Fritz Saaby Pedersen

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SPITZ IN l-GROUPS

F. Pedersen, Carbondale

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For the purpose of this paper, $(G, +)$ will denote an abelian lattice ordered group (l-group). The class of l-groups under consideration is all l-groups which can be represented by an l-group of real valued functions on a set X. For such a representation the set X may be chosen to be any subset of the maximal l-ideals of G with the property that $\bigcap X = \{0\}$. This study is motivated by the question, how much connection is there between the topologies on X induced by G and the structure of G? Thus, assume G has a representation as an l-group of real valued functions on a set X, where X is a set of maximal l-ideals of G.

Definition. (Stone Topology) For $A \subseteq G$, let $\Delta(A) = \{M \in X : A \not\subseteq M\}$. The topology Δ for X is given by $\{\Delta(A) : A \subseteq G\}$.

The only difficulty in verifying that Δ is a topology is showing that $\Delta(A) \cap \Delta(B) = \Delta(A \cap B)$. This follows from:

i) $\Delta(A) = \Delta(c(A))$ where $c(A)$ is the l-ideal generated by A.

ii) Maximal l-ideals are prime l-ideals [see 1, Theorem 3.2].

The following are some additional facts about Δ.

a) *A base for Δ is given by $\{\Delta(g) : g \in G^+\}$ and each $\Delta(g)$ is just the co-zero set of the function g.*

b) *If $g, h \in G^+$, then $\Delta(g) \cap \Delta(h) = \Delta(g \wedge h)$.*

c) *If $x, y \in X$, then there exists $g, h \in G^+$ such that $x \in \Delta(g)$ $y \in \Delta(h)$ and $g \wedge h = 0$. Thus, the topology Δ is Hausdorff.*

d) *If X is the set of all maximal l-ideals of G, it does not follow that X is compact in the Δ topology. Just consider the l-group of all real valued functions on N (the natural numbers).*

e) *If G contains the constant functions on X, then all $f \in G$ are continuous in the Δ topology. Moreover, if $g \in G^+$, then $g^{-1}(a, b)$ is the co-zero set of some bounded function in G^+.*

Outline of a proof. Let $0 \leqq a < b$ and denote constant functions by constants. $\Delta((g - a) \vee 0) = \{M \in X : g(M) > a\}$. $\Delta((g - b) \wedge 0) = \{M \in X : g(M) < b\}$. Let $h_1 = (g - a) \vee 0$ and $h_2 = |(g - b) \wedge 0|$. Since $h_1, h_2 \geqq 0$, $\Delta(h_1) \cap \Delta(h_2) = \Delta(h_1 \wedge h_2)$. Moreover, $\Delta(h_1 \wedge h_2) = g^{-1}(a, b)$.

Note. This result gives a condition for Δ to be the same topology as the weak topology.

f) *A compactification of Δ can be obtained by using R the ring generated by G and considering the maximal ideal space of R.*

g) *If the space of all maximal l-ideals is connected in the Δ topology, then G is cardinally indecomposable as an l-group.*

Definition. G has a *basis* if for each $0 < g \in G$, there exists $0 < b \in G$ such that $c(b)$ is totally ordered and $b \leqq g$. If $c(b)$ is totally ordered and $0 < b$, then b is called a *basis element.*

h) *If G has a basis, then there exists a space X of maximal l-ideals where the Δ topology is discrete.*

i) *If G can be represented on a space X of maximal l-ideals where the Δ topology is discrete and G contains a non-zero constant function over X, then G has a basis.*

Outline of a proof. One may assume that G is divisible and G contains the rational constants. The weak topology is stronger than Δ, thus the weak topology is discrete.

$$\{x_0\} = \{x \in X : |f_i(x) - f_i(x_0)| < 1/n \text{ for } i = 1, \ldots, k\} .$$

