Ivan Netuka

The third boundary value problem in potential theory

*Czechoslovak Mathematical Journal*, Vol. 22 (1972), No. 4, 554–580

Persistent URL: http://dml.cz/dmlcz/101126

Czechoslovak Mathematical Journal, 22 (1972), Praha

# THE THIRD BOUNDARY VALUE PROBLEM IN POTENTIAL THEORY

IVAN NETUKA, Praha

(Received July 24, 1971)

**Introduction.** This paper deals with further properties of the operator $\mathcal{T}$ introduced in [7] and studied in [7] and [8]. Let $G$ be an open set in the Euclidean $m$-space $R^m$, $m > 2$, and suppose that the boundary $B$ of $G$ is compact and $B \neq \emptyset$. For every $\mu \in \mathfrak{B}$ (= the Banach space of all finite signed Borel measures with support in $B$), the corresponding Newtonian potential $U\mu$ is defined by

$$U\mu(x) = \int_B p(x - y)\, \mathrm{d}\mu(y)\,, \quad x \in R^m\,,$$

where $p(z) = |z|^{2-m}/(m-2)$. In what follows, $\lambda$ will be a fixed non-negative element of $\mathfrak{B}$ and we shall assume that

$$\sup_{y \in B} [v_\infty(y) + U\lambda(y)] < \infty \tag{1}$$

where the quantity $v_\infty(y)$ which is closely connected with the geometrical shape of $G$ was introduced by J. KRÁL in [4] (for the definition see also [7] or [8]).

Under the condition (1), for each $\mu \in \mathfrak{B}$, the distribution $\mathcal{T}\mu$ defined in [7] by

$$\mathcal{T}\mu(\varphi) = \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\mu(x)\, \mathrm{d}x + \int_B \varphi(x)\, U\mu(x)\, \mathrm{d}\lambda(x) \tag{2}$$

over the class $\mathcal{D}$ of all infinitely differentiable functions with compact support in $R^m$ can be identified with a uniquely determined element $\mathcal{T}\mu$ of $\mathfrak{B}$ and the operator $\mathcal{T} : \mu \mapsto \mathcal{T}\mu$ acting on $\mathfrak{B}$ is a bounded linear operator (see [7], theorem 5).

In this paper we are going to apply the Riesz-Schauder theory to the third boundary value problem in the following formulation: Given $\nu \in \mathfrak{B}$, find $\mu \in \mathfrak{B}$ with $\mathcal{T}\mu = \nu$. In connection with the applicability of the mentioned theory it is useful to consider the decomposition

$$\mathcal{T} = \alpha A \mathcal{I} + \mathcal{T}_\alpha$$

(where $\alpha$ is a real number, $A$ is the area of the unit $m$-sphere and $\mathscr{I}$ stands for the identity operator on $\mathfrak{B}$) and to investigate the quantity

$$\omega' \mathscr{T}_\alpha = \inf_Q \|\mathscr{T}_\alpha - Q\| ,$$

$Q$ ranging over the class of all operators acting on $\mathfrak{B}$ of the form

$$Q \ldots = \sum_{j=1}^{n} \langle f_j, \ldots \rangle \, m_j$$

where $n$ is a positive integer, $m_j \in \mathfrak{B}$ and $f_j$'s are bounded Baire functions on $B$.

Indeed, the condition

$$a' = \inf_{\alpha \neq 0} \frac{\omega' \mathscr{T}_\alpha}{A|\alpha|} < 1 \tag{3}$$

guarantees the applicability of the Fredholm theorem to the operator equation

$$\mathscr{T} \mu = \nu \quad \text{over} \quad \mathfrak{B} . \tag{4}$$

It should be noted here that general conditions securing the validity of (3) have been given in [8] in terms of quantities connected with the shape of $G$ and the distribution $\lambda$ over $B$. In [8] a detailed discussion of questions related to the quantities $a'$ and $\omega' \mathscr{T}_\alpha$ may be found.

Using some ideas of J. Radon [10] we are able to give a proof of the following theorem which is a basic tool for investigations of the null-space of the operator $\mathscr{T}$

**Theorem I.** *Let $\alpha$, $\beta$ be real numbers, $A|\beta| > \omega' \mathscr{T}_\alpha$, and denote by $d(y)$ the $m$-density of $G$ at $y$. Suppose that*

$$d(y) \neq \alpha - \beta$$

*for every $y \in B$. If $\mu \in \mathfrak{B}$ satisfies*

$$[A\beta \mathscr{I} + \mathscr{T}_\alpha] \, \mu = 0 ,$$

*then the corresponding potential $U\mu$ is quasi-everywhere bounded.*

This proposition enables us to prove the following

**Theorem II.** *Assume $G$ to be a domain ($=$ connected and open set) with $d(y) \neq 0$ for every $y \in B$ and suppose that (3) holds good. Then*

$$\mathscr{T}(\mathfrak{B}) = \mathfrak{B}$$

*with the only exception which occurs if $G$ is bounded and $\lambda = 0$. In this case the range of $\mathscr{T}$ consists precisely of those $\nu \in \mathfrak{B}$ with $\nu(B) = 0$.*

The theorems stated above were announced without proofs in [6].

**1. Preliminaries.** The purpose of this section is to recall the basic notation adopted in [7] and [8]. Throughout this paper we keep the notation from the introduction. The set $B$ will be supposed to be infinite, because the case of finite $B$ is included in the investigations of [4] (see section 1 of [8]).

For $M \subset R^m$ we shall denote by cl $M$ and fr $M$ the closure and the boundary of $M$, respectively; dist $(z, M)$ will denote the distance of $\{z\}$ and $M$. $H_k$ will stand for the $k$-dimensional Hausdorff measure in $R^m$ (for definition see [7]) and $\Omega_r(x)$ will denote the open ball centered at $x \in R^m$ with radius $r > 0$.

Recall that results of [4] imply, for each $y \in R^m$, the existence of a uniquely determined $v_y \in \mathfrak{B}$ such that

$$(5) \qquad Ad(y)\,\varphi(y) + \langle \varphi, v_y \rangle = \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\delta_y(x)\, \mathrm{d}x\,,$$

provided $\varphi \in \mathscr{D}$ where $\delta_y$ denotes the Dirac measure concentrated at $y$ (compare [7], section 2).

Let $\mathscr{B}$ denote the Banach space of all bounded Baire functions defined on $B$ with the usual supremum norm and $\mathscr{C}$ will be the subspace of all continuous functions in $\mathscr{B}$. The symbol $\mathscr{B}^*$ stands for the dual space of $\mathscr{B}$ and for $\mu \in \mathfrak{B}$ we shall denote by $|\mu|$ the indefinite variation of $\mu$; of course, $\|\mu\| = |\mu|\,(B)$ is the norm of a $\mu$ in $\mathfrak{B}$.

Let us also recall the definitions of the operators $\tilde{W}, V$ acting on $\mathscr{B}$ defined as follows:

$$Vf(y) = Uf\lambda(y) \left[ = \int_B f(x)\, p(x - y)\, \mathrm{d}\lambda(x) \right],$$

$$\tilde{W}f(y) = Ad(y) f(y) + \langle f, v_y \rangle\,, \quad y \in B\,, \quad f \in \mathscr{B}\,.$$

There is a close connection between the operator $T = V + \tilde{W}$ and the operator $\mathscr{T}$, namely, the restriction to $\mathfrak{B}$ of the dual operator $T^*$ of $T$ coincides with the operator $\mathscr{T}$ (see [7], proposition 8).

Denoting by $\tilde{W}^*, V^*$ the dual operator of $\tilde{W}, V$, respectively, we observe that

$$\tilde{W}^*\mathfrak{B} \subset \mathfrak{B}\,, \quad V^*\mathfrak{B} \subset \mathfrak{B}\,.$$

Indeed, as mentioned above, $T^*\mathfrak{B} = \mathscr{T}\mathfrak{B} \subset \mathfrak{B}$. Observing that $T = \tilde{W}$ for $\lambda = 0$ we conclude that $\tilde{W}^*\mathfrak{B} \subset \mathfrak{B}$ and the inclusion $V^*\mathfrak{B} \subset \mathfrak{B}$ follows immediately from the relation $V^* = T^* - \tilde{W}^*$. In particular, given $\mu \in \mathfrak{B}$, it has a good sense to speak of the potential $U\tilde{W}^*\mu$, $U|\tilde{W}^*\mu|$ and, similarly, $UV^*\mu$, $U|V^*\mu|$.

We shall start with the following lemma.

