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RELATIVE BOUNDEDNESS CONDITIONS AND THE PERTURBATION OF NONLINEAR OPERATORS

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1. INTRODUCTION

The objective of this paper is to investigate the perturbation of nonlinear operators with respect to such properties as closedness, Lipschitz invertibility, resolvent, and spectrum. Our results are primarily nonlinear analogues to the linear case as set forth by T. KATO in [9]. As in the linear case we require criteria for the "size" of the perturbing operator and for this purpose we will use nonlinear analogues to the linear notion of relative boundedness. Our main attention will be given to discontinuous nonlinear operators which correspond to unbounded linear ones.

In Section 2 we will describe notation and terminology. In Section 3 we will consider the stability of closedness under relatively Lipschitz perturbations and in Section 4 the stability of demiclosedness and weakly closedness under relatively bounded perturbations. In Section 5 we will investigate resolvent and spectral properties with respect to relatively Lipschitz perturbations. In Section 6 we shall analyze the continuity and bifurcation of the spectrum of operators Lipschitz in the graph norm of a closed linear operator.

2. NOTATION AND TERMINOLOGY

In what follows X and Y will denote Banach spaces (real or complex), X^* will denote the dual space of X, and X_w will denote X with the weak topology. We will use $\rightarrow$ to denote strong convergence and $\overset{w}{\rightarrow}$ to denote weak convergence. If S is a subset of X then $\bar{S}$ is the strong closure of S and $|S| = \inf_{x \in S} \|x\|$. Furthermore, $X \times Y$ will mean the product space with $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$.

Because of the recent interest in nonlinear accretive operator theory, where multi-valued operators play an important role and where relative boundedness conditions have been used in its perturbation theory (see, e.g., [1], [3], [4], [5], [10]), we prefer to allow for operators to be multi-valued in our investigation. Accordingly,

$m(X, Y)$ will denote the set of multi-valued operators from X to Y, or, equivalently, subsets of $X \times Y$. If $A \in m(X, Y)$ we will identify A with its graph $G(A)$ in $X \times Y$. For $A \in m(X, Y)$ we have

$$D(A) = \{x \in X : \text{there exists some } y \in Y \text{ such that } (x, y) \in A\} ; \tag{2.1}$$

$$R(A) = \{y \in Y : \text{there exists some } x \in X \text{ such that } (x, y) \in A\} ; \tag{2.2}$$

$$\text{if } x \in D(A), \quad Ax = \{y \in Y : (x, y) \in A\} . \tag{2.3}$$

For A, $T \in m(X, Y)$, $B \in m(Y, X)$, and c a scalar we define

$$cA = \{(x, cy) : (x, y) \in A\} ; \tag{2.4}$$

$$A + T = \{(x, y + z) : (x, y) \in A \text{ and } (x, z) \in T\} ; \tag{2.5}$$

$$BA = \{(x, z) : \text{there exists } y \in Y \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\} . \tag{2.6}$$

We will let $s(X, Y)$ denote the subset of $m(X, Y)$ consisting of single-valued operators, $1(X, Y)$ the subset of $s(X, Y)$ consisting of linear operators, and $b1(X, Y)$ the subset of $1(X, Y)$ consisting of bounded everywhere defined linear operators. By $\text{lip}\,(X, Y)$ we denote the subset of $s(X, Y)$ consisting of Lipschitz continuous operators. If $A \in \text{lip}\,(X, Y)$ then $|A|$ denotes the Lipschitz constant for A, that is,

$$|A| = \sup_{x,\, y \in D(A),\, x \neq y} \|Ax - Ay\| / \|x - y\| . \tag{2.7}$$

Finally, for $A \in m(X, Y)$, "A^{-1} exists" means that for each $y \in R(A)$ there exists a unique $x \in D(A)$ such that $y \in Ax$. In this case $A^{-1} \in s(Y, X)$ is given by

$$A^{-1} = \{(y, x) : (x, y) \in A\} . \tag{2.8}$$

3. RELATIVE LIPSCHITZ CONDITIONS

Definition 3.1. $A \in m(X, Y)$ is *closed* if and only if A (or $G(A)$) is closed in $X \times Y$. Equivalently, $A \in m(X, Y)$ is closed if and only if

$$\text{whenever } \{x_n\} \subset D(A), \quad x_n \to x \in X, \quad y_n \in Ax_n, \quad \text{and} \tag{3.1}$$
$$y_n \to y \in Y, \quad \text{then} \quad x \in D(A) \quad \text{and} \quad y \in Ax .$$

The following proposition asserts that "closedness is stable under continuous perturbations".

Proposition 3.1. *Let $A \in s(X, Y)$, $T \in m(X, Y)$, $D(T) \subset D(A)$, and let A be continuous with $D(A)$ closed. Then $A + T$ is closed if and only if T is.*

Proof. Let T be closed, $x_n \to x$, $u_n \in (A + T)\, x_n$, $u_n \to u$, $u_n = Ax_n + y_n$, $y_n \in Tx_n$. Then $Ax_n \to Ax$ whence $y_n \to u - Ax$. Then $x \in D(T)$ and $u - Ax \in Tx$ since T is closed. But then $A + T$ is closed since $u = Ax + (u - Ax) \in (A + T)\,x$. Now let $A + T$ be closed, $x_n \to x$, $y_n \in Tx_n$, $y_n \to y$. Then $u_n = Ax_n + y_n \in (A + T)\,x_n$ and $u_n \to Ax + y$. Since $A + T$ is closed, $(x, Ax + y) \in A + T$. Then $x \in D(T)$ and $y \in Tx$, because $(x, z) \in A + T$ if and only if $z = Ax + w$ uniquely where $(x, w) \in T$. Therefore T is closed.

It is obvious that every $A \in m(X, Y)$ has a closed extension in $m(X, Y)$, namely, A_c where $G(A_c) = \overline{G(A)}$. It is of interest, however, to determine when $A \in s(X, Y)$ has a closed extension in $s(X, Y)$.

