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PARTITION OF NONDENUMERABLE CLOSED SETS OF REALS

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In what follows every set mentioned is a subset of the set of all real numbers. Moreover, every item pertaining to measure is in the sense of Lebesgue. As usual, $m(S)$, $m^*(S)$ and $m_*(S)$ denote respectively the *measure*, the *outer* and the ***inner** measures* of the set S.

Lemma 1. *Let C be a closed set. Let B be a subset of C such that B has a nonempty intersection with every closed subset of C of positive measure. Then*

$$m^*(B) = m(C) . \tag{1}$$

Proof. Assume on the contrary that $m^*(B) < m(C)$. But then there exists a covering H of B by pairwise disjoint open intervals such that $C - \bigcup H$ is a closed subset of C of positive measure. Clearly, B has no point in common with $C - \bigcup H$ which is a contradiction. Thus, (1) is established.

As usual, we identify every cardinal k with the set of all ordinals preceding k. Thus, k is a well ordered set and k is greater than the cardinality of every initial segment of k. Moreover, if $\bar{\bar{S}} = k$ then S is well ordered by virtue of the equipollence between S and k. Based on this, we prove:

Lemma 2. *Let n be a cardinal and c be an infinite cardinal such that*

$$n \leqq c . \tag{2}$$

Let $(A_i)_{i<n}$ be a (not necessarily disjoint) family of sets A_i such that

$$\bar{\bar{A}}_i = c \quad \textit{for every} \quad i < n . \tag{3}$$

Then there exists a family $(a_i)_{i<n}$ of pairwise distinct real numbers a_i such that

$$a_i \in A_i \quad \textit{for every} \quad i < n . \tag{4}$$

Proof. Clearly, every A_i is well ordered by virtue of (3). We assert the existence of the family $(a_i)_{i<n}$ based on transfinite induction given by:

$$a_i = \textit{the first element of} \quad A_i - \bigcup_{j<i}\{a_j\} \quad \textit{for every} \quad i < n . \tag{5}$$

The above definition is justified since by (2), we see that $i < n$ implies $i < c$ and therefore $c - i = c$, which by (3), implies that $A_i - \bigcup_{j<i}\{a_j\}$, in (5), is nonempty. But then clearly (5) implies (4), as desired.

Remark. In what follows we let c denote the cardinality of the continuum (i.e., the set of all real numbers). We recall that every closed set P of positive measure (or for that matter every nondenumerable closed set) is of cardinality c. Moreover, the family of all the closed subsets of P of positive measure is also of cardinality c. Based on this, we prove:

Theorem 1. *Let P be a closed set of positive measure. Let c be the cardinal of the continuum and let k be any positive cardinal such that $k \leqq c$. Then P is a disjoint union of k-many subsets B_j of P such that*

$$m^*(B_j) = m(P) \quad \textit{for every} \quad j < k . \tag{6}$$ [1])

Proof. Since c is infinite and $k \leqq c$, we see that

$$kc = c . \tag{7}$$

In view of the Remark, we let $(P_i)_{i<c}$ denote the family of all the closed subsets P_i of P of positive measure. Again, in view of the Remark, we have $\overline{\overline{P}}_i = c$ for every $i < c$ which, by (7) implies that every P_i is a disjoint union of k-many subsets A_{ij} such that

$$A_{ij} \subseteq P_i \quad \textit{and} \quad \overline{\overline{A}}_{ij} = c \quad \textit{for every} \quad i < c \quad \textit{and} \quad j < k . \tag{8}$$

Let us consider the family A given by

$$A = \{A_{ij} \mid i < c \ \textit{and} \ j < k\} . \tag{9}$$

From (7) it follows that $kc \leqq c$ and therefore, from (9) and (8), by Lemma 2 we see that there exists a family $(a_{ij})_{i<c}$ with $j < k$ of pairwise distinct real numbers a_{ij}