**2. Lemma.** *There are numbers $c_1, c_2 \in R^1$ such that the inequalities*

$$(6) \qquad U|V^*\mu| \leqq c_1 U|\mu|\,,$$

$$(7) \qquad U|\tilde{W}^*\mu| \leqq c_2 U|\mu|$$

*hold for any $\mu \in \mathfrak{B}$.*

Proof. We first show (6). By the definition of the operator $V$ we have

$$\langle f, V^*\mu\rangle = \langle Uf\lambda, \mu\rangle = \int_B \left(\int_B p(z-y) f(z)\,\mathrm{d}\lambda(z)\right) \mathrm{d}\mu(y)$$

for any $f \in \mathscr{B}$, $\mu \in \mathfrak{B}$.

Fix an $x \in R^m$ with $U|\mu|(x) < \infty$ and put

$$\mathscr{I} = \int_{B\times B} p(z-y)\, p(z-x)\,\mathrm{d}\lambda(z)\,\mathrm{d}|\mu|(y)\,. \tag{8}$$

One easily verifies that

$$U|V^*\mu|(x) \leqq \mathscr{I}\,. \tag{9}$$

Fix a $y \neq x$ and denote

$$Z_1 = \{z; |z-y| \geqq \tfrac{1}{2}|x-y|\}, \quad Z_2 = \{z; |z-y| < \tfrac{1}{2}|x-y|\},$$
$$c_1 = 2^{m-1} \sup_{x\in R^m} U\lambda(x)\,.$$

Since $\sup\limits_{x\in B} U\lambda(x) < \infty$ we conclude by the maximum principle for potentials that $c_1$ s finite. If $z \in Z_1$, then

$$p(z-y) \leqq 2^{m-2} p(x-y)\,,$$

which yields

$$\int_{B\cap Z_1} p(z-y)\, p(z-x)\,\mathrm{d}\lambda(z) \leqq 2^{m-2} p(x-y)\, U\lambda(x) \leqq \tfrac{1}{2} c_1 p(x-y)\,, \tag{10}$$

while for $z \in Z_2$

$$|z-y| < \tfrac{1}{2}|x-y|\,, \quad |z-x| \geqq |x-y| - |y-z| > \tfrac{1}{2}|x-y|\,,$$
$$p(z-x) \leqq 2^{m-2} p(x-y)\,,$$

so that

$$\int_{B\cap Z_2} p(z-y)\, p(z-x)\,\mathrm{d}\lambda(z) \leqq 2^{m-2} p(x-y)\, U\lambda(y) \leqq \tfrac{1}{2} c_1 p(x-y)\,. \tag{11}$$

Making the sum of (10) and (11) we get

$$\int_B p(z-y)\, p(z-x)\,\mathrm{d}\lambda(z) \leqq c_1 p(x-y)\,.$$

Consequently,

$$\mathcal{J} \leqq c_1 \, U|\mu| \, (x) \, . \tag{12}$$

The inequality in (6) follows now by (12) and (9).

We are going to prove (7). By the definition of $\tilde{W}$,

$$\langle f, \tilde{W}^*\mu \rangle = \langle \tilde{W}f, \mu \rangle = \int_B \left[ Ad(x) f(x) + \int_B f(z) \, \mathrm{d}v_x(z) \right] \mathrm{d}\mu(x) \, ,$$

provided $f \in \mathcal{B}$ and $\mu \in \mathfrak{B}$. If, moreover, $f \geqq 0$, then

$$\langle f, |\tilde{W}^*\mu| \rangle \leqq A\langle f, |\mu| \rangle + \int_{B \times B} f(z) \, \mathrm{d}|v_x| \, (z) \, \mathrm{d}|\mu| \, (x) \, .$$

Referring to the formula (5) in [8] we may write for $y \in R^m$

$$U|\tilde{W}^*\mu| \, (y) \leqq AU|\mu| \, (y) + \int_{B \times B} p(y - z) \frac{|n(z) \, . \, (z - x)|}{|z - x|^m} \, \mathrm{d}H_{m-1}(z) \, \mathrm{d}|\mu| \, (x) \tag{13}$$

where $n(z)$ stands for the exterior normal of $G$ at $z$ in the sense of Federer (for definition see [7]). Fix an $x \neq y$ and put

$$K = \int_B p(y - z) \frac{|n(z) \, . \, (z - x)|}{|z - x|^m} \, \mathrm{d}H_{m-1}(z) \, . \tag{14}$$

Then, with the same notation as above,

$$K_1 = \int_{B \cap Z_1} p(y - z) \frac{|n(z) \, . \, (z - x)|}{|z - x|^m} \, \mathrm{d}H_{m-1}(z) \leqq$$
$$\leqq 2^{m-2} p(x - y) \, . \int_B \frac{|n(z) \, . \, (z - x)|}{|z - x|^m} \, \mathrm{d}H_{m-1}(z) =$$
$$= 2^{m-2} p(x - y) \, v_\infty(x) \leqq 2^{m-2} p(x - y) \sup_{z \in R^m} v_\infty(z)$$

(in the last equatity we have used the expression for $v_\infty(x)$ established in [4], lemma 2.12). Recalling that $n(z) = 0$ outside of the reduced boundary $\hat{B}$ we have

$$K_2 = \int_{B \cap Z_2} p(y - z) \, . \, \frac{|n(z) \, . \, (z - x)|}{|z - x|^m} \, \mathrm{d}H_{m-1}(z) \leqq$$
$$\leqq 2^{m-1} |x - y|^{1-m} \int_{B \cap Z_2} p(y - z) \, \mathrm{d}H(z)$$

where $H$ denotes the restriction of $H_{m-1}$ to $\hat{B}$. Letting in lemma 21 in [8] $l_1 = 1$ on $B$, $\beta = 1$, $r = \frac{1}{2}|x - y|$, $y_0 = y$, we have $Z_2 = \Omega_r(y_0)$ and by the formula (58) in [8] we arrive at

$$\int_{B \cap Z_2} p(y - z) \, \mathrm{d}H(z) \leqq 2\gamma \cdot \tfrac{1}{2}|x - y| ,$$

so that

$$K_2 \leqq 2^{m-1}\gamma(m - 2) \, p(x - y)$$

where the constant $\gamma$ was defined in the above mentioned lemma. Since $\sup_{z \cdot B} v_\infty(z) < < \infty$, it is $\sup_{z \in R^m} v_\infty(z) < \infty$ by theorem 2.13 in [4].

Putting

$$c_2' = 2^{m-2}( \sup_{z \in R^m} v_\infty(z) + 2\gamma(m - 2))$$

and observing that $K = K_1 + K_2$ we get

$$K \leqq c_2' p(x - y) \tag{15}$$

and, by (14) and (13),

$$U|W^*\mu| \, (y) \leqq (A + c_2') \, U|\mu| \, (y) \, .$$

Thus (7) is established.

**3. Notation.** Let $C_0$ stand for the class of all Borel subsets of $R^m$ having the Newtonian capacity zero. It should be noted here that $H_{m-1}(M) = 0$ for any $M \in C_0$ ([5], theorem 3.13) and $\lambda(M) = 0$ as well, because $\lambda$ has a bounded potential ([5], theorem 2.1). We shall say that a property holds quasi-everywhere in $Q \subset R^m$ if it holds for all points in $Q$ except possibly those in a set $M \in C_0$.

Let us denote by $\mathfrak{B}_*$ the set of all $\mu \in \mathfrak{B}$ with the following property: There are $M \in C_0$ and $c \in R_1$ such that the difference $U\mu(x) = U\mu^+(x) - U\mu^-(x)$ is meaningful for each $x \in R^m - M$ and $|U\mu(x)| \leqq c$ holds provided $x \in R^m - M$ (as usual, $\mu = = \mu^+ - \mu^-$ is the Jordan decomposition of $\mu$). Clearly, $\mathfrak{B}_*$ is a linear subspace of $\mathfrak{B}$.

The function $g$ is said to belong to the class $\tilde{\mathscr{B}}_0$, if it is defined quasi-everywhere in $B$ and there is a function $\tilde{g} \in \mathscr{B}$ such that $g = \tilde{g}$ quasi-everywhere in $B$. For $g \in \tilde{\mathscr{B}}_0$ denote by $\mathbf{g}$ the class of all $h \in \tilde{\mathscr{B}}_0$ that coincide with $g$ quasi-everywhere in $B$. Let us denote by $\mathscr{B}_0$ the Banach space of such classes $\mathbf{g}$ with the norm defined by

$$\|\mathbf{g}\|_0 = \operatorname*{quasisup}_B |g| , \quad g \in \mathbf{g} ,$$

where $\operatorname*{quasisup}_B |g|$ equals the infimum of all $c$'s for which

$$\{x \in B; \ |g(x)| > c\} \in C_0$$

provided $B \notin C_0$; in the case that $B \in C_0$ we set $\operatorname*{quasisup}_B |g| = 0$.