Definition 3.2. $A \in s(X, Y)$ is called *closable* if and only if A has a closed extension in $s(X, Y)$. Evidently, $A \in s(X, Y)$ is closable if and only if $\overline{G(A)} = G(A_c)$ where $A_c \in s(X, Y)$. A_c is clearly the smallest closed extension of A in $s(X. Y)$. Moreover, $A = A_c$ if and only if A is closed and A is closable if and only if

$$\text{(3.2)} \qquad \text{whenever } \{u_n\}\,,\ \{v_n\} \subset D(A)\,,\quad u_n \to x\,,\quad v_n \to x\,,\quad Au_n \to u\,,$$
$$\text{and} \quad Av_n \to v\,, \quad \text{then} \quad u = v\,.$$

Propositions 3.1 asserts the stability of closedness without any assumption on the "size" of the perturbing operator A, but imposes the strong condition that A be continuous and have closed domain. In order to weaken this condition we have to make sure that the perturbation is "small". A useful notion for this purpose generalizes the concept of relative boundedness in the linear case (see [9], p. 190) and is provided by

Definition 3.3. Let $A \in s(X, Y)$, $T \in m(X, Y)$, $D(T) \subset D(A)$. A is *relatively Lipschitz* with respect to T (or T-Lipschitz) if and only if there exist constants a, $b \geqq 0$ such that

$$\text{(3.3)} \qquad \|Ax - Ay\| \leqq a\|x - y\| + b|Tx - Ty| \quad \text{for all} \quad x, y \in D(T)\,.$$

The infimum of all possible constants b in (3.3) is the *T-Lipschitz constant* for A.

Example 3.1. Let A, $T \in 1(X, Y)$, $D(T) \subset D(A)$ such that (3.3) is satisfied (for examples in the linear case see [9], p. 191–194). Let $F \in \mathrm{lip}\,(Y, X)$ with $D(F) = Y$ and define $B = FA$ where $D(B) = D(A)$. Then $D(T) \subset D(B)$ and for all x, $y \in D(T)$

$$\|Bx - By\| \leqq |F|\,\|Ax - Ay\| \leqq a|F|\,\|x - y\| + b|F|\,\|Tx - Ty\|$$

so that B is T-Lipschitz with T-Lipschitz constant $\leqq b|F|$.

Example 3.2. Let A, T be as in Example 3.1 and let $F \in \mathrm{lip}\,(X, X)$ such that $F(D(A)) \subset D(A)$ and $F(D(T)) \subset D(T)$. Define $B = AF$ where $D(B) = D(A)$ and $S = TF$ where $D(S) = D(T)$. Then $D(S) \subset D(B)$ and for all x, $y \in D(S)$

$$\|Bx - By\| = \|A(Fx - Fy)\| \leqq a|F|\,\|x - y\| + b\|Sx - Sy\|$$

so that B is S-Lipschitz with S-Lipschitz constant $\leqq b$.

Example 3.3. Let $A \in s(X, Y)$, $F \in s(Y, X)$, $F^{-1} \in \operatorname{lip}(X, Y)$, and let $R(A) \subset D(F)$. Define $T = FA$ where $D(T) = D(A)$. Then for all $x, y \in D(T)$

$$\|Ax - Ay\| = \|F^{-1}FAx - F^{-1}FAy\| \leqq |F^{-1}| \, \|Tx - Ty\|$$

so that A is T-Lipschitz with T-Lipschitz constant $\leqq |F^{-1}|$.

Proposition 3.2. *Let $A \in s(X, Y)$, A closed, $T \in m(X, Y)$ such that $D(T) \subset D(A)$ and (3.3) is satisfied with $b < 1$. Then $A + T$ is closed if and only if T is.*

Proof. Let $\{x_n\} \subset D(A + T) = D(T)$ and let $u_n \in (A + T)\, x_n$ where $u_n = Ax_n + y_n$ with $y_n \in Tx_n$. From (3.3) we obtain

$$-a\|x_n - x_m\| + (1 - b)\,\|y_n - y_m\| \leqq \|u_n - u_m\| \leqq \leqq a\|x_n - x_m\| + (1 + b)\,\|y_n - y_m\| \,. \tag{3.4}$$

Suppose T is closed, $x_n \to x$, and $u_n \to u$. By (3.4) $y_n \to$ some y and $Ax \to u - y$. Then $(x, y) \in T$ and $Ax = u - y$, which yields $(x, u) \in A + T$, that is $A + T$ is closed. Now suppose $A + T$ is closed, $x_n \to x$, $y_n \to y$. By (3.4) $u_n \to$ some u and $Ax_n \to u - y$. Then $(x, u) \in A + T$ and $Ax = u - y$ since $A + T$ and A are both closed. Thus $x \in D(T) = D(A + T)$ and $y \in Tx$ since $u = Ax + z$ where $(x, z) \in T$ and z is unique. Therefore T is closed.

Proposition 3.3. *Let $A, T \in s(X, Y)$, $D(T) \subset D(A)$ such that (3.3) is satisfied with* b < 1. *Then $S = A + T$ is closable if and only if T is. In this case $D(T_c) = D(S_c)$. In particular, S is closed if and only if T is.*

Proof. From (3.3) we have for all $x, y \in D(T)$

$$-a\|x - y\| + (1 - b)\,\|Tx - Ty\| \leqq \|Sx - Sy\| \leqq \leqq a\|x - y\| + (1 + b)\,\|Tx - Ty\| \,. \tag{3.5}$$

