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ON EXACTLY COVERING SYSTEMS OF CONGRUENCES HAVING MODULI OCCURRING AT MOST TWICE

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1. The concept of *Covering Systems of Congruences* (in short Covering Systems) was introduced by P. ERDÖS [2].

A *Covering System* (abbreviated CS) is a set of ordered pairs of integers (a_i, m_i), $i = 1,2, \ldots, k$, $a_i \geqq 0$, $m_i > 1$ and

$$m_i \neq m_j \quad \text{for} \quad i \neq j\,, \tag{1}$$

such that every integer satisfies *at least* one of the congruences $x \equiv a_i (\bmod m_i)$.

An *Exactly Covering System* (abbreviated ECS) is defined [2] similarly, but omitting condition (1) and requiring that every integer satisfies *exactly* one of the congruences $x \equiv a_i (\bmod m_i)$.

Denote by ECS(M) an ECS each modulus of which occurs at most M times.

2. Erdös [2] posed the following question:

Question 1. *Does there exist for each natural number n a CS whose* $\min (m_i) \geqq n$?

The problem is rather difficult and as known, Erdös [3] has offered a \$50 reward for a proof. As Erdös notified the authors, the largest $\min (m_i)$, namely 20, has been obtained by CHOI. The former record 18 is due to SELFRIDGE, and KRUKENBERG gave examples for $\min (m_i) = 2,3, \ldots, 18$ in his thesis [4].

The analogous question for ECS's in general has little meaning, since the system (j, b_j) $j = 1,2, \ldots, m$, $b_1 = b_2 = \ldots = b_m = m$ is exactly covering and m may be as large as we wish. Hence we have to restrict somehow the question; namely we will consider it for ECS's(2). This is the strongest possible restriction on the number of occurrences of a modulus, since a well known theorem of Davenport, Mirsky, Newman and Radó [2], states that in every ECS the largest modulus occurs at least twice.

Thus the following question arises:

Question 2. *Does there exist for each natural number n an* ECS(2) *whose* $\min(m_i) \geqq n$?

Unfortunately, this paper does not provide an answer to the question. The result obtained is rather similar to Choi's, Selfridge's and Krukenberg's result in the CS case, namely an example (ex. 1 sec. 4) of an ECS(2) with $\min(m_i) = 21$ is presented. The authors also possess examples with $\min(m_i) = n$ for every n, $2 \leqq n \leqq 21$, except for $n = 11, 13, 17, 19$, and for these values it can be shown using the method of [1], that there do not exist ECS's (2).

3. Znám [9] gave the following definition:

Definition 1. The only simple 1-tuple is the number 1. Let all simple h-tuples be defined for $1 \leqq h < t$, then the t-tuple

$$\{n_1, n_2, \ldots, n_t\} \tag{2}$$

of naturals is said to be simple if it contains such numbers

$$n_{i_1} = n_{i_2} = \ldots = n_{i_r} \quad (r \geqq 2), \quad 1 \leqq i_1 < i_2 < \ldots < i_r \leqq t$$

that n_{i_1}/r is an integer and substituting $\{n_{i_1}, n_{i_2}, \ldots, n_{i_r}\}$ in the t-tuple (2) by the single number n_{i_1}/r we get a simple $(t - r + 1)$-tuple.

The following lemma points out a property of simple k-tuples which will be used later.

Lemma 1. *If the numbers* $\{n_1, n_2, \ldots, n_t\}$ $t > 1$, *form a simple tuple, then their greatest common divisor is greater than 1.*

Proof. Suppose the contrary, then there exists a minimal simple s-tuple $\{m_1, m_2, \ldots \ldots, m_s\}$ such that

$$(m_1, m_2, \ldots, m_s) = 1\,, \quad s > 1\,. \tag{3}$$

The algorithm of definition 1 would lead to a simple $(s - r + 1)$-tuple with $s - r + 1 > 1$, such that, by the minimality of s, the greatest common divisor of the numbers occurring in it, is greater than 1. But this would imply a fortiori $(m_1, m_2, \ldots, m_s) > 1$ contradicting (3).

Znám [9] also proved the following theorem:

Theorem 1. *If* $\{n_1, n_2, \ldots, n_t\}$ *is a simple t-tuple, then* $n_1, n_2, \ldots$, *nt are the moduli of an ECS.*

The converse of theorem 1 is false as shown by the following example which is due to Š. Porubský [6]:

$$\{6, 10, 15 \text{ and twenty times } 30\}\,. \tag{4}$$

The numbers in (4) are the moduli of an ECS, but the corresponding numbers do not form a simple tuple. This follows by lemma 1, since $(6, 10, 15, 30) = 1$.

In (4) the largest modulus occurs twenty times. The natural question is then raised:

Question 3. *Does there exist an ECS(2) whose moduli do not form a simple tuple?*

The answer is affirmative and we give an example in section 4 (ex. 2).

