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Branko Grünbaum; Ernest Jucovič

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ON NON-INSCRIBABLE POLYTOPES

BRANKO GRÜNBAUM, Seattle and ERNEST JUCOVIČ, Košice

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1. Introduction. The purpose of this note is to establish the existence of polytopes which behave very poorly with respect to the possibility of inscribing any isomorphic polytope into a sphere. To formulate the result and its background more precisely we need some definitions.

Let P be a 3-polytope (that is, 3-dimensional convex polytope) with $v(P)$ vertices. We shall denote by $s(P)$ the largest integer s with the property: There exists a 3-polytope P' isomorphic to P, and a sphere S that encloses P', such that s vertices of P' are on S.

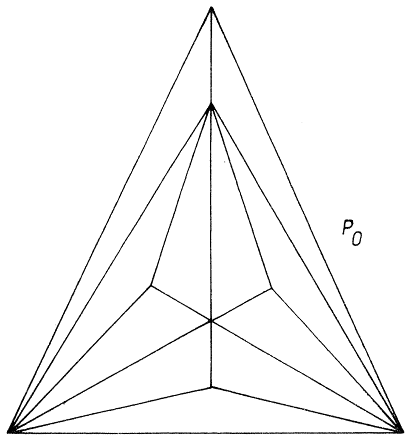

Fig. 1

STEINER [7] asked whether all 3-polytopes are inscribable, where a 3-polytope P is called *inscribable* provided $s(P) = v(P)$. BRÜCKNER [1, p. 163, footnote 4] remarked that all simplicial 3-polytopes are necessarily inscribable; we recall that a 3-polytope P is *simplicial* if all the faces of P are triangles. However, as was observed by STEINITZ [8], this remark of Brückner's is mistaken; indeed, Steinitz proceeded to prove by a peculiarly elegant geometric argument that there are infinitely many types of 3-polytopes (among them simplicial ones) that are non-inscribable. The non-inscribable simplicial 3-polytope P_0 with fewest vertices obtainable by Steinitz's method is represented in Figure 1 by its Schlegel diagram. Note that P_0 may be interpreted as the polytope arising from the tetrahedron by placing a sufficiently flat 3-sided pyramid on each of the four faces of the tetrahedron.

Steinitz's method was extended by JUCOVIČ [6]; a corollary of that paper which we shall use here is: If S is a sphere that contains a polytope P'_0 isomorphic to P_0, then not all four 3-valent vertices of P'_0 are on S.

Another step in investigating how badly non-inscribable may 3-polytopes be was taken in Jucovič [5]. His result may be interpreted as asserting the existence of simplicial 3-polytopes P with an arbitrarily prescribed number of vertices such that

$$s(P) \leqq [(2v(P) + 13)/3] .$$

The main aim of the present note is to improve upon those results by establishing the following

Theorem. *If P ranges over all types of simplicial 3-polytopes then*

$$\liminf \frac{\log s(P)}{\log v(P)} \leqq \frac{\log 2}{\log 3} = 0.630930\ldots .$$

In other words, for every $\alpha > \log 2/\log 3$ there exists a constant β, and a sequence of non-isomorphic simplicial 3-polytopes P_n such that

$$s(P_n) \leqq \beta \,.\, (v(P_n))^{\alpha} \quad \text{for all} \quad n .$$

We shall present the proof of the Theorem in Section 2; various remarks, open problems and conjectures are presented in Section 3.

2. Proof of the Theorem. Before starting the actual proof, we shall establish a lemma. In order to ease its formulation, we shall first explain the terminology and notation.

Let Q be a simplicial 3-polytope in which we have singled out a set T of 3-valent vertices no two of which share an edge. Although the lemma would be valid also without this assumption, we shall simplify its formulation by assuming that T is the set of all 3-valent vertices of Q, the number of which is denoted by $v_3(Q)$. We denote by W the set of $v(Q) - v_3(Q)$ vertices of Q that have valence 4 or more. We shall denote by $t(Q)$ and by $w(Q)$ the maximal possible numbers of 3-valent vertices, and of 4 or higher valent vertices, of any 3-polytope Q' isomorphic to Q, that belong to a sphere that contains Q'. Therefore $s(Q) \leqq t(Q) + w(Q)$. Note that if $V \in T$, the three faces of Q incident with V form a complex isomorphic to the complex N indicated in Figure 2. The vertices U_1, U_2, U_3 of N belong to W. We shall denote by Q^* the simplicial 3-polytope obtained from Q by replacing every vertex $V \in T$ by a copy of the complex H shown in Figure 3. It follows easily that the polytopes Q and Q^*