Suppose T is closable, $\{u_n\}, \{v_n\} \subset D(S)$, $u_n \to x$, $v_n \to x$, $Su_n \to u$, $Sv_n \to v$. From (3.5) $Tu_n \to$ some y, $Tv_n \to$ some z and by (3.2) $y = z$. Again from (3.5) $Su_n - Sv_n \to u - v = 0$ and so by (3.2) S is closable. Now suppose S is closable, $\{u_n\}, \{v_n\} \subset D(T)$, $u_n \to x$, $v_n \to x$, $Tu_n \to u$, $Tv_n \to v$. By (3.5) $Su_n \to$ some y, $Sv_n \to$ some z. Since S is closable, (3.2) gives $y = z$. Again by (3.5) $Tu_n - Tv_n \to 0$. Then $u = v$ and so T is closable by (3.2). Let T be closable and let $u \in D(T_c)$. There exists $\{u_n\} \subset D(T)$ such that $u_n \to u$ and $\{Tu_n\}$ is Cauchy. By (3.5) $\{Su_n\}$ is Cauchy. Hence $u \in D(S_c)$ and $D(T_c) \subset D(S_c)$. The reverse inclusion is proven similarly. Finally, if T is closed, it is closable; hence S is closable and we have $D(S) = D(T) = D(T_c) = D(S_c)$. This shows that S is closed. The converse is proven similarly.

Definition 4.1. $T \in m(X, Y)$ is called *demiclosed* (respectively, *weakly closed*) if and only if $\{x_n\} \subset D(T)$, $x_n \to x \in X$ (respectively, $x_n \overset{w}{\to} x \in X$), $y_n \in Tx_n$, $y_n \overset{w}{\to} y \in$ $\in Y$, imply $x \in D(T)$ and $y \in Tx$.

It is immediate that T weakly closed implies T demiclosed implies T closed. The converse, however, is not true in general, as may be seen by the following examples.

Example 4.1. Let $X = Y = 1^2(R)$, let $D(A)$ consist of all sequences $\{x_m\} \in 1^2(R)$ such that at least one $x_m = 0$ and

$$\text{(4.1)} \qquad \text{if} \quad k_0 = \min\{k : x_k = 0\} \quad \text{then} \quad x_j \geqq 1/j \quad \text{for} \quad 1 \leqq j < k_0 \,.$$

Define $A\{x_m\} = \{\delta_{mk_0}\}$. To see that A is closed let $\{x_n\} \subset D(A)$, $x_n \to x$, $Ax_n \to y$. Then there exists n_0 such that $Ax_n = y$ for all $n \geqq n_0$. Then for all $n \geqq n_0$, x_n has its first zero component at some place k_0. Since $x_n \to$ x, x has a zero component at k_0 and (4.1) assures that it is the first one. Then $x \in D(A)$ and $Ax = y$. But A is not demiclosed. Define $u_n = \{1, 1/2, \ldots, 1/(n-1), 0, 1/(n+1), \ldots\}$. Then $u_n \in D(A)$, $u_n \to u = \{1, 1/2, 1/3, \ldots\}$, $Au_n \overset{w}{\to} 0$, but $u \notin D(A)$.

Example 4.2. Let $X = Y = 1^2(R)$, let $e_n = \{\delta_{mn}\}$, let $D(A) = \{e_n : n = 1, 2, \ldots\}$, and define $Ae_n = e_n$. If $\{x_n\} \subset D(A)$, $x_n \to x$, $Ax_n \overset{w}{\to} y$, then $x_n = x$ for all but finitely many n. Then $x \in D(A)$ and $Ax = y$. Hence A is demiclosed. But $\{e_n\} \subset D(A)$, $e_n \overset{w}{\to} 0$, $Ae_n \overset{w}{\to} 0$, and $0 \notin D(A)$. Thus A is not weakly closed.

We observe that if $A \in 1(X, Y)$ then A closed implies A weakly closed by virtue of the fact that every strongly closed subspace of $X \times Y$ is necessarily closed in the weak topology (see [7], Theorem 2.9.2, p. 36). The next proposition, which is proved analogously to Proposition 3.1, asserts the stability of demiclosedness and weakly closedness under continuous perturbations.

Proposition 4.1. *Let $A \in s(X, Y)$, $T \in m(X, Y)$ such that $D(T) \subset D(A)$. If A is demicontinuous (that is, x_n, $x \in D(A)$, $x_n \to x$ implies $Ax_n \overset{w}{\to} Ax$) and $D(A)$ is closed, then $A + T$ is demiclosed if and only if T is. If A is weakly continuous (that is, x_n, $x \in D(A)$, $x_n \overset{w}{\to} x$ implies $Ax_n \overset{w}{\to} Ax$) and $D(A)$ is weakly closed, then $A + T$ is weakly closed if and only if T is.*

Again, as in Section 3, we want to relax the strong assumption that the perturbing operator be continuous. We also want to weaken the relative Lipschitz condition. This gives rise to

Definition 4.2. Let $A \in s(X, Y)$, $T \in m(X, Y)$ such that $D(T) \subset D(A)$. A is *relatively bounded* with respect to T, or *T-bounded*, if and only if there exist constants a, $b \geqq 0$ such that

$$\text{(4.2)} \qquad \|Ax\| \leqq a\|x\| + b|Tx| \quad \text{for all} \quad x \in D(T)\,.$$

The infimum of all possible constants b in (4.2) is called the *T-bound* of A.

Proposition 4.2. *Let Y be reflexive, $A \in s(X, Y)$, $T \in s(X, Y)$, $D(T) \subset D(A)$, and let A be demiclosed (respectively, weakly closed) and T-bounded with T-bound < 1. Then $A + T$ is demiclosed (respectively, weakly closed) if and only if T is.*

Proof. The proof is analogous to that of Proposition 3.2. One uses the following nequalities obtained from (4.2):

$$-a\|x\| + (1 - b)\|y\| \leqq \|u\| \leqq a\|x\| + (1 + b)\|y\| \tag{4.3}$$

where $x \in D(A + T) = D(T)$, $u \in (A + T)x$, $u = Ax + y$, $y \in Tx$. One also uses the fact that in reflexive spaces bounded sequences have weakly convergent subsequences (see [17], Theorem 1, p. 126) and in any Banach space weakly convergent sequences are norm bounded (see [9], p. 137).