Let r_i be a rational constant such that $|f_i(x_0) - r_i| < 1/2nk$. Define $g_i = f_i - r_i$, then $|g_i(x_0)| < 1/2nk$ for $i = 1, \ldots, k$. Moreover $|g_i(x) - g_i(x_0)| = |f_i(x) - f_i(x_0)|$. Let $h(x) = |g_1(x)| + \ldots + |g_k(x)|$. $h(x_0) < 1/2n$. For $x \neq x_0$, there exists i such that $|g_i(x)| \geqq 1/2n$. Thus, it follows that $h(x) \geqq 1/2n$ for $x \neq x_0$. $B = 1/2n - [1/2n \wedge h]$ will have the property that $B(x_0)$ is greater than zero and $B(x)$ equals zero for all $x \neq x_0$.

SPITZ AND LOCAL CONNECTEDNESS

Definition. An element $0 \neq g \in G^+$ is a *spitz* if g cannot be written as the join of two positive disjoint elements of G [see 2, section 1].

Lemma 1. *If $0 < g$ and $\Delta(g)$ is connected, then g is a spitz.*

Proof. $g = g_1 \vee g_2$, $g_1 \wedge g_2 = 0$. $\Delta(g_1) \cup \Delta(g_2) = \Delta(g_1 \vee g_2) = \Delta(g)$, and $\Delta(g_1) \cap \Delta(g_2) = \emptyset$. Therefore either g_1 or g_2 is zero.

Lemma 2. *If G is a complete l-group, then $\Delta(g)$ is connected for each spitz $g \in G$.*

Proof. $\Delta(g) = U \cup V$, $U \cap V = \emptyset$. For each $x \in U$, let $g_x > 0$, $\Delta(g_x) \subseteq U$, and $g_x(x) > 0$. Since $ng_x(x) \geqq g(x)$ for some $n \in N$, we may assume that $g_x(x) \geqq g(x)$. Since $(g_x \wedge g)$ meets the same requirements, we may assume that $g_x \leqq g$. Let $\bigvee_{x \in U} g_x = h_1$ and $\bigvee_{x \in V} g_x = h_2$. Then $h_1 \vee h_2 = g$ and $h_1 \wedge h_2 = 0$, which implies that g is not a spitz, unless U or V is empty.

Theorem. *If G is a complete l-group, then the following are equivalent:*

(a) *The Δ topology is locally connected.*

(b) *Every $0 \neq g \in G^+$ is the join of spitz.*

Proof. Suppose the Δ topology is locally connected. Let $\Delta(g) = U$. For every $y \in U$ let V_y be chosen so that $y \in V_y$, V_y is connected, and $V_y \subseteq U$. Consider a fixed y. For each $x \in V_y$ let $g_x > 0$, $\Delta(g_x) \subseteq V_y$ and $g_x(x) > 0$. Let $g_x \wedge g = h_x$. $\bigvee_{x \in V} h_x = h$ exists and $\Delta(h) = V_y$. h is a spitz by Lemma 1. Now for each $y \in U$ we choose h_y such that h_y is a spitz $h_y(y) = g(y)$ and $h_y \leqq g$. Then $\bigvee_{y \in U} h_y = g$.

Suppose each $g \in G^+$ is the join of spitz. Each $\Delta(h)$ for h a spitz is connected by Lemma 2. Therefore $\Delta(g)$ is the union of connected open sets, for each $0 < g \in G$. Since the $\Delta(g)$ form a base for the topology Δ, Δ is locally connected.

References

[1] *P. Conrad,* The lattice of all convex *l*-subgroups of a lattice ordered group, Czech. Math. J. *15 (90)*, 1965, pp. 101–123.

[2] *F. Šik,* Über die Beziehungen zwischen eigenen Spitzen und minimalen Kompomenten einer *l*-Gruppe, Acta. Math. Acad. Sci. Hungaricae *13* (1962) 171–178. [MR 26, 242].

Author's address: Southern Illinois University, Department of mathematics, Carbondale, Illinois 62901, U.S.A.