An operator $P$ acting on $\mathscr{B}$ is said to operate in $\mathscr{B}_0$ if $Pf = 0$ quasi-everywhere whenever $f = 0$ quasi-everywhere. Such an operator defines in an obvious manner an operator acting on $\mathscr{B}_0$ which will be denoted by $\boldsymbol{P}$.

Let $L$ be a linear space over the field of real numbers. We shall denote by $^\wedge L$ the set of all elements of the form $x + iy$ where $x, y \in L$. If the sum of two elements of $^\wedge L$ and the multiplication of an element of $^\wedge L$ by a complex number are defined in an obvious way, then $^\wedge L$ becomes a linear space over the field of complex numbers. Let $Q$ be a linear operator acting on $L$. The same symbol will denote the extension of $Q$ to $^\wedge L$ defined by

$$Q(x + iy) = Q(x) + iQ(y).$$

If an operator $Q$ on $L$ possesses an inverse operator $Q^{-1}$, then the extension of $Q^{-1}$ to $^\wedge L$ is an inverse operator (on $^\wedge L$) of the extension of $Q$ to $^\wedge L$. If, moreover, $^\wedge L$ is a normed linear space with the norm $\|\ldots\|'$ and $Q$ is a bounded linear operator on $^\wedge L$, then $\|Q\|'$ denotes its norm. Similarly, $\|l\|'$ denotes the norm of a linear functional $l$ on $^\wedge L$. We shall write $^\wedge L^*$ in place of $(^\wedge L)^*$ (the dual space of $^\wedge L$).

For $f \in {}^\wedge\mathscr{B}$, $\mathbf{g} \in {}^\wedge\mathscr{B}_0$ put

$$\|f\|' = \sup_{x \in B} |f(x)|,$$

$$\|\mathbf{g}\|'_0 = \operatorname*{quasisup}_B |g|, \quad g \in \mathbf{g}.$$

Note that $^\wedge\mathscr{B}$, $^\wedge\mathscr{B}_0$ with the above defined norms are Banach spaces and for any $\mu \in {}^\wedge\mathfrak{B}$

$$\|\mu\|' = \sup \left| \int_B f \, d\mu \right|$$

where the supremum is taken over all $f \in {}^\wedge\mathscr{B}$ with $\|f\|' \leqq 1$. If $\mu \in {}^\wedge\mathfrak{B}$, $\mu = \mu^1 + i\mu^2$, then

$$\max(\|\mu_1\|, \|\mu_2\|) \leqq \|\mu\|'. \tag{16}$$

Similarly as above, an operator $Q$ acting on $^\wedge\mathscr{B}$ is said to operate in $^\wedge\mathscr{B}_0$, if $Qf = 0$ quasi-everywhere whenever $f = 0$ quasi-everywhere. Such an operator defines an operator on $^\wedge\mathscr{B}_0$ that will be denoted by $\mathbf{Q}$. The inequality $\|\mathbf{Q}\|'_0 \leqq \|Q\|'$ holds good. Note that if an operator $P$ on $\mathscr{B}$ operates in $\mathscr{B}_0$, then its extension to $^\wedge\mathscr{B}$ operates in $^\wedge\mathscr{B}_0$.

For any $\mu \in {}^\wedge\mathfrak{B}_*$, $\mu = \mu^1 + i\mu^2$, $U\mu^j$ determines the only element of $\mathscr{B}_0$ which will be denoted by $\boldsymbol{U}\mu^j$ $(j = 1, 2)$. Defining

$$\boldsymbol{U}\mu = \boldsymbol{U}\mu^1 + i\boldsymbol{U}\mu^2$$

we have $\boldsymbol{U}\mu \in {}^\wedge\mathscr{B}_0$ and the mapping

$$\boldsymbol{U} : \mu \mapsto \boldsymbol{U}\mu$$

is a linear mapping of ${}^\wedge\mathfrak{B}_*$ into ${}^\wedge\mathscr{B}_0$.

In what follows, fix a $\gamma \in R^1$ and put $T_\gamma = T - \gamma A I$ where $I$ stands for the identity operator on $\mathscr{B}$.

According to our definitions, $T$, $T_\gamma$ will also denote the above defined extension of $T$, $T_\gamma$ to ${}^\wedge\mathscr{B}$, respectively.

The following lemma is in fact a variant of Plemelj's "Symmetriegesetz" ([9], § 13; compare also [10], IV, section 4).

**4. Lemma.** *The operators $T$, $T_\gamma$ acting on ${}^\wedge\mathscr{B}$ operate in ${}^\wedge\mathscr{B}_0$, $T^* {}^\wedge\mathfrak{B}_* \subset {}^\wedge\mathfrak{B}_*$, $T_\gamma^* {}^\wedge\mathfrak{B}_* \subset {}^\wedge\mathfrak{B}_*$ and*

$$\boldsymbol{T}\boldsymbol{U}\mu = \boldsymbol{U}T^*\mu\,, \quad \boldsymbol{T}_\gamma\boldsymbol{U}\mu = \boldsymbol{U}T_\gamma^*\mu \tag{17}$$

*whenever $\mu \in {}^\wedge\mathfrak{B}_*$.*

Proof. It is easily seen that it suffices to verify the following assertion: The operators $V$, $\tilde{W}$ (on $\mathscr{B}$) operate in $\mathscr{B}_0$, $V^*\mathfrak{B}_* \subset \mathfrak{B}_*$, $\tilde{W}^*\mathfrak{B}_* \subset \mathfrak{B}_*$ and

$$\boldsymbol{U}V^*\mu = \boldsymbol{V}\boldsymbol{U}\mu\,, \tag{18}$$

$$\boldsymbol{U}\tilde{W}\mu = \tilde{\boldsymbol{W}}\boldsymbol{U}\mu \tag{19}$$

for any $\mu \in \mathfrak{B}_*$.

Let $h \in \mathscr{B}$ be a function vanishing quasi-everywhere on $B$. Consequently, $\int_B h \, d\lambda = 0$ and we see at once that $V : f \mapsto Uf\lambda$ operates in $\mathscr{B}_0$. Since $v_y$ is absolutely continuous with respect to $H_{m-1}$ (see the formula (5) in [8]) we get $\langle h, v_y \rangle = 0$ and

$$\tilde{W} h(y) = Ad(y)\, h(y)$$

for each $y \in B$, so that $\tilde{W}$ operates in $\mathscr{B}_0$ as well.

Suppose now that $\mu \in \mathfrak{B}_*$ and let $M \in C_0$ and $c \in R^1$ be chosen such that $U|\mu|(z) < \infty$ and $|U\mu(z)| \leqq c$ for any $z \in R^m - M$.

Fix an $x \in R^m - M$. Using (8), (9) and (12) we can assert that

$$U|V^*\mu|(x) \leqq \int_{B \times B} p(z - y)\, p(x - z)\, d\lambda(z)\, d|\mu|(y) < \infty$$

whence

$$UV^*\mu(x) = \int_{B \times B} p(z - y)\, p(x - z)\, d\lambda(z)\, d\mu(y) =$$

$$= \int_B \left( \int_B p(z - y)\, d\mu(y) \right) p(x - z)\, d\lambda(z) = Ug\lambda(x)$$

where $g = U\mu$ quasi-everywhere. Since the inequalities

$$|UV^*\mu(x)| \leqq c \,.\, U\lambda(x) \leqq c \,.\, \sup_{z \in R^m} U\lambda(z)$$

are true for any $x \in R^m - M$, we conclude that $V^*\mu \in \mathfrak{B}_*$ and (18) holds.