4. Example 1. An ECS(2) with $\min(m_i) = 21$. The pairs

(16, 21)	(38, 108)	(5, 288)	(238, 672)	(474, 1440)	(1135, 3780)
(19, 21)	(74, 108)	(149, 288)	(574, 672)	(1434, 1440)	(3025, 3780)
(11, 24)	(24, 120)	(72, 300)	(207, 675)	(463, 1512)	(1330, 4032)
(23, 24)	(84, 120)	(222, 300)	(432, 675)	(1471, 1512)	(3346, 4032)
(3, 30)	(22, 126)	(67, 315)	(240, 720)	(403, 1680)	(2007, 4050)
(18, 30)	(64, 126)	(172, 315)	(600, 720)	(823, 1680)	(4032, 4050)
(14, 36)	(36, 135)	(8, 324)	(295, 756)	(0, 1800)	(1197, 4800)
(32, 36)	(81, 135)	(170, 324)	(673, 756)	(450, 1800)	(3597, 4800)
(13, 42)	(53, 144)	(70, 336)	(396, 810)	(505, 1890)	(1261, 5040)
(34, 42)	(101, 144)	(154, 336)	(801, 810)	(1765, 1890)	(3781, 5040)
(6, 45)	(42, 150)	(60, 360)	(193, 840)	(658, 2016)	(1350, 5400)
(21, 45)	(117, 150)	(150, 360)	(613, 840)	(2002, 2016)	(3150, 5400)
(17, 48)	(62, 162)	(43, 378)	(270, 900)	(657, 2025)	(2797, 5670)
(41, 48)	(116, 162)	(169, 378)	(720, 900)	(1332, 2025)	(5632, 5670)
(26, 54)	(28, 168)	(126, 405)	(277, 945)	(690, 2160)	(1663, 6720)
(44, 54)	(112, 168)	(261, 405)	(592, 945)	(1770, 2160)	(5023, 6720)
(9, 60)	(75, 180)	(88, 420)	(234, 960)	(897, 2400)	(2397, 7200)
(39, 60)	(165, 180)	(298, 420)	(714, 960)	(2097, 2400)	(7197, 7200)
(10, 63)	(52, 189)	(90, 450)	(337, 1008)	(1, 2520)	(1891, 7560)
(31, 63)	(115, 189)	(180, 450)	(841, 1008)	(631, 2520)	(4411, 7560)
(20, 72)	(46, 210)	(174, 480)	(330, 1080)	(810, 2700)	(3343, 10080)
(56, 72)	(151, 210)	(414, 480)	(1050, 1080)	(2160, 2700)	(10063, 10080)
(12, 75)	(2, 216)	(85, 504)	(556, 1134)	(907, 2835)	(4950, 10800)
(27, 75)	(110, 216)	(211, 504)	(1123, 1134)	(1852, 2835)	(10350, 10800)
(7, 84)	(57, 225)	(210, 540)	(297, 1200)	(954, 2880)	(4797, 14400)
(49, 84)	(132, 225)	(480, 540)	(597, 1200)	(2394, 2880)	(11997, 14400)
(15, 90)	(54, 240)	(178, 567)	(379, 1260)	(967, 3024)	(6931, 15120)
(45, 90)	(114, 240)	(367, 567)	(1009, 1260)	(2479, 3024)	(14491, 15120)
(29, 96)	(106, 252)	(147, 600)	(322, 1344)	(1243, 3360)	(6703, 20160)
(77, 96)	(232, 252)	(447, 600)	(994, 1344)	(2923, 3360)	(16783, 20160)
(4, 105)	(30, 270)	(127, 630)	(360, 1350)	(900, 3600)	
(25, 105)	(120, 270)	(253, 630)	(1260, 1350)	(2700, 3600)	

form an ECS(2) and the moduli a simple tuple. The reader could easily verify both assertions.

Example 2. An ECS(2) with moduli not forming a simple tuple. The pairs

(1, 6)	(10, 150)	(60, 540)	(400, 1350)	(947, 2880)	(4000, 8100)
(3, 6)	(40, 150)	(240, 540)	(850, 1350)	(2867, 2880)	(8050, 8100)
(2, 10)	(30, 180)	(130, 600)	(480, 1440)	(0, 3600)	(2380, 9600)
(4, 10)	(90, 180)	(430, 600)	(1200, 1440)	(900, 3600)	(7180, 9600)
(5, 15)	(47, 240)	(120, 720)	(796, 1620)	(1300, 4050)	(2700, 10800)
(11, 15)	(167, 240)	(300, 720)	(1606, 1620)	(2650, 4050)	(6300, 10800)
(8, 20)	(76, 270)	(256, 810)	(540, 1800)	(1380, 4320)	(4780, 14400)
(18, 20)	(166, 270)	(526, 810)	(1440, 1800)	(3540, 4320)	(14380, 14400)
(23, 30)	(70, 300)	(180, 900)	(467, 1920)	(1780, 4800)	(9900, 21600)
(29, 30)	(220, 300)	(360, 900)	(1427, 1920)	(4180, 4800)	(20700, 21600)
(6, 60)	(150, 360)	(347, 960)	(660, 2160)	(1620, 5400)	(9580, 28800)
(36, 60)	(330, 360)	(827, 960)	(2100, 2160)	(4320, 5400)	(23980, 28800)
(16, 90)	(100, 450)	(420, 1080)	(580, 2400)	(1907, 5760)	
(46, 90)	(250, 450)	(960, 1080)	(1180, 2400)	(4787, 5760)	
(17, 120)	(107, 480)	(280, 1200)	(720, 2700)	(1800, 7200)	
(77, 120)	(227, 480)	(880, 1200)	(2520, 2700)	(5400, 7200)	

form an ECS(2), but the moduli do not form a simple tuple. The first part of the assertion is easy to check, for the second part, observe that the greatest common divisor of the moduli is 1 and therefore the second part is a consequence of Lemma 1.

Remark. Znám [8], conjectured that in an ECS(2) all moduli are of the form $m_i = 2^{\alpha_i}3^{\beta_i}$, where α_i and β_i are non-negative integers. The authors [7] disproved this conjecture by means of a counter example. Examples 1 and 2 constitute further counter examples.

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