As in the "closed" case every $A \in m(X, Y)$ has a demiclosed (respectively, weakly closed) extension in $m(X, Y)$, namely, A_d (respectively, A_w) where $G(A_d)$ (respectively, $G(A_w)$) is the closure of $G(A)$ in $X \times Y_w$ (respectively, $X_w \times Y_w$). If one tries to define demiclosable or weakly closable operators in $s(X, Y)$, that is, ones with demiclosed or weakly closed extensions, the following difficulty is encountered: contrary to the "closed" case it is this time not possible to just "close the graph" in $X \times Y_w$ or $X_w \times Y_w$. The reason is that X_w and Y_w do not necessarily satisfy the axiom of first countability. In order to retain the definition of a demiclosed operator in terms of sequences we arrive at the problem of finding useful criteria of demiclosability and weakly closability:

$$\begin{aligned} &A \in s(X, Y) \quad \text{and if} \quad \{x_n\}, \{y_n\} \subset D(A), \quad x_n \to x, \\ &y_n \to x, \quad Ax_n \overset{w}{\to} y, \quad Ay_n \overset{w}{\to} z, \quad \text{then} \quad y = z. \end{aligned} \tag{4.4}$$

$$\begin{aligned} &A \in s(X, Y) \quad \text{and if} \quad \{x_n\}, \{y_n\} \subset D(A), \quad x_n \overset{w}{\to} x, \\ &y_n \overset{w}{\to} x, \quad Ax_n \overset{w}{\to} y, \quad Ay_n \overset{w}{\to} z, \quad \text{then} \quad y = z. \end{aligned} \tag{4.5}$$

It is easy to see that if $A \in s(X, Y)$ has a demiclosed (respectively, weakly closed) extension in $s(X, Y)$, then A satisfies (4.4) (respectively, (4.5)). We do not know if the converse holds, but we can prove the following proposition.

Proposition 4.3. *Let $A \in s(X, Y)$ satisfy (4.4) and let A map bounded sequences into bounded sequences. Let Y^* be separable. Then A has a demiclosed extension $\hat{A}$ in $s(X, Y)$.*

Proof. Let $D(\hat{A}) = \{x \in X$: there exists $\{x_n\} \subset D(A)$, $y \in Y$ such that $x_n \to x$ and $Ax_n \overset{w}{\to} y\}$. Define $\hat{A}x = y$ and by (4.4) the definition is independent of $\{x_n\}$. It is obvious that $\hat{A}$ extends A. Let $\{f_n\} \subset Y^*$ be dense. Let $\{u_n\} \subset D(\hat{A})$, $u_n \to u$, $\hat{A}u_n \overset{w}{\to} v$. We have to show that $u \in D(\hat{A})$ and $\hat{A}u = v$. Let $\{u_n(m)\} \subset D(A)$ such that $\lim_{m\to\infty} u_n(m) = u_n$ and $Au_n(m) \overset{w}{\to} \hat{A}u_n = v_n$ as $m \to \infty$. Fix f_1 and choose n_1 such that

$$\|u_{n_1} - u\| < 1 \quad \text{and} \quad |f_1(v_{n_1}) - f_1(v)| < 1\,.$$

Choose p_1 such that

$$\|u_{n_1}(p_1) - u_{n_1}\| < 1 \quad \text{and} \quad |f_1(Au_{n_1}(p_1) - f_1(v_{n_1})| < 1\,.$$

In general, for each k choose n_k such that

$$\|u_{n_k} - u\| < 1/k\,, \quad |f_i(v_{n_k}) - f_i(v)| < 1/k \quad \text{for} \quad 1 \leqq i \leqq k\,.$$

Choose p_k such that

$$\|u_{n_k}(p_k) - u_{n_k}\| < 1/k\,, \quad |f_i(Au_{n_k}(p_k)) - f_i(v_{n_k})| < 1/k$$

for $1 \leqq i \leqq k$. It is apparent that $\lim_{k\to\infty} u_{n_k}(p_k) = u$ and since $\{u_{n_k}(p_k)\}$ is bounded, so is $\{Au_{n_k}(p_k)\}$. For arbitrary i we have $f_i(Au_{n_k}(p_k)) \to f_i(v)$. By the denseness of $\{f_i\}$ we obtain $Au_{n_k}(p_k) \overset{w}{\to} v$ (see [9], Lemma 1.31, p. 137) and the proof is complete.

We observe that the operator $\hat{A}$ is clearly the smallest demiclosed extension of A. The proof of the following proposition is analogous to the one above.

Proposition 4.4. *Let $A \in s(X, Y)$ satisfy (4.5) and let $D(A)$ and $R(A)$ be bounded. Let X^* and Y^* be separable. Then A has a weakly closed extension $\hat{A}$ in $s(X, Y)$.*

Proposition 4.5. *Let A, $T \in s(X, Y)$, $D(T) \subset D(A)$, let A and T satisfy (4.4), let T map bounded sequences into bounded sequences, and let A be T-bounded with T-bound < 1. Let Y be reflexive and Y^* separable. Then $S = A + T$ has a smallest demiclosed extension $\hat{S}$ in $s(X, Y)$. In this case $D(\hat{S}) = D(\hat{T})$. In particular, T is demiclosed if and only if S is.*

Proof. The proof is similar to Proposition 3.3. One uses the facts that weakly convergent sequences are norm bounded and in reflexive spaces bounded sequences have weakly convergent subsequences. From (4.2) one obtains that S satisfies (4.4) and maps bounded sequences into bounded sequences. Hence, by Proposition 4.3, S has smallest demiclosed extension $\hat{S}$ in $s(X, Y)$. If $x \in D(\hat{T})$, there exists $\{x_n\} \subset D(T)$ such that $x_n \to x$ and $Tx_n \overset{w}{\to} \hat{T}x$. By (4.2) $\{Sx_n\}$ is bounded and thus has a weakly convergent subsequence. Hence, $x \in D(\hat{S})$. The reverse inclusion is proven similarly. If T is demiclosed, $D(S) = D(T) = D(\hat{T}) = D(\hat{S})$ and S is demiclosed, and conversely.

