a complemented lattice in which one of the following conditions is satisfied:*

(i′) *For any elements x, y of L with $x \nparallel y$ and for any complements x', y', $(x \cap y)'$ of the elements $x, y, x \cap y$, respectively, $(x \cap y)' = x' \cup y'$.*

(ii′) *For any comparable pair x, y of elements of L with $x \neq y$ and for any complements x', y', $(x \cap y)'$ of the elements $x, y, x \cap y$, respectively, $(x \cap y)' = x' \cup y'$.*

(iv′) *The dual of* (i′).

(v′) *The dual of* (ii′).

Then L is uniquely complemented.

Proof. Suppose (ii′) and let a be any element of L different from the greatest one. Then there exists an element b in L such that $a < b$. It follows, by (ii′), that

$$a' = (a \cap b)' = a' \cup b' ,$$

i.e. $a' \geqq b'$ for each complement a' of a and for each complement b' of b. Let a_1, a_2 be two arbitrary complements of a. Then $a_2 \geqq b'$ and by setting $(a \cap b)' = a_1$ and $a' = a_2$, (ii′) implies

$$a_1 = a_2 \cup b' = a_2 .$$

This means that the element a is uniquely complemented. Since the greatest element has the same property, L is uniquely complemented, indeed.

Suppose now (i′). Then (ii′) is satisfied, a fortiori, and L is uniquely complemented.

The proof can be completed by the dualization of the preceding considerations.

Proposition 2 shows that only (iii) and (vi) can be effectively generalized with the requirement that they are true for arbitrary complements. Accordingly, we introduce the following definition:

Definition 2. A complemented lattice L is said to *satisfy the De Morgan meet-formula universally* if:

UMM. For each incomparable pair x, y of L and for each complements x', y', $(x \cap y)'$ of the elements $x, y, x \cap y$, respectively, $(x \cap y)' = x' \cup y'$.

The (universal) De Morgan join-formula UMJ is defined dually.

Remark. The conditions are very weak. For example, any lattice given in Diagram 2 satisfies both UMM and UMJ (but not UA). These lattices are not modular (cf. Theorem 4).

Proposition 3. *The system of conditions* UMM, UMJ *and* UA *is independent.*

Proof. The lattice given in Diagram 1 satisfies UA and UMM but not UMJ.[3]) Its dual satisfies UA and UMJ but not UMM. Finally, the independence of UA follows from the Remark before this proposition.

Generalizations of the second type (i.e., with the requirement "for suitably chosen complements") seem to be more interesting. We formulate them in Definitions 3 and 4 below.

Definition 3. A complemented lattice L is said to *satisfy the De Morgan meet-formula* [resp. *the restricted De Morgan meet-formula*] *partially* if the following condition PMM [RPMM] is fulfilled:

PMM [RPMM]. For any [incomparable] elements x, y of L there exist complements $x', y', (x \cap y)'$ of the elements $x, y, x \cap y$, respectively, such that $(x \cap y)' = x' \cup y'$.

The partial De Morgan formulae PMJ and RPMJ are defined dually.

Definition 4. A complemented lattice L is said to *satisfy the antitony of complements partially* if

PA. For any elements x, y of L with $x < y$ there exist complements x', y', respectively, such that $x' \geqq y'$.

Remark. Replacing the word "incomparable" by "comparable" in RPMM or RPMJ we given a condition that is equivalent to PA.

Proposition 4. PMM [*or even* UMM] *does not imply* PMJ, *and* PMJ [*or even* UMJ] *does not imply* PMM.

Proposition 5. PA [*or even* UA] *implies neither* PMM *nor* PMJ.

Proof. Consider Diagram 1 and Footnote 3.

Nonetheless, we have the following affirmative theorem:

Theorem 3. PMM *or* PMJ *implies* PA.