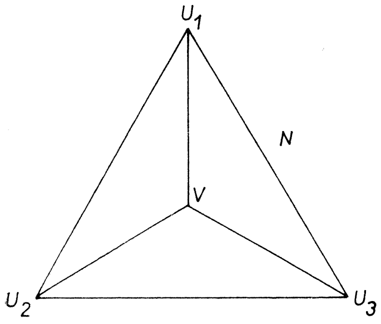

Fig. 2

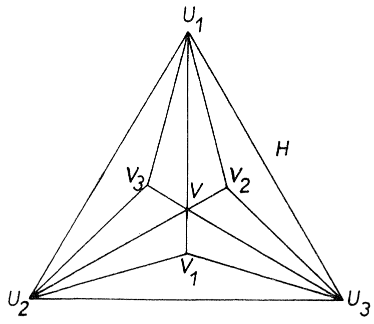

Fig. 3

satisfy $v(Q^*) = v(Q) + 3\, v_3(Q)$ and $v_3(Q^*) = 3\, v_3(Q)$. Furthermore, we shall establish the following

Lemma. $t(Q^*) \leqq 2\, t(Q)$ *and* $w(Q^*) \leqq w(Q) + t(Q)$.

Proof. If any one of the four "interior" vertices V, V_1, V_2, V_3 of a copy of H belongs to a sphere S containing a polytope $Q^{*\prime}$ isomorphic to Q^*, the three triangles determined by that vertex and by the three "outer" vertices U_1, U_2, U_3 of H yield a copy of N, and the convexity of $Q^{*\prime}$ is not affected by the replacement of that copy of H by the just constructed copy of N. Therefore, if in more than $t(Q)$ copies of H a vertex would belong to a containing sphere S, we should be able to form a polytope isomorphic to Q in which more than $t(Q)$ of its 3-valent vertices belong to the sphere S that contains it. As this is impossible, we must have $w(Q^*) \leqq w(Q) + t(Q)$. On the other hand, in each of the at most $t(Q)$ copies of H that have any of the interior vertices belonging to S, at most 2 of the three 3-valent vertices present may belong to the sphere (unless *all* the vertices of $Q^{*\prime}$ that belong to S are among the seven vertices of that one copy of H). Indeed, if for some copy of H the three vertices V_1, V_2, V_3 were on the sphere S that contains $Q^{*\prime}$, the convex hull of that copy of H and of an arbitrary vertex of $Q^{*\prime}$ outside of H that belongs to S would yield a polytope isomorphic to the polytope P_0 of Section 1, which has its four 3-valent vertices on a sphere that contains P_0. As that is impossible, we have $t(Q^*) \leqq 2\, t(Q)$, and the proof of the Lemma is completed.

We shall now prove our Theorem by exhibiting a sequence P_n, $n = 0, 1, \ldots,$ of simplicial 3-polytopes such that $v(P_n) = 2 + 6.3^n$ and $s(P_n) \leqq 1 + 6.2^n$. It is obvious that

$$\liminf \frac{\log s(P)}{\log v(P)} \leqq \lim_{n\to\infty} \frac{\log s(P_n)}{\log v(P_n)} \leqq \frac{\log 2}{\log 3}\,.$$

We construct the sequence P_n by induction, taking as P_0 the polytope described in Section 1 and shown in Figure 1. If any P_n is already constructed, we form P_{n+1} as P_n^*, by replacing the configuration N around each 3-valent vertex of P_n by a copy of the complex H. Since $v(P_0) = 8$ and $v_3(P_0) = 4$, the construction clearly implies that $v(P_n) = 2 + 6.3^n$ and $v_3(P_n) = 4.3^n$. Using the Lemma and $t(P_0) = 3$, $w(P_0) = 4$, it then follows at once that $t(P_n) \leqq 3.2^n$, and $w(P_n) \leqq 1 + 3.2^n$, so that $s(P_n) \leqq 1 + 6.2^n$, as asserted.

This completes the proof of the Theorem.

3. Remarks. (1) Following the terminology introduced in GRÜNBAUM-WALTHER [4], the number

$$s^{(d)} = \liminf \frac{\log s(P)}{\log v(P)}$$

where P ranges over all d-polytopes, could be called the "*inscribability exponent*" of the family of all d-polytopes. Our Theorem asserts that $s^{(3)} \leqq \log 2/\log 3$; we make the following

Conjecture. $s^{(3)} = \log 2/\log 3$.

It should be pointed out that our conjecture could be wildly wrong. Not only do we ignore whether $s^{(3)} > 0$, but all our present knowledge is even compatible with the following rather outrageous statement: There exists an absolute constant k and 3-polytopes P with arbitrarily large $v(P)$ such that $s(P) \leqq k$.