The proof of the following proposition is analogous to the one above.

Proposition 4.6. *Let A, $T \in s(X, Y)$, $D(T) \subset D(A)$, let A and T satisfy (4.5), let $D(T)$ and $R(T)$ be bounded, and let A be T-bounded with T-bound < 1. Let Y be reflexive and X^* and Y^* separable. Then $S = A + T$ has a smallest weakly closed extension $\hat{S}$ in $s(X, Y)$. In this case $D(\hat{S}) = D(\hat{T})$. In particular, T is weakly closed if and only if S is.*

Definition 5.1. Let $A \in m(X, X)$. We define the scalar λ to be in the *resolvent* of A, denoted by $\varrho(A)$, provided $(\lambda I - A)^{-1}$ exists, $D((\lambda I - A)^{-1}) = X$, and $(\lambda I - A)^{-1} \in$ $\in \operatorname{lip}(X, X)$. The complement of $\varrho(A)$ is called the *spectrum* of A and is denoted by $\sigma(A)$.

Before investigating the resolvent and spectrum of nonlinear operators we first need to consider Lipschitz invertibility and its stability under perturbation. The following proposition is well known but we prove it here for the sake of completeness. (This result, and, indeed, a thorough discussion of spaces of Lipschitz operators may be found in Chapter 2 of [12].)

Proposition 5.1. *Let* $A \in \operatorname{lip}(X, X)$, $D(A) = X$ *and let* $|I - A| < 1$. *Then* A^{-1} *exists,* $D(A^{-1}) = X$, $A^{-1} \in \operatorname{lip}(X, X)$, *and*

$$|A^{-1}| \leqq 1/(1 - |I - A|)\,. \tag{5.1}$$

Proof. Define for all $x, y \in X$, $A_x(y) = (I - A)\,y + x$. Then

$$\|A_x(y) - A_x(z)\| \leqq |I - A|\,\|y - z\|$$

and so by the contraction mapping principle has precisely one fixed point z_x, that is, $A_x(z_x) = z_x$ or $Az_x = x$. This shows that A is surjective and the uniqueness of z_x implies that A is injective. Hence, A^{-1} exists and $D(A^{-1}) = X$. Furthermore, for all $x, y \in X$

$$\|A^{-1}x - A^{-1}y\| = \|z_x - z_y\| = \|A_x(z_x) - A_y(z_y)\| =$$
$$= \|(I - A)\,z_x - (I - A)\,z_y + (x - y)\| \leqq |I - A|\,\|A^{-1}x - A^{-1}y\| + \|x - y\|$$

which yields (5.1).

The next proposition is analogous to the linear case (see [9], p. 196).

Proposition 5.2. *Let* $A \in s(X, X)$, $T \in m(X, X)$ *such that* T^{-1} *exists in* $\operatorname{lip}(X, X)$, *and* $D(T^{-1}) = X$. *Suppose* $D(T) \subset D(A)$ *and* (3.3) *is satisfied with* $a|T^{-1}| + b < 1$. *Then* $S^{-1} = (A + T)^{-1}$ *exists in* $\operatorname{lip}(X, X)$, $D(S^{-1}) = X$, *and*

$$|S^{-1}| \leqq |T^{-1}|/(1 - a|T^{-1}| - b)\,. \tag{5.2}$$

Proof. First, observe that $S = (I + AT^{-1})\,T$. From (3.3) we obtain for $x, y \in X$,

$$\|AT^{-1}x - AT^{-1}y\| \leqq a\|T^{-1}x - T^{-1}y\| + b|TT^{-1}x - TT^{-1}y| \leqq$$
$$\leqq (a|T^{-1}| + b)\,\|x - y\|\,.$$

By Proposition 5.1 $(I + AT^{-1})^{-1}$ exists in $\operatorname{lip}(X, X)$ with domain X and satisfies

$$|(I + AT^{-1})^{-1}| \leqq 1/(1 - |AT^{-1}|)\,. \tag{5.3}$$

The conclusions now follow.

Proposition 5.3. *Let $A \in m(X, Y)$. The following are true:*

(5.4) $\varrho(A)$ *is open;*

(5.5) $(\lambda I - A)^{-1} x$ *is continuous as a function from $\varrho(A)$ to X for each fixed $x \in X$;*

(5.6) *if $\lambda \in \varrho(A)$ and $(\lambda I - A)^{-1}$ is Fréchet differentiable at some $x \in X$, then $(\lambda I - A)^{-1} x$ is differentiable in some neighborhood of λ.*

Proof. To establish (5.4) let $\lambda \in \varrho(A)$ and observe that for any μ, $(\mu I - A) = (I + (\mu - \lambda)(\lambda I - A)^{-1})(\lambda I - A)$. Let $B_\mu = I + (\mu - \lambda)(\lambda I - A)^{-1}$. For μ such that $|\mu - \lambda| < 1/|(\lambda I - A)^{-1}|$, B_μ^{-1} exists in $\text{lip}(X, X)$ with domain X by virtue of Proposition 5.1. For these μ, $(x, y) \in (\mu I - A)$ if and only if $(y, x) \in (\lambda I - A)^{-1} B_\mu^{-1}$ and thus $\mu \in \varrho(A)$. From (5.1) we also obtain

$$|(\mu I - A)^{-1}| \leqq |(\lambda I - A)^{-1}|/(1 - |\mu - \lambda|\, |(\lambda I - A)^{-1}|)\,.$$