Going back to (13), (14) and (15) we have for each $y \in R^m - M$

$$U|\tilde{W}^*\mu|(y) \leqq AU|\mu|(y) + \int_{B \times B} p(y - z) \, \mathrm{d}|\nu_x|(z) \, \mathrm{d}|\mu|(x) < \infty$$

so that Fubini's theorem may be applied to assert

$$U\tilde{W}^*\mu(y) = A \int_B d(x) \, p(y - x) \, \mathrm{d}\mu(x) +$$

$$+ \int_{B \times B} p(y - z) \, \mathrm{d}\nu_x(z) \, \mathrm{d}\mu(x) = \int_B K(y, x) \, \mathrm{d}\mu(x)$$

where we have put

$$K(y, x) = Ad(x) \, p(y - x) + \int_B p(y - z) \, \mathrm{d}\nu_x(z) \,.$$

We are now going to prove the following implication

(20) $$(x, y \in R^m, \; x \neq y) \Rightarrow K(y, x) = K(x, y) \,.$$

Fix $x, y \in R^m$, $x \neq y$, and for every non-negative integer $n$ put

$$f_y^n(z) = \min\,(n, p(y - z)).$$

Since $f_y^n$ is Lipschitzian, it follows from (5)

$$Ad(x) f_y^n(x) + \int_B f_y^n(z) \, \mathrm{d}\nu_x(z) = \int_G \mathrm{grad}_z f_y^n(z) \,.\, \mathrm{grad}\, U\delta_x(z) \, \mathrm{d}z \,.$$

Since by (14) and (15)

$$\int_B p(z - y) \, d|\nu_x|(z) < \infty$$

we conclude that

$$\lim_{n \to \infty} \int_B f_y^n(z) \, \mathrm{d}\nu_x(z) = \int_B p(z - y) \, \mathrm{d}\nu_x(z) \,.$$

For $H_m$-almost all points $z \in R^m$ and for each $n$ we have

$$|\mathrm{grad}_z f_y^n(z) \,.\, \mathrm{grad}\, U\delta_x(z)| \leqq |\mathrm{grad}_z \, p(y - z) \,.\, \mathrm{grad}\, U\delta_x(z)|$$

and the function on the right-hand side of the last inequality is $H_m$-integrable with respect to $z$ over $R^m$. The last fact can be verified by a simple direct calculation (compare [4], remark 1.3). Now we can write

$$\lim_{n\to\infty}\int_G \operatorname{grad}_z f_y^n(z) \cdot \operatorname{grad} U\delta_x(z)\,\mathrm{d}z = \int_G \operatorname{grad}_z p(y-z) \cdot \operatorname{grad} U\delta_x(z)\,\mathrm{d}z\,.$$

We see that

$$K(y, x) = \int_G \operatorname{grad}_z p(y-z) \cdot \operatorname{grad} U\delta_x(z)\,\mathrm{d}z =$$
$$= \int_G \operatorname{grad} U\delta_y(z) \cdot \operatorname{grad} U\delta_x(z)\,\mathrm{d}z = K(x, y)\,,$$

which proves (20).

Fix now a $y \in R^m - M$. By (14) and (15) (with the role of $x$, $y$ interchanged),

$$\int_B p(x-z)\,\mathrm{d}|\nu_y|(z) \leqq c_2' p(y-x)$$

so that

$$\int_{B\times B} p(x-z)\,\mathrm{d}|\nu_y|(z)\,\mathrm{d}|\mu|(x) < \infty\,.$$

Using (20) we get

$$U\tilde{W}^*\mu(y) = \int_B K(y, x)\,\mathrm{d}\mu(x) = \int_B K(x, y)\,\mathrm{d}\mu(x) =$$
$$= Ad(y)\,.\int_B p(y-x)\,\mathrm{d}\mu(x) + \int_{B\times B} p(x-z)\,\mathrm{d}\nu_y(z)\,\mathrm{d}\mu(x) =$$
$$= Ad(y)\,U\mu(y) + \langle g, \nu_y\rangle$$

where $g = U\mu$ quasi-everywhere. According to the inequality

$$|U\tilde{W}^*\mu(y)| \leqq c(A + \sup_{y\in R^m} v_\infty(y)) < \infty$$

we conclude that $\tilde{W}^*\mu \in \mathfrak{B}_*$ and (19) holds.

The proof of the lemma is complete.

**5. Lemma.** *Suppose that* $\mu_n \in {}^\wedge\mathfrak{B}_*$, $\sum_{n=1}^{\infty} \|\mu_n\|' < \infty$, $\sum_{n=1}^{\infty} \|\boldsymbol{U}\mu_n\|_0' < \infty$. *Then* $\mu = \sum_{n=1}^{\infty} \mu_n \in {}^\wedge\mathfrak{B}_*$ and $\boldsymbol{U}\mu = \sum_{n=1}^{\infty} \boldsymbol{U}\mu_n$.

Proof. It is sufficient to prove the following assertion only: If $\nu_n \in \mathfrak{B}_*$, $\sum_{n=1}^{\infty} \|\nu_n\| < \infty$, $\sum_{n=1}^{\infty} \|\boldsymbol{U}\nu_n\|_0$, then $\nu = \sum_{n=1}^{\infty} \nu_n \in \mathfrak{B}_*$ and $\boldsymbol{U}\nu = \sum_{n=1}^{\infty} \boldsymbol{U}\nu_n$. Indeed, both the real and imaginary part of $\mu_n$ satisfy the assumptions formulated above for $\nu_n$ (compare (16)).

Since the space $\mathfrak{B}$ is complete, there is a $\nu \in \mathfrak{B}$ with $\sum_{n=1}^{\infty} \nu_n = \nu$. Denoting by $\nu_n = \nu_n^+ - \nu_n^-$ the Jordan decomposition of $\nu_n$, we have

$$\nu = \sum_{n=1}^{\infty} \nu_n^+ - \sum_{n=1}^{\infty} \nu_n^-$$

and the equality

$$U\nu = U(\sum_{n=1}^{\infty} \nu_n^+) - U(\sum_{n=1}^{\infty} \nu_n^-)$$

holds quasi-everywhere in $R^m$.

One easily verifies (compare [5], p. 86) that

$$U(\sum_{n=1}^{\infty} \nu_n^+)(x) = \sum_{n=1}^{\infty} U\nu_n^+(x),$$

$$U(\sum_{n=1}^{\infty} \nu_n^-)(x) = \sum_{n=1}^{\infty} U\nu_n^-(x)$$

for any $x \in R^m$ and we conclude that

$$U\nu = \sum_{n=1}^{\infty} U\nu_n$$

quasi-everywhere. Observing that

$$\|\mathbf{U}\nu\|_0 \leqq \sum_{n=1}^{\infty} \|\mathbf{U}\nu_n\|_0 < \infty$$

we see that the potential $U\nu$ is bounded quasi-everywhere. Since $U\nu = U\nu^+ - U\nu^-$ is meaningful quasi-everywhere in $R^m$ we conclude that $\nu \in \mathfrak{B}_*$ and

$$\mathbf{U}\nu = \sum_{n=1}^{\infty} \mathbf{U}\nu_n .$$

**6. Notation.** Let $Q$ be a bounded operator acting on $\mathscr{B}$. The quantity $\tilde{\omega}Q$ is defined by

$$\tilde{\omega}Q = \inf_Y \|Q - Y\|$$

where $Y$ runs over the class of all compact operators acting on $\mathscr{B}$.

Let $\Omega$ be the set of all complex numbers $\beta$ with $|\beta| > \tilde{\omega}T_\gamma$. It is well-known (see e.g. [11]) that there is a countable set $N \subset \Omega$ consisting of isolated points such that for any $\beta \in \Omega - N$ the operators $\beta I + T_\gamma$ (on $^\wedge\mathscr{B}$) and $\beta I^* + T_\gamma^*$ (on $^\wedge\mathscr{B}^*$) possess inverse operators $I_{\beta\gamma} = (\beta I + T_\gamma)^{-1}$ and $(\beta I^* + T_\gamma^*)^{-1} = I_{\beta\gamma}^*$, respectively.

An operator $Q$ acting on $^\wedge\mathscr{B}$ is said to have the property $(\Phi)$, if it satisfies the following conditions:

$$Q \quad \text{operates in} \quad {}^\wedge\mathscr{B}_0\,,$$

$$Q^*\,{}^\wedge\mathfrak{B}_* \subset {}^\wedge\mathfrak{B}_*\,,$$

$$\mathbf{U}Q^*\mu = \mathbf{Q}\mathbf{U}\mu \quad \text{whenever} \quad \mu \in {}^\wedge\mathfrak{B}_*\,.$$

In this terminology, lemma 4 states that $T$, $T_\gamma$ have the property $(\Phi)$.

We shall denote by $\Omega_0$ the set of all $\beta \in \Omega - N$ for which $I_{\beta\gamma}$ has the property $(\Phi)$.