(2) The results of Steinitz [8] and Jucovič [6] mentioned in Section 1 may easily be extended to higher dimensions. It follows, for example, that $P_0^{(d)}$, the "Kleetope" (see GRÜNBAUM. [3, p. 217]) over the d-dimensional simplex has the following property: If a polytope P' isomorphic to $P_0^{(d)}$ is contained in the ball determined by some $(d-1)$-dimensional sphere S, then not all $d+1$ of the d-valent vertices of P' can be on S. By a simple extension of the proof of the Theorem in Section 2 we may construct a sequence $P_n^{(d)}$, $n = 0, 1, \ldots$, of simplicial d-polytopes such that

$$\lim_{n\to\infty} \frac{\log s(P_n^{(d)})}{\log v(P_n^{(d)})} = \frac{\log (d-1)}{\log d}.$$

We venture:

Conjecture. The inscribability exponent of the family of simplicial d-polytopes, $d \geqq 3$,

$$\bar{s}^{(d)} = \liminf \frac{\log s(P)}{\log v(P)}$$

where P ranges over the simplicial d-polytopes, satisfies

$$\bar{s}^{(d)} = \log (d-1)/\log d .$$

It should be noted that, as shown by the 4-dimensional pyramids based on the 3-polytopes P_n constructed in Section 2, the inscribability exponent $s^{(4)}$ of *all* 4-polytopes satisfies $s^{(4)} \leqq \log 2/\log 3$; similarly $s^{(d)} \leqq \log 2/\log 3$ for *all* $d \geqq 3$. It is well possible that strict inequality holds in this relation, at least for sufficiently large d; this is in contrast to the conjectured relation $s^{(3)} = \bar{s}^{(3)}$.

(3) It was shown in Grünbaum [2] that there exist simple 3-polytopes P which are not inscribable. (A d-polytope is called *simple* if each vertex is incident with precisely d different edges.) It would be of interest to investigate the inscribability exponent $\hat{s}^{(d)}$ of simple d-polytopes. So far we were unable to show that $\hat{s}^{(d)} < 1$ for any $d \geqq 3$.

(4) If P is a d-polytope, let $c(P)$ denote the maximal number of co-spherical vertices in any polytope isomorphic with P. Then clearly $c(P) \geqq s(P)$, but if P is non-inscribable then $c(P) < v(P)$. It would be of interest to investigate whether

$$\liminf \frac{\log c(P)}{\log v(P)} < 1$$

when P ranges over all d-polytopes, or over certain classes of them.

(5) If P is a d-polytope with $f(P)$ facets (that is, $(d-1)$-dimensional faces), let $i(P)$ denote the greatest integer i such that there exists a polytope P' isomorphic with P, with the property that i facets of P' are inscribable $(d-1)$-spheres. Clearly $i(P) = f(P)$ whenever P is a simplicial polytope; on the other hand, if P is simple then $s(P) < v(P)$ if and only if $i(P) < f(P)$. The 3-dimensional case of this observation was used in Grünbaum [2] to derive the existence of simple non-inscribable 3-polytopes from the existence of 3-polytopes "without circumcircles". Although it is easy to find a sequence of 3-polytopes P_n such that

$$\lim_{n\to\infty} \frac{i(P_n)}{f(P_n)} = 2/3$$

it is not known whether other sequences would give smaller limits. Similarly, the higher dimensional analogues are completely unexplored.

(6) It is probably true that if P is a 3-polytope with $v(P) \leqq 7$ then P is inscribable; this is easily checked for simplicial P. However, in general we have no method of determining whether a given 3-polytope is inscribable or not. Several criteria establishing non-inscribability of 3-polytopes are known (Steinitz [8], Grünbaum [2]) but they certainly do not cover all cases.

(7) As explained in detail in Steinitz [8], the question whether a polytope P is *circumscribable* (that is, isomorphic to a polytope all facets of which are tangent to a fixed sphere) is equivalent to the question whether the polytope P^* dual to P is inscribable. However, the following problems are still open:

(i) Does every 3-polytope P have an isomorphic polytope P' such that every facet of P' is a polygon with an incircle?

(ii) Is every 3-polytope isomorphic to a cage for the sphere? (Here a polytope Q is called a *cage* for the sphere provided there exists a sphere that touches all the edges of Q.)

(iii) Is every 3-polytope isomorphic to a normal one? (We call a 3-polytope Q *normal* if there exists a point O such that for each face F of Q the foot of the perpendicular from O to the plane of F belongs to the interior of F.)

Clearly, such and similar questions may be formulated also for higher dimensional polytopes, and in many other variations.

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Authors' adresses: B. Grünbaum, University of Washington, Seattle (USA); E. Jucovič, P. J. Šafárik University, Košice (ČSSR).