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[1]) The results presented strengthens some former results of Professor W. SIERPINSKI (*L* equivalence par decomposition finite et la measure extérieure des ensembles, Fund. Math. *XXXVII* (1950), 209–212). In this paper Sierpinski proved for example the following assertion: *If* $\aleph_1 = 2^{\aleph_0}$ *and* $E \subset R_m$ *has positive measure, n is positive integer, then* $E = \bigcup_{j=1}^{n} E_j$ *(disjoint union) and the outer measure of each of the sets* E_j *is equal to the measure of the set E.* (The reviewer's remark.)

such that

(10) $$a_{ij} \in A_{ij} \quad for\ every \quad i < c \quad and \quad j < k\,.$$

Let

(11) $$B_0 = \{a_{i0} \mid i < c\} \cup (P - \{a_{ij} \mid i < c\ and\ j < k\})$$

and

(12) $$B_j = \{a_{ij} \mid i < c\} \quad with \quad 0 < j < k\,.$$

From (11) and (12) we see that $(B_j)_{j<k}$ is a family of pairwise disjoint subsets B_j of P such that

(13) $$P = \bigcup_{j<k} B_j\,.$$

Moreover, from (10) and (8), it follows that for every $j < k$ it is the case that B_j has a nonempty intersection with every closed subset P_i of P of positive measure. Hence, from Lemma 1 it follows that

(14) $$m^*(B_j) = m(P) \quad for\ every \quad j < k\,.$$

Thus, from (13) and (14) it follows that P is a disjoint union of k-many subsets B_j of P satisfying (6). Hence the Theorem is proved.

Corollary. *Let P be a closed set of positive measure. Let c be the cardinal of the continuum and k any cardinal such that $2 \leqq k \leqq c$. Then P is a disjoint union of k-many nonmeasurable subsets B_j of P such that*

(15) $$m^*(B_j) = m(P) \quad and \quad m_*(B_j) = 0 \quad for\ every \quad j < k\,.$$

Proof. In view of the hypothesis of the Corollary, from Theorem 1 it follows that P is a disjoint union of k-many subsets B_j of P satisfying (6). On the other hand, since $k \geqq 2$ we see that for every $j < k$ there exists $i < k$ such that $j \neq i$ and $B_i \subseteq (P - B_j)$ with

$$m^*(B_j) = m^*(P - B_j) = m(P)$$

which implies (15) and the nonmeasurability of B_j for every $j < k$.

Thus the Corollary is proved.

We observe that if C is a closed set of positive measure $m(C)$ then for every *nonnegative extended real number* r (i.e., $0 \leqq r \leqq +\infty$) such that $r \leqq m(C)$ there exists a closed subset P of C such that $m(P) = r$.

Based on the above observation we prove:

Theorem 2. *Let C be a nondenumerable closed set. Let r be a nonnegative extended real number such that $r \leqq m(C)$. Then C is a disjointed union of continuumly many subsets C_j of C such that $m^*(C_j) = r$.*

Proof. If $r = 0$ then the conclusion of the Theorem follows immediately since C is a disjoint union of (see the Remark) continuumly many of its singletons. Next, let $0 < r \leqq m(C)$. Thus, C is a closed set of positive measure and we let (in view of the above observation) P be a closed subset of C such that $m(P) = r$. Let c be the cardinal of the continuum then since $c \leqq c$, from Theorem 1 it follows that P is the union of a family $(B_j)_{j<c}$ of continuumly many pairwise disjoint subsets B_j of P such that $m^*(B_j) = m(P) = r$. Clearly, $\overline{\overline{C - P}} = e \leqq c$ and therefore $C - P$ is equal to the family $(b_j)_{j<e}$ of pairwise distinct real numbers b_j. But then letting

$$C_j = B_j \cup \{b_j\} \quad \textit{if} \quad j < e \quad \textit{and} \quad C_j = B_j \quad \textit{if} \quad e \leqq j < c$$

we see that the above C_j's satisfy the conclusion of the Theorem. Thus, the Theorem is proved.

For related results see the reference below.

Reference

Oxtoby, J. C.: Measure and Category, Springer-Verlag (1970), p. 79.

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