**7. Lemma.** *Suppose that $\beta \in \Omega_0$ and $\|I^*_{\beta\gamma}\|' < K$. Then $\Omega_0$ contains the open disc with center $\beta$ and radius $1/K$. If $\alpha$ satisfies $|\alpha| > \|T_\gamma\|'$, then $\alpha \in \Omega_0$.*

Proof. Using the equality

$$\alpha I^* + T^*_\gamma = (\beta I^* + T^*_\gamma)\,(I^* + (\alpha - \beta)\,I^*_{\beta\gamma})$$

we get for $\alpha$ satisfying $|\alpha - \beta| < 1/K$

$$I^*_{\alpha\gamma} = \sum_{n=0}^{\infty} (\beta - \alpha)^n\,(I^*_{\beta\gamma})^{n+1}\,, \quad I_{\alpha\gamma} = \sum_{n=0}^{\infty} (\beta - \alpha)^n\,(I_{\beta\gamma})^{n+1}\,.$$

Since $\beta \in \Omega_0$, the operator $I_{\beta\gamma}$ operates in $^\wedge\mathscr{B}_0$ and the equality

$$\mathbf{U}(I^*_{\beta\gamma})^{n+1}\,\mu = \mathbf{I}^{n+1}_{\beta\gamma}\mathbf{U}\mu$$

holds for each $\mu \in {}^\wedge\mathfrak{B}_*$ and each $n$. Consequently,

$$\|\mathbf{U}[(\beta - \alpha)^n\,(I^*_{\beta\gamma})^{n+1}\,\mu]\|'_0 \leqq (\|I^*_{\beta\gamma}\|')^{n+1}\,.\,|\beta - \alpha|^n\,\|\mathbf{U}\mu\|'_0 \leqq |\beta - \alpha|^n\,K^{n+1}\|\mathbf{U}\mu\|'_0\,.$$

We conclude that

$$\sum_{n=0}^{\infty} \|\mathbf{U}[(\beta - \alpha)^n\,(I^*_{\beta\gamma})^{n+1}\,\mu]\|'_0 < \infty\,.$$

Applying lemma 5 we get

$$I^*_{\alpha\gamma}\mu \in {}^\wedge\mathfrak{B}_*\,,$$

$$\mathbf{U}I^*_{\alpha\gamma}\mu = \sum_{n=0}^{\infty} \mathbf{U}[(\beta - \alpha)^n\,(I^*_{\beta\gamma})^{n+1}\,\mu] = \sum_{n=0}^{\infty} (\beta - \alpha)^n\,\mathbf{I}^{n+1}_{\beta\gamma}\mathbf{U}\mu = \mathbf{I}_{\alpha\gamma}\mathbf{U}\mu$$

for any $\mu \in {}^\wedge\mathfrak{B}_*$. Since $I_{\alpha\gamma}$ operates in $^\wedge\mathscr{B}_0$ we have $\alpha \in \Omega_0$.

Suppose now that $|\alpha| > \|T_\gamma\|'$. Then

$$(\alpha I^* + T^*_\gamma)^{-1} = \sum_{n=0}^{\infty} (-\alpha)^{n+1}\,(T^*_\gamma)^n\,,$$

$$(\alpha I + T_\gamma)^{-1} = \sum_{n=0}^{\infty} (-\alpha)^{n+1}\,T^n_\gamma\,.$$

The last equality together with lemma 4 implies that $I_{\alpha\gamma}$ operates in ${}^\wedge\mathscr{B}_0$. Fix a $\mu \in {}^\wedge\mathfrak{B}_*$. By lemma 4 we have $(T_\gamma^*)^n \mu \in {}^\wedge\mathfrak{B}_*$ for each $n$ and $\boldsymbol{U}T_\gamma^*\mu = \boldsymbol{T}_\gamma\boldsymbol{U}\mu$. In a similar way as above we establish

$$\sum_{n=0}^{\infty} \left\| \boldsymbol{U}[(-\alpha)^{n+1}(T_\gamma^*)^n \mu] \right\|_0' < \infty$$

and lemma 5 may be used to assert that

$$I_{\alpha\gamma}^* \mu \in {}^\wedge\mathfrak{B}_* ,$$

$$\boldsymbol{U}I_{\alpha\gamma}^*\mu = \sum_{n=0}^{\infty} \boldsymbol{U}[(-\alpha)^{n+1}(T_\gamma^*)^n \mu] = \sum_{n=0}^{\infty} (-\alpha)^{n+1} (\boldsymbol{T}_\gamma)^n \boldsymbol{U}\mu = \boldsymbol{I}_{\alpha\gamma}\boldsymbol{U}\mu .$$

Consequently, $\alpha \in \Omega_0$ and the proof is complete.

**8. Lemma.** *The set $\Omega_0$ is relatively closed in $\Omega - N$.*

Proof. Let $\beta_0 \in \operatorname{cl} \Omega_0 \cap (\Omega - N)$. Since $I_{\alpha\gamma}^*$ is a continuous function of the variable $\alpha$ on $\Omega - N$, there is $K > 0$ and a neighborhood $M$ of the point $\beta_0$ such that $\|I_{\alpha\gamma}^*\|' \leqq K$ holds for any $\alpha \in M$. Choosing $\beta \in \Omega_0 \cap M$ in such a way that $|\beta - \beta_0| < 1/K$ we conclude by lemma 7 that $\beta_0 \in \Omega_0$.

**9. Lemma.** *The sets $\Omega_0$ and $\Omega - N$ coincide.*

Proof. It follows from lemma 7 that $\Omega_0$ is open in $\Omega - N$ and $\Omega_0 \neq \emptyset$. Since $\Omega_0$ is relatively closed by lemma 8 we conclude $\Omega_0 = \Omega - N$, because $\Omega - N$ is connected.

**10. Notation.** Fix $\alpha_0 \in N$ and $r > 0$ such that the closed disc $K$ centered at $\alpha_0$ with radius $r$ is contained in $\Omega$ and $K \cap \Omega = \{\alpha_0\}$. Let $C$ be the boundary of $K$. (It is $C \subset \Omega_0$ by lemma 9.) The operator $A_{-1}$ acting on ${}^\wedge\mathscr{B}$ is defined by

$$A_{-1} = (2\pi i)^{-1} \int_C I_{\alpha\gamma} \, d\alpha \tag{21}$$

where the integral is taken over positively oriented circumference $C$ (compare [15], chap. VIII).

**11. Lemma.** *The operator $A_{-1}$ has the property $(\Phi)$.*

Proof. Since $I_{\alpha\gamma}$ is a continuous function of the variable $\alpha$, the integral occurring in (21) is the limit of the Riemann sums $S_n$ and each $S_n$ is a finite linear combination of operators $I_{\alpha_j\gamma}$ with complex coefficients and $\alpha_j \in C$. Consequently, each $S_n$ has the property $(\Phi)$.

We may suppose $\sum_{n=1}^{\infty} \|S_n - S_{n+1}\|' < \infty$ by passing, if necessary, to a suitably chosen subsequence. Put $T_1 = S_1$, $T_{n+1} = S_{n+1} - S_n$ $(n = 1, 2, \ldots)$. Then each $T_n$ has the property $(\Phi)$, $A_{-1} = \sum_{n=1}^{\infty} T_n$, $\mathbf{A}_{-1} = \sum_{n=1}^{\infty} \mathbf{T}_n$ and $A_{-1}$ operates in $^\wedge\mathscr{B}_0$.

Fix a $\mu \in {}^\wedge\mathfrak{B}_*$ and put $\mu_n = T_n^*\mu$. Since $\mu_n \in {}^\wedge\mathfrak{B}_*$ and $\mathbf{U}\mu_n = \mathbf{T}_n\mathbf{U}\mu$ we get easily

$$\|\mathbf{U}\mu_n\|_0' \leqq \|T_n\|' \, \|\mathbf{U}\mu\|_0'$$

whence

$$\sum_{n=1}^{\infty} \|\mathbf{U}\mu_n\|_0' < \infty .$$

Observing that

$$\sum_{n=1}^{\infty} \|\mu_n\|' \leqq \Big(\sum_{n=1}^{\infty} \|T_n\|'\Big) \|\mu\|' < \infty$$

we may conclude by lemma 5 that $A_{-1}^*\mu \in {}^\wedge\mathfrak{B}_*$ and

$$\mathbf{U}A_{-1}^*\mu = \sum_{n=1}^{\infty} \mathbf{U}T_n^*\mu = \sum_{n=1}^{\infty} \mathbf{T}_n\mathbf{U}\mu = \mathbf{A}_{-1}\mathbf{U}\mu .$$

The proof is complete.