To establish (5.5) observe that for μ sufficiently close to λ,

$$\begin{aligned}\|(\lambda I - A)^{-1} x - (\mu I - A)^{-1} x\| &= \|(\lambda I - A)^{-1} x - (\lambda I - A)^{-1} B_\mu^{-1} x\| \leqq \\ &\leqq |(\lambda I - A)^{-1}|\, \|B_\mu^{-1} B_\mu x - B_\mu^{-1} x\| \leqq \\ &\leqq |(\lambda I - A)^{-1}|\, \|B_\mu x - x\|/(1 - |\mu - \lambda|\, |(\lambda I - A)^{-1}|) \leqq \\ &\leqq |(\lambda I - A)^{-1}|\, |\mu - \lambda|\, \|(\lambda I - A)^{-1} x\|/(1 - |\mu - \lambda|\, |(\lambda I - A)^{-1}|)\,.\end{aligned}$$

Finally, (5.5) is due to J. NEUBERGER and a proof can be found in [15].

It is well known that if $A \in \text{lip}(X, X)$, $D(A) = X$, then $\lambda \in \varrho(A)$ provided $|\lambda| > |A|$ and a proof of this fact may be found in [11], p. 144, [16], p. 39, or [13], p. 211 (actually, this also follows from Proposition 5.2). The authors do not know if in this case the spectrum of A is nonempty (where the space X is over a complex field). Our notion of spectrum agrees with R. KAČUROVSKII [8] and E. ZARANTONELLO [18], [19]. Another notion of spectrum is investigated by S. BURÝŠEK in [2] and a localization of the spectrum is developed by L. MAY in [13]. In [15] J. Neuberger defines λ to be in the resolvent of A provided $(\lambda I - A)^{-1}$ exists with domain X and $(\lambda I - A)^{-1}$ is Fréchet differentiable at every $x \in X$ (where X is complex). He then establishes the existence of a point in the complement of the resolvent of A provided A is locally Lipschitz in some neighborhood of 0. This differentiability requirement will not be satisfied in general, as may be seen by the following example.

Example 5.1. Let $A \in s(R, R)$ where $Ax = x$ if $x \leqq 2$ and $Ax = 2$ if $x > 2$. Then, for $\lambda \neq [0, 1]$, $(\lambda I - A)^{-1} x = x/(\lambda - 1)$ if $x \leqq 2(\lambda - 1)$ and $(\lambda I - A)^{-1} x = (x + 2)/\lambda$ if $x \geqq 2(\lambda - 1)$. For $x = 2$ one sees that $(\lambda I - A)^{-1} x$ is not diferentiable in λ at $\lambda = 2$. Also $(\lambda I - A)^{-1}$ is not continuous as a function from $\varrho(A)$

to the Banach space $\mathrm{lip}_0(X, X)$ consisting of all $T \in \mathrm{lip}(X, X)$ with $D(T) = X$, $T0 = 0$. and norm given by (2.7), as may be seen by

$$|((2I - A)^{-1} - ((2 + 1/n) I - A)^{-1}) 2 -$$
$$- ((2I - A)^{-1} - ((2 + 1/n) I - A)^{-1}) (2 + 2/n)| =$$
$$= (\tfrac{1}{2}) |(n - 1)/(n + 1)| |2 - (2 + 2/n)| ,$$

which implies that $|(2I - A)^{-1} - ((2 + 1/n) I - A)^{-1}| \geqq \frac{1}{3}$ for all $n \geqq 2$.

Example 5.2. Let $A \in m(R, R)$ where $Ax = 0$ if $x < 0$, $A0 = [-1, 0]$, and $Ax = -1$ if $x > 0$. Then $\lambda \in \varrho(A)$ provided that $\lambda > 0$.

Proposition 5.4. *Let $A \in s(X, X)$, $T \in m(X, X)$ such that $D(T) \subset D(A)$. Suppose $\lambda \in \varrho(T)$ and (3.3) is satisfied where*

$$(5.7) \qquad (a + |\lambda| b) |(\lambda I - T)^{-1}| + b < 1 .$$

Then $\lambda \in \varrho(A + T)$ and

$$(5.8) \quad |(\lambda I - (A + T))^{-1}| \leqq |(\lambda I - T)^{-1}|/(1 - (a + |\lambda| b) |(\lambda I - T)^{-1}| - b) .$$

Proof. Since $(\lambda I - (A + T)) = (I - A(\lambda I - T)^{-1}) (\lambda I - T)$, it suffices to show that $(I - A(\lambda I - T)^{-1})^{-1}$ exists in $\mathrm{lip}(X, X)$ with domain X. We observe that for all $x \in X$, $\lambda(\lambda I - T)^{-1} x - x \in T(\lambda I - T)^{-1} x$ and thus for all $x, y \in X$,

$$\|A(\lambda I - T)^{-1} x - A(\lambda I - T)^{-1} y\| \leqq$$
$$\leqq a\|(\lambda I - T)^{-1} x - (\lambda I - T)^{-1} y\| + b|T(\lambda I - T)^{-1}x - T(\lambda I - T)^{-1} y| \leqq$$
$$\leqq (a|(\lambda I - T)^{-1}| + |\lambda| b|(\lambda I - T)^{-1}| + b) \|x - y\| .$$

Then (5.7) and Proposition 5.1 yield the desired result.