Proof. Suppose PMM. Then, for any elements x, y of the lattice in question, there exist complements $(x \cap y)'$, x_1 and y_1 of $x \cap y$, x and y, respectively, such that $(x \cap y)' = x_1 \cup y_1$. Let $x < y$. Then we have $x' = x_1 \cup y_1 \geqq y_1$. Thus we have found a complement x' of x and a complement y_1 of y with $x' \geqq y_1$, implying PA.

Example. We conclude this section by presenting a complemented lattice in which PA does not hold. In the lattice shown in Diagram 3, the elements b and e

[3]) In fact, $(a \cup b)' = i' = o$ and $a' \cap b' = c \cap c = c$ in this lattice.

have only one complement each: d is the unique complement of b and c is that of e. Since $b < e$ but $d \parallel c$, PA is not satisfied in this lattice. We call the attention of the reader to the fact that this lattice is not modular (cf. Theorem 5 below).

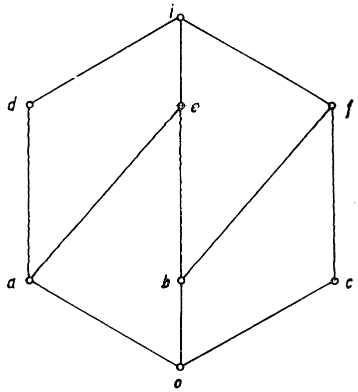

Diagram 3

4. On generalized conditions in modular lattices. The five-element non-modular lattice furnishes an example of a lattice in which UMM, UMJ and UA are satisfied, without the complementation being unique. In modular lattices, however, the situation is quite different:

Theorem 4. *If a complemented modular lattice satisfies* UMM *or* UMJ, *then it is uniquely complemented and, consequently, distributive.*[4])

Proof. Let L be a complemented modular lattice in which UMM holds. Assume that there exists an element c in L which has two different complements p and q. It is well-known that any two different complements of an element are incomparable in a modular lattice (see, e.g., [1], p. 90). Hence $p \parallel q$ and $p \cap q < p$. Then, by making use of UMM with $p' = q' = c$, we get $(p \cap q)' = p' \cup q' = c$ for each complement of $p \cap q$. This means that c would have two different comparable complements, $p \cap q$ and p, in contradiction to the theorem cited above. By this contradiction, our assumption that c has two different complements has been shown to be false and the theorem is proved.

Remark. The five-element modular, but non-distributive lattice shows that neither UA, nor PMM and PMJ imply the distributivity. On the other hand, there exist

[4]) It is known that a uniquely complemented modular lattice is distributive, too (see, e.g., [1], p. 113.

complemented modular lattices in which UA does not hold: consider, e.g., the subspace lattice of the three-dimensional projective space.

In contrast with the second part of this remark, we have

Theorem 5. *Every complemented modular lattice satisfies* PA.

Proof. Let L be a complemented modular lattice and $a < b$ $(a, b \in L)$. Then there exists $r \in L$ such that

$$a \cap r = o \quad \text{and} \quad a \cup r = b$$

because L is relatively complemented, too (see, e.g., [1], p. 112). Let b' be a complement of b. We prove the theorem by showing that the element $t = r \cup b'$ $(\geqq b')$ is a complement of a.

First, $a \cup t = a \cup (r \cup b') = (a \cup r) \cup b' = b \cup b' = i$. On the other hand,

$$a \cap t \leqq a \tag{4}$$

and, by the isomorphism theorem of modular lattices (see, e.g., [1], p. 95), the mapping

$$x \to x \cup b' \quad (x \in [o, b])$$

is an isomorphism of the interval $[o, b]$ onto $[b', i]$. By this isomorphism, $a \cap r = o$ implies $(a \cup b') \cap t = b'$ whence

$$a \cap t \leqq b'. \tag{5}$$

By (4) and (5) we get $a \cap t \leqq a \cap b' \leqq b \cap b' = o$, completing the proof.

I do not know whether every complemented modular lattice satisfies PMM and PMJ or not.

Reference

[1] *G. Szász:* Introduction to lattice theory, Akadémiai Kiadó, Budapest and Academic Press, New York—London, 1963.

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