**12. Notation.** Let $X$ be a Banach space and $Q$ be a linear mapping on $X$. The null-space and the range of $Q$ will be denoted by $\mathscr{K}(Q)$ and $\mathscr{R}(Q)$, respectively. The dimension of $X$ will be denoted by $\dim X$ $(0 \leqq \dim X \leqq \infty)$.

**13. Lemma.** *Let $p$ be a positive integer and $Q$ be an operator on $^\wedge\mathscr{B}$ such that* $\dim \mathscr{K}(Q) < \infty$. *Then* $\dim \mathscr{K}(Q^p) < \infty$.

Proof. The proof is by induction on $p$. The $p = 1$ case is obvious. Assume that $p > 1$ and $\dim \mathscr{K}(Q^{p-1}) < \infty$. Put $\tilde{Q} = Q^{p-1}$, $\mathscr{B}_1 = \mathscr{R}(\tilde{Q}) \cap \mathscr{K}(Q)$ and let $y_1, \ldots, y_r$ and $z_1, \ldots, z_s$ be a basis of $\mathscr{K}(\tilde{Q})$ and $\mathscr{B}_1$, respectively. Fix an $x_i \in {}^\wedge\mathscr{B}$ such that $\tilde{Q}x_i = z_i$ $(i = 1, 2, \ldots, s)$ and denote by $\mathscr{B}_2$ the linear space generated by $x_1, \ldots$ $\ldots, x_s, y_1, \ldots, y_r$. If $x_0 \in \mathscr{K}(Q^p)$, then $x_0 \in \mathscr{B}_2$. Indeed, since $Q\tilde{Q}x_0 = 0$, we have $\tilde{Q}x_0 = \sum_{i=1}^{s} \alpha_i z_i$ and $\tilde{x} = x_0 - \sum_{i=1}^{s} \alpha_i x_i$ satisfies $\tilde{Q}\tilde{x} = 0$. Consequently, $\tilde{x} = \sum_{j=1}^{r} \beta_j y_j$. We see that $\dim \mathscr{K}(Q^p) \leqq r + s$ and the proof is complete.

**14. Lemma.** *Let us denote*

$$N(\alpha_0) = \{y \in B;\ d(y) = \gamma - \alpha_0 A^{-1}\}$$

*and let $p$ be any positive integer. Then the set $N(\alpha_0)$ is finite and each $f \in {}^\wedge\mathscr{B}$*

*satisfying*

$$(\alpha_0 I + T_\gamma)^p f = 0\,, \tag{22}$$

$$\langle f, \mu \rangle = 0 \quad \textit{for each} \quad \mu \in {}^\wedge\mathfrak{B}_* \tag{23}$$

*has its support contained in* $N(\alpha_0)$.

Proof. Denoting by $f_z$ the characteristic function of the set $\{z\} \subset B$ we get for any $y \in B$

$$(\alpha_0 I + T_\gamma)^p f_z(y) = [\alpha_0 - \gamma A + Ad(y)]^p f_z(y)\,.$$

We see that $f_z$ is a solution of (22) if and only if $z \in N(\alpha_0)$. Since $|\alpha_0| > \tilde{\omega} T_\gamma$ it is $\dim \mathscr{K}(\alpha_0 I + T_\gamma) < \infty$ and also $\dim \mathscr{K}([\alpha_0 I + T_\gamma]^p) < \infty$ by lemma 13. Consequently, the set $N(\alpha_0)$ is finite.

Recall that we have denoted by $H$ the restriction of $H_{m-1}$ to the reduced boundary $\hat{B}$. Let (22) and (23) hold for an $f \in {}^\wedge\mathscr{B}$. Given a Borel set $M \subset B$ we denote by $\lambda_M$ and $H_M$ the restriction of $\lambda$ and $H$ to $M$, respectively. For such an $M$ we have $\lambda_M \in {}^\wedge\mathfrak{B}_*$, $H_M \in {}^\wedge\mathfrak{B}_*$. Indeed, $\lambda$ has bounded potential by hypothesis and the potential of $H$ is continuous by [8], corollary 22. Since the relations

$$\langle f, \lambda_M \rangle = 0\,, \quad \langle f, H_M \rangle = 0$$

hold for each Borel set $M \subset B$, we conclude that $f = 0$ $\lambda$-almost everywhere and $f = 0$ $H$-almost everywhere as well. Now it is easily seen by the definition of $T$ that

$$0 = (\alpha_0 I + T_\gamma)^p f(y) = [\alpha_0 - \gamma A + Ad(y)]^p f(y)\,.$$

If $y \notin N(\alpha_0)$, then $f(y) = 0$. Consequently, the support of $f$ is contained in $N(\alpha_0)$.

The proof of the lemma is complete.

**15. Lemma.** *Suppose that* $N(\alpha_0) = \emptyset$ *and let* $f_1, \ldots, f_q$ *be linearly independent solutions of* (22). *Then there exist* $\mu_1, \ldots, \mu_q \in {}^\wedge\mathfrak{B}_*$ *such that* $\langle f_i, \mu_j \rangle = \delta_{ij}$ ($\delta_{ij} = 0$ *for* $i \neq j$, $\delta_{ii} = 1$) *for* $1 \leqq i, j \leqq q$.

Proof. The proof is by induction on $q$. If $q = 1$, then there is $\mu_1 \in {}^\wedge\mathfrak{B}_*$ with $\langle f_1, \mu_1 \rangle = 1$. Indeed, if there were no such $\mu_1$, then the hypothesis $N(\alpha_0) = \emptyset$ together with lemma 14 would imply $f_1 = 0$, a contradiction.

Suppose that $q > 1$ and let the assertion be true for $q - 1$. We shall first prove that there is $\mu_1 \in {}^\wedge\mathfrak{B}_*$ such that $\langle f_j, \mu_1 \rangle = \delta_{j1}$ for $j = 1, \ldots, q$. Denote by $\{\mu'_2, \ldots, \mu'_q\}$ a biorthonormal system to $\{f_2, \ldots, f_q\}$. Then, for each $\mu \in {}^\wedge\mathfrak{B}_*$, the element

$$\mu - \sum_{k=2}^{q} \langle f_k, \mu \rangle\, \mu'_k$$

is orthogonal to $f_2, \ldots, f_q$. If the same is true for $f_1$, then $f_1 = \sum_{k=2}^{q} \langle f_1, \mu'_k \rangle f_k$ by lemma

14, which is a contradiction with the linear independence of $f_1, \ldots, f_q$. Consequently, there exists a $\mu \in {}^\wedge\mathfrak{B}_*$ such that

$$\mu_1 = \mu - \sum_{k=2}^{q} \langle f_k, \mu \rangle \mu'_k$$

satisfies $\langle f_1, \mu_1 \rangle = 1$ and, of course, $\langle f_j, \mu_1 \rangle = 0$ for $j = 2, \ldots, q$. In a similar way we can construct $\mu_j$'s with $\langle f_k, \mu_j \rangle = \delta_{kj}$ $(1 \leqq k \leqq q)$ for $j = 2, \ldots, q$.

**16. Lemma.** *Let us put $N(\alpha) = \emptyset$ for $\alpha \notin N$. Suppose that $\alpha_0 \in \Omega$ and $N(\alpha_0) = \emptyset$. If $p$ is a positive integer and $\mu \in {}^\wedge\mathscr{B}^*$ satisfies*

$$(\alpha_0 I^* + T_\gamma^*)^p \mu = 0 \,, \tag{24}$$

*then $\mu \in {}^\wedge\mathfrak{B}_*$.*

Proof. The assertion is trivial for $\alpha_0 \in \Omega - N$ by the definition of $\Omega_0$. Suppose that $\alpha_0 \in N$. It is well-known that the resolvents of the operators $\alpha I^* + T_\gamma^*$, $\alpha I + T_\gamma$ have a pole at $\alpha_0$ (compare [11]) and these poles have the same order (compare [15], chap. VIII, 6, 8), say $p_0$. Clearly, we may assume that $p \geqq p_0$.

Similarly as in 10, define the operator $\mathscr{A}_{-1}$ on ${}^\wedge\mathscr{B}^*$ by

$$\mathscr{A}_{-1} = (2\pi \mathrm{i})^{-1} \int_C I_{\alpha\gamma}^* \, \mathrm{d}\alpha$$

where $C$ has the same meaning as in 10. Then the set $Y$ of all solutions of the equation (24) coincides with $\mathscr{R}(\mathscr{A}_{-1})$ ([15], chap. VIII, 8). Since $\mathscr{A}_{-1} = A_{-1}^*$ ([15], chap. VIII, 7), we have $Y = \mathscr{R}(A_{-1}^*)$. Similarly, denoting by $X$ the set of all solutions of the equation (22), we get $X = \mathscr{R}(A_{-1})$.