We note that in the case $T \in s(X, X)$, (5.7) can be replaced by

$$(5.9) \qquad a|(\lambda I - T)^{-1}| + b|T(\lambda I - T)^{-1}| < 1$$

and (5.8) can be replaced by

$$(5.10) \qquad |(\lambda I - (A + T))^{-1}| \leqq$$
$$\leqq |(\lambda I - T)^{-1}|/(1 - a|(\lambda I - T)^{-1}| - b|T(\lambda I - T)^{-1}|) .$$

Proposition 5.5. *Suppose $A \in m(X, X)$, $\mu \in \varrho(A)$ and $(\mu I - A)^{-1}$ is compact (weakly compact). Then $(\lambda I - A)^{-1}$ is compact (weakly compact) for all $\lambda \in \varrho(A)$.*

Proof. For $\lambda \in \varrho(A)$ we have $(\mu I - A) = (I + (\mu - \lambda) (\lambda I - A)^{-1}) (\lambda I - A)$, which implies $(\lambda I - A)^{-1} = (\mu I - A)^{-1} (I + (\mu - \lambda) (\lambda I - A)^{-1})$ (this "nonlinear resolvent formula" may be found in [5], Lemma 1.2). The conclusion follows since the composition of a Lipschitz operator and a compact (weakly compact) operator is compact (weakly compact).

Proposition 5.6. *Suppose the hypothesis of Proposition* 5.4 *and in addition suppose that* $(\lambda I - T)^{-1}$ *is compact* (*weakly compact*). *Then* $(\lambda I - (A + T))^{-1}$ *is compact* (*weakly compact*).

Proof. The conclusion follows as in the proof of Proposition 5.5 from the identity $(\lambda I - (A + T))^{-1} = (\lambda I - T)^{-1} (I - A(\lambda I - T)^{-1})^{-1}$ established in the proof of Proposition 5.4.

6. CONTINUITY AND BIFURCATION OF THE SPECTRUM

In what follows we shall let Y be a Banach space with norm $\| \ \|$, T a closed operator in $1(Y, Y)$, and X the Banach space $D(T)$ with norm $|||x||| = a\|x\| + b\|Tx\|$, $x \in X = D(T)$, where a, b are some fixed positive constants. We let I_0 denote the operator in $b\ 1(X, Y)$ given by $I_0 x = x$ for all $x \in X$.

Definition 6.1. Let $A \in m(X, Y)$. We say the scalar λ is in the *point spectrum* of A if and only if there exists a nonzero $x \in D(A)$ such that $0 \in (\lambda I_0 - A)\, x$. In this case we call λ an *eigenvalue* of A and x an *eigenvector* of A.

Proposition 6.1. *Let X and Y be as above. Suppose that D is an open convex subset of X and let $A \in s(X, Y)$ such that $D \subset D(A)$. If A has a bounded Fréchet derivative on D, that is, there exists some constant c such that $|A'(x)| \leqq c$ for all $x \in D$, then A satisfies*

$$\|Ax - Ay\| \leqq ac\|x - y\| + bc\|Tx - Ty\| \quad \text{for all} \quad x, y \in D\,. \tag{6.1}$$

Proof. By Corollary 5.4.1 of [14], p. 176, A satisfies a Lipschitz condition on D, that is, there exists some positive constant c such that

$$\|Ax - Ay\| \leqq c|||x - y||| \quad \text{for all} \quad x, y \in D\,. \tag{6.2}$$

Then (6.1) follows immediately.

Proposition 6.2. *Let X and Y be as above. Suppose that D is an open subset of X, $A \in s(X, Y)$, $D \subset D(A)$, and A is continuously Fréchet differentiable on D. Let $x_0 \in D$ and let λ_0 be a scalar such that $(\lambda_0 I_0 - A)\, x_0 = 0$. If $(\lambda_0 I_0 - A'(x_0))^{-1} \in b\ 1(Y, X)$, then there exists a neighborhood U_0 of λ_0 such that for every neighborhood U of λ_0, contained in U_0, there exists a unique continuous mapping x of U into X such that $x(\lambda_0) = x_0$ and for all $\lambda \in U$ and $x(\lambda) \in D$, $(\lambda I_0 - A)\, x(\lambda) = 0$.*

Proof. Define $F(\lambda, x) = (\lambda I_0 - A)x$ where F has domain $R \times D$ (or $C \times D$). Then F is continuously Fréchet differentiable on its domain, $F(\lambda_0, x_0) = 0$, and $F_2(\lambda_0, x_0) = \lambda_0 I - A'(x_0)$ is a linear homeomorphism of X onto Y. By the implicit function theorem (see [6], p. 270) the conclusions follow.

Definition 6.2. Following KRASNOSELSKII [11], p. 149, we define the point spectrum of A to be *continuous* provided it contains an interval. (In the case of Proposition 6.2, if $x_0 \neq 0$, then the point spectrum of A is continuous.)

Proposition 6.3. *Let X and Y be as above. Let $A \in s(X, Y)$ such that A is defined on some neighbourhod of* 0, $A0 = 0$, *and A is Fréchet differentiable at* 0. *Suppose $\lambda_0 \neq 0$ is a scalar such that $(I_0 - \lambda_0 A'(0))^{-1} \in b\,1(Y, X)$. Then there exists a neighborhood U about λ_0 and a neighborhood V about* 0 *in X such that if $\lambda \in U$, $x \in V$, and $(I_0 - \lambda A)x = 0$, then $x = 0$.*

Proof. Let $Ax - A0 = Ax = A'(0)x + w(x)$ where $\|w(x)\|/|||x||| \to 0$ as $|||x||| \to 0$. Let V be a neighbourhood of 0 in X and let U be a neighborhood of λ_0 such that if $x \in V$ and $\lambda \in U$ then

$$\|w(x)\| \leqq |||x|||/3|(I - \lambda_0 A'(0))^{-1}|\,|\lambda|, \tag{6.3}$$

and

$$|\lambda - \lambda_0| \leqq \tfrac{1}{3}|(I - \lambda_0 A'(0))^{-1}|\,|A'(0)|. \tag{6.4}$$

Suppose $x \in V$, $\lambda \in U$, and $(I_0 - \lambda A)x = 0$. From (6.3) and (6.4) we have

$$\begin{aligned} |||x||| &= |||(I_0 - \lambda_0 A'(0))^{-1}(I_0 - \lambda_0 A'(0))x||| \leqq \\ &\leqq |(I_0 - \lambda_0 A'(0))^{-1}|\,\|\lambda A x - \lambda_0 A'(0)x\| = \\ &= |(I_0 - \lambda_0 A'(0))^{-1}|\,\|(\lambda - \lambda_0)A'(0)x + \lambda w(x)\| \leqq 2|||x|||/3 \end{aligned}$$

and the proof is complete.