Let $f_1, \ldots, f_q$ be a basis of $X$. Then the operator $A_{-1}$ possesses the form

$$A_{-1} \ldots = \sum_{k=1}^{q} \langle \ldots, \mu_k \rangle f_k$$

where $\mu_k \in {}^\wedge\mathscr{B}^*$. Consequently,

$$A_{-1}^* \ldots = \sum_{k=1}^{q} \langle f_k, \ldots \rangle \mu_k \,. \tag{25}$$

By virtue of lemma 15 we construct $\mu'_1, \ldots, \mu'_q \in {}^\wedge\mathfrak{B}_*$ such that $\langle f_j, \mu_i \rangle = \delta_{ij}$, $1 \leqq i, j \leqq q$. It follows from (25) that $A_{-1}^* \mu'_k = \mu_k$ for $k = 1, \ldots, q$ and we conclude by lemma 11 that $\mu_k \in {}^\wedge\mathfrak{B}_*$. Since $Y = \mathscr{R}(A_{-1}^*)$, we have $Y \subset {}^\wedge\mathfrak{B}_*$ and the proof is complete.

Let us summarize our results in the following theorem stated in the introduction.

**17. Theorem.** *Let $\beta \in R^1$ satisfy the inequality $A|\beta| > \tilde{\omega} T_\gamma$. Suppose that*

$$d(y) \neq \gamma - \beta$$

*for each $y \in B$. If $\mu \in \mathscr{B}^*$ satisfies*

$$(A\beta I^* + T_\gamma^*) \mu = 0 ,$$

*then $\mu \in \mathfrak{B}_*$.*

*In particular, any solution of*

$$[A(\beta - \gamma)\mathscr{I} + \mathscr{T}] \mu = 0$$

*belongs to $\mathfrak{B}_*$.*

Proof. Putting $\alpha_0 = \beta A$, $p = 1$, the assertion of the theorem follows by lemma 16 and by the definition of $N(\alpha_0)$.

**18. Example.** We are going to show that the hypothesis $d(y) \neq \gamma - \beta$ is essential for the validity of theorem 17. Put $G = \{x \in R^m;\ 0 < |x| < 1\}$, $\gamma = \frac{1}{2}$, $\beta = -\frac{1}{2}$ and let $\bar{\lambda}$ stand for the restriction of $H_{m-1}$ to fr $G$ and $\lambda = (m-2)\,\bar{\lambda}$. Using (56) in [8] one easily verifies that $\omega T_\gamma = 0$. Consequently, $\tilde{\omega} T_\gamma = 0$ and $A|\beta| > \tilde{\omega} T_\gamma$. Note that $U\lambda$ is continuous on $R^m$ by corollary 22 in [8].

An easy calculation shows that

$$\int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\delta_0(x)\, \mathrm{d}x = A\varphi(0) - \int_{\mathrm{fr}G} \varphi \,\mathrm{d}H_{m-1} ,$$

$$\mathscr{T}\delta_0(\varphi) = A\varphi(0) - \int_{\mathrm{fr}G} \varphi \,\mathrm{d}H_{m-1} + (m-2)^{-1} \int_{\mathrm{fr}G} \varphi \,\mathrm{d}\lambda = A\varphi(0) .$$

We conclude that

$$(-A\mathscr{I} + \mathscr{T})\,\delta_0 = 0$$

but $\delta_0 \notin \mathfrak{B}_*$.

For our further purposes the following special case of theorem 17 will be useful. Recall that the quantity $a'$ has been defined in the introduction.

**19. Theorem.** *Suppose that $d(y) \neq 0$ for each $y \in B$ and*

$$\tilde{a} = \inf_{\alpha \neq 0} \frac{\tilde{\omega} T_\alpha}{A|\alpha|} < 1 . \tag{26}$$

*Then*

$$T^* \nu = 1$$

*implies $\nu \in \mathfrak{B}_*$. In particular, if $a' < 1$ and $\nu \in \mathfrak{B}$ satisfies*

$$\mathscr{T}\nu = 0 ,$$

*then $\nu \in \mathfrak{B}_*$.*

Proof. As for the first part, choose a $\beta \in R^1$ with $A|\beta| > \tilde{\omega}T_\beta$ and apply theorem 17 with $\beta = \gamma$.

Noting that $a' \geqq \tilde{a}$ (see the definition of $\tilde{\omega}T_\alpha$ and lemma 33 in [8]), the second part is a consequence of the first assertion.

**20. Remark.** The method of proofs of last theorems is in part a variant of Radon's ideas developed in [10]. J. Radon has considered in place of $\mathfrak{V}_*$ a class of charges (distributed on the plane curves of bounded rotation) inducing a potential having the same interior and exterior limits. In the case that $U\lambda$ is continuous, the Radon results may be modified without an essential change for spaces of higher dimension (see [3] and [13] for $R^3$, [2] for $R^n$). In our case it was not possible to use the same way, because, in general, the inclusion $T\mathscr{C} \subset \mathscr{C}$ fails (see proposition 9 in [8]).

We are now going to show that under a suitable condition the potential $U\mu$ possesses finite Dirichlet integral provided $\mu \in \mathfrak{V}_*$.

**21. Notation.** Let us define the function $\theta$ on $R^m$ as follows:

$$\theta(x) = \exp\left(|x|^2 - 1\right)^{-1} \quad \text{for} \quad |x| < 1\,,$$
$$\theta(x) = 0 \qquad\qquad \text{for} \quad |x| \geqq 1\,.$$

For $\delta > 0$ put

$$\theta_\delta(x) = h_\delta \theta(x/\delta)$$

with $h_\delta$ so chosen that

$$\int_{R^m} \theta_\delta(x)\, \mathrm{d}H_m(x) = 1\,.$$

Clearly, $\theta_\delta \in \mathscr{D}$ for each $\delta$.

If $D$ is a distribution over $\mathscr{D}$, then the convolution $D*\theta_\delta$ will be denoted by $R_\delta D$ (see [14], chap. VI). In particular, if $f$ is locally integrable over $R^m$, then

$$R_\delta f(x) = \int_{R^m} f(t)\, \theta_\delta(x - t)\, \mathrm{d}H_m(t)\,, \quad x \in R^m\,.$$

Let us suppose that for such an $f$ there is $\beta \in R^1$ such that $|f(t)| \leqq \beta$ holds for $H_m$-almost all $t \in R^m$. Then the inequality

$$|R_\delta f(x)| \leqq \beta \tag{27}$$

is true for any $x \in R^m$.

Finally, for each $\varepsilon > 0$ let

$$B^\varepsilon = \{x \in R^m;\ \mathrm{dist}\,(x, B) > \varepsilon\}\,.$$

**22. Lemma.** *Suppose that* $\mu \in \mathfrak{B}$ *and* $\varepsilon > 0$. *Then*

$$\lim_{\delta \to 0+} R_\delta U\mu = U\mu \tag{28}$$

*holds quasi-everywhere in* $R^m$ *and for each* $\delta \in (0, \varepsilon)$ *we have*

$$R_\delta U\mu = U\mu \quad on \quad B^\varepsilon . \tag{29}$$

Proof. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of $\mu$. Then the equality $U\mu = U\mu^+ - U\mu^-$ holds quasi-everywhere (see [5]). Consequently, it is sufficient to prove (28), (29) under the additional assumption that $\mu$ is a non-negative element of $\mathfrak{B}$.

If this is the case, then $U\mu$ is a superharmonic function in $R^m$, harmonic in $R^m - B$ and locally integrable in $R^m$ (see [5]).

Since $U\mu$ is superharmonic, it is easy to verify the inequalities

$$\begin{gathered} R_\delta U\mu(x) \leqq U\mu(x) , \\ \limsup_{\delta \to 0+} R_\delta U\mu(x) \leqq U\mu(x) , \quad x \in R^m . \end{gathered} \tag{30}$$

Suppose that $\delta \in (0, \varepsilon)$ and $x \in B^\varepsilon$. Since the ball centered at $x$ with radius $\delta$ is contained in $R^m - B$, the mean-value property of harmonic functions implies immediately

$$R_\delta U\mu(x) = U\mu(x) .$$

Thus (29) is established.