Proposition 6.3, which is modeled after Lemma 2.1 of [11], p. 192, says that the eigenvectors of A with small norms can correspond only to eigenvalues of A which are close to scalars λ_0 such that $(I_0 - \lambda_0 A'(0))^{-1} \notin b\,1(Y, X)$. Following [11], p. 181, we make the following definition.

Definition 6.3. Let X and Y be as above. A scalar $\lambda_0 \neq 0$ is called a *bifurcation point* of the point spectrum of $A \in m(X, Y)$ provided that for all $\varepsilon, \delta > 0$ there exists a scalar λ such that $|\lambda - \lambda_0| < \varepsilon$ and there exists some $x \neq 0$ such that $x \in D(A)$, $0 \in (I_0 - \lambda A)x$, and $|||x||| < \delta$. (In the case of Proposition 6,3, if λ_0 is a bifurcation point of the point spectrum of A, then $(I_0 - \lambda_0 A'(0))^{-1} \notin b\,1(Y, X)$).

We conclude with two examples to which Propositions 6.1, 6.2, and 6.3 can be applied.

Example 6.1. Let $Y = C_0([0, 1]; R)$, the Banach space of continuous real-valued functions on $[0, 1]$ which are 0 at 0 and with supremum norm. Let $Tx = x'$, $D(T) = \{x \in Y : x' \in Y\}$, let $|||x||| = \|x\| + \|Tx\|$, and let $F : R \to R$ such that F is continuously differentiable and $F(0) = 0$. Define $Ax = F(x) + x'$ with $D(A) = X = D(T)$. A is Fréchet differentiable at each $x \in X$, since for all $h \in X$, $A'(x)\, h = F'(x)\, h + h'$ and $\|A'(x)\, h\| \leqq c\|h\| + \|Th\|$ for some constant c depending on x. Also A' is continuous from X to $b\ 1(X, Y)$, since

$$\|(A'(x) - A'(y))\, h\| \leqq \|F'(x) - F'(y)\| \,.\, |||h||| \,.$$

We will establish that for all $x_0 \in X$ and $\lambda_0 \in R$, $(\lambda_0 I_0 - A'(x_0))^{-1} \in b\ 1(Y, X)$. It suffices to show that $(\lambda_0 I_0 - A'(x_0))$ is closed, injective, and onto by virtue of the closed graph theorem (see [9], Theorem 5.20, p. 166). Obviously $(\lambda_0 I_0 - A'(x_0))$ is closed since it is continuous and everywhere defined. Moreover, if $g \in Y$ and $p = F'(x_0) - \lambda_0$, then $h' + ph = g$ has the unique solution $h \in X$ given by

$$h(u) = \exp\left(-\int_0^u p(s)\, ds\right) \int_0^u \exp\left(\int_0^v p(s)\, ds\right) g(v)\, dv \,.$$

We observe that A does not have a continuous spectrum nor does A have any bifurcation points, since $(I_0 - \lambda_0 A)\, x_0 = 0$ implies $x_0 \equiv 0$.

Example 6.2. Let Y, T, and $|||\ |||$ be as in Example 6.1. Define $Ax = xx'$ with $D(A) = X = D(T)$. A is Fréchet differentiable at each $x \in X$, since for each $h \in X$, $A'(x)\, h = xh' + x'h$, $A(x + h) - A(x) - A'(x)\, h = w(x, h) = hh'$, $\|w(x, h)\|/|||h||| \leqq \|h\| \,.\, \|h'\|/\|h\| + \|h'\| \to 0$ as $|||h||| \to 0$, and $\|A'(x)\, h\| \leqq |||x||| \,.\, |||h|||$. Also A' is continuous form X to $b\ 1(X, Y)$, since

$$\|(A'(x) - A'(y))\, h\| \leqq |||x - y||| \,.\, |||h||| \,.$$

Let λ_0 be any scalar and let $x_0 \in X$ be defined by $x_0(s) = \lambda_0 s$. Then $(\lambda_0 I_0 - A)\, x_0 = 0$ and $(\lambda_0 I_0 - A'(x_0))\, h = g$ is equivalent to $-\lambda_0 s h'(s) = g(s)$. If $\lambda_0 = 0$ then $\lambda_0 I_0 - A'(x_0) \equiv 0$ and thus not onto. If $\lambda_0 \neq 0$ then $\lambda_0 I_0 - A'(x_0)$ is also not onto (since for $g(s) = \sqrt{s}$, $h(s) = (-1/\lambda_0) \int_0^s g(u)/u\, du$ and $h \notin X$). Thus for any choice for λ_0, $(\lambda_0 I_0 - A'(x_0))^{-1}$ does not exist in $b\ 1(Y, X)$. Notice that $x(s) = \lambda s$ is an eigenvector of A whose norm can be made arbitrarily small. Thus there does not exist a unique solution of $(\lambda I_0 - A)\, x = 0$ in any neighbourhood about $\lambda_0 = 0$, $x_0 = 0$. Also A has no bifurcation points, since if $(I_0 - \lambda_0 A)\, x_0 = 0$, $\lambda_0 \neq 0$, then either $x_0 \equiv 0$ or $x_0(s) = s/\lambda_0$. This demonstrates that the converse of Proposition 6.3 is not true.

Addendum. The authors have recently learned that (5.4) and (5.5) of Proposition 5.3 were proven by G. DaPrato "Some d'applications non linéaires et solutions globales d'équations quasi-linéaires dans des espaces de Banach", Boll. U. M. I. *4*, No. 2 (1969), 229–240.

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