Since $U\mu$ is lower semicontinuous on $R^m$ we get

$$U\mu(x) \leqq \liminf_{\delta \to 0+} R_\delta U\mu(x) , \quad x \in R^m .$$

This together with (30) yields (28).

**23. Proposition.** *Suppose that* $\mu \in \mathfrak{B}_*$ *and* $H_m(B) = 0$. *Then*

$$\int_{R^m} |\text{grad } U\mu(x)|^2 \, dH_m(x) < \infty .$$

Proof. Fix $R > 1$ such that $B \subset \Omega_R(0)$ and let $\beta \in R^1$ be chosen such that $|U\mu| \leqq \beta$ quasi-everywhere in $R^m$. Suppose that $r > 2R$, $\delta \in (0, 1)$, and write $\Omega_r$ instead of $\Omega_r(0)$. By the Gauss-Green theorem we get

$$\begin{gathered} \int_{\mathrm{fr}\Omega_r} R_\delta U\mu(z) \, . \, n_{\Omega_r}(z) \, . \, \text{grad } R_\delta U\mu(z) \, dH_{m-1}(z) = \\ = \int_{\Omega_r} |\text{grad } R_\delta U\mu(x)|^2 \, dH_m(x) + \int_{\Omega_r} R_\delta U\mu(x) \, . \, \Delta R_\delta U\mu(x) \, dH_m(x) \end{gathered} \tag{31}$$

where $n_{\Omega_r}(z)$ denotes the exterior normal of $\Omega_r$ at $z$. Let $\varphi \in \mathscr{D}$ satisfy $|\varphi| \leqq 1$ on $R^m$ and $\varphi = 1$ on $\Omega_{2R}(0)$. By lemma 22 the function $R_\delta U\mu$ is harmonic on $R^m - \Omega_{2R}$ and we conclude that

$$\int_{\Omega_r} R_\delta U\mu(x) \,.\, \Delta R_\delta U\mu(x) \, \mathrm{d}H_m(x) = \tag{32}$$

$$= \int_{R^m} \varphi(x) \, R_\delta \, U\mu(x) \, \Delta R_\delta \, U\mu(x) \, \mathrm{d}H_m(x) \,.$$

Let us now consider the distributions $U^\mu$, $M^\mu$ over $\mathscr{D}$ defined as follows:

$$\langle \psi, U^\mu \rangle = \int_{R^m} \varphi(x) \, U\mu(x) \, \mathrm{d}H_m(x) \,,$$

$$\langle \psi, M^\mu \rangle = \int_{R^m} \psi(x) \, \mathrm{d}\mu(x) \,, \quad \psi \in \mathscr{D} \,.$$

It is well-known that $\Delta U^\mu = -AM^\mu$ and we get for any $\delta > 0$ the equality $\Delta R_\delta U^\mu = -AR_\delta M^\mu$ (compare [14]). Since $\varphi \,.\, R_\delta U\mu \in \mathscr{D}$, we have

$$\int_{R^m} \varphi(x) \, R_\delta U\mu(x) \,.\, \Delta R_\delta U\mu(x) \, \mathrm{d}H_m(x) = \tag{33}$$

$$= -A\langle \varphi \,.\, R_\delta U\mu, R_\delta M^\mu \rangle = -A \int_{R^m} R_\delta(\varphi R_\delta U\mu) \, (x) \, \mathrm{d}\mu(x) \,.$$

Applying (27) (with $f = U\mu$) we get from (31), (32) and (33) for $r > 2R$ and $\delta \in (0, 1)$ the estimate

$$\int_{\Omega_r} |\text{grad } R_\delta U\mu(x)|^2 \, \mathrm{d}H_m(x) \leqq A\beta \|\mu\| + \mathscr{I}(r, \delta) \tag{34}$$

where we have put

$$\mathscr{I}(r, \delta) = \int_{\mathrm{fr}\,\Omega_r} R_\delta \, U\mu(x) \,.\, n_{\Omega_r}(x) \,.\, \text{grad } R_\delta \, U\mu(x) \, \mathrm{d}H_m(x) \,.$$

By lemma 22, for $z \in \mathrm{fr}\,\Omega_r$, the equalities $R_\delta U\mu(z) = U\mu(z)$ and $\text{grad } R_\delta U\mu(z) = \text{grad } U\mu(z)$ hold and one easily verifies that $\mathscr{I}(r, \delta)$ admits the estimate

$$|\mathscr{I}(r,\delta)| \leqq \frac{1}{m-2} \cdot \frac{\|\mu\|}{(r-R)^{m-2}} \cdot \frac{\|\mu\|}{(r-R)^{m-1}} \, Ar^{m-1} \,.$$

Now from (34) it follows for $\delta \in (0, 1)$

$$\int_{R^m} |\text{grad } R_\delta U\mu(x)|^2 \, \mathrm{d}H_m(x) \leqq A\beta \|\mu\| \tag{35}$$

and lemma 22 yields

$$\lim_{\delta \to 0+} \text{grad } R_\delta U\mu(x) = \text{grad } U\mu(x)$$

whenever $x \in R^m - B$. Since $H_m(B) = 0$, Fatou's lemma may be applied to assert

$$\int_{R^m} |\text{grad } U\mu|^2 \leqq A\beta \|\mu\| < \infty .$$

The proof is complete.

**24. Lemma.** *Suppose that* $\mu \in \mathfrak{B}_*$ *and* $H_m(B) = 0$. *Then there exist functions* $\varphi_n \in \mathscr{D}$ *such that*

$$\lim_{n\to\infty} \int_G \text{grad } \varphi_n(x) \,.\, \text{grad } U\mu(x) \, \mathrm{d}H_m(x) =$$

$$= \int_G |\text{grad } U\mu(x)|^2 \, \mathrm{d}H_m(x) ,$$

$$\lim_{n\to\infty} \int_B \varphi_n(x) \, U\mu(x) \, \mathrm{d}\lambda(x) = \int_B [U\mu(x)]^2 \, \mathrm{d}\lambda(x) .$$

Proof. Let $\beta$, $R$, $\delta$ have the same meaning as in the last proof. Denote by $\gamma$ a function defined in $R^1$ having the following properties: $\gamma$ is symmetric infinitely differentiable function in $R^1$, $|\gamma| \leqq 1$, $\gamma(t) = 1$ for $t \in (0, 1)$ and $\gamma(t) = 0$ for $t \in (2, \infty)$. Defining the function $\psi_\delta$ in $R^m$ by

$$\psi_\delta(x) = \gamma(\delta|x|) , \quad x \in R^m ,$$

we see that $\psi_\delta \in \mathscr{D}$ and

(36) $$|\text{grad } \psi_\delta(x)| \leqq \sigma\delta , \quad x \in R^m ,$$

where $\sigma = \sup \{\gamma'(t); t \in R^1\}$. Finally, let $\varphi_\delta = \psi_\delta \,.\, R_\delta U\mu$. Then $\varphi_\delta \in \mathscr{D}$ and

$$\left(\int_{R^m} |\text{grad } \varphi_\delta(x)|^2 \, \mathrm{d}H_m(x)\right)^{1/2} \leqq \mathscr{I}_1(\delta) + \mathscr{I}_2(\delta)$$

where we have put

$$\mathscr{I}_1(\delta) = \left(\int_{R^m} |\psi_\delta(x) \,.\, \text{grad } R_\delta U\mu(x)|^2 \, \mathrm{d}H_m(x)\right)^{1/2} ,$$

$$\mathscr{I}_2(\delta) = \left(\int_{R^m} |R_\delta U\mu(x) \,.\, \text{grad } \psi_\delta(x)|^2 \, \mathrm{d}H_m(x)\right)^{1/2} .$$

It is $\mathscr{J}_1(\delta) \leqq (A\beta\|\mu\|)^{1/2}$ by (35). Fix $\delta \in (0, (2R)^{-1})$. Then $|x| > \delta^{-1}$ implies $R_\delta U\mu(x) = U\mu(x)$ and

$$|U\mu(x)| \leqq \frac{1}{m-2} \cdot \frac{\|\mu\|}{(\delta^{-1} - R)^{m-2}} .$$

As it follows easily by the definition of $\psi_\delta$ and by (36),

$$\mathscr{J}_2(\delta) \leqq \left[ H_m[\Omega_{2\delta^{-1}}(0) - \Omega_{\delta^{-1}}(0)] \cdot \frac{\sigma^2 \|\mu\|^2 \delta^2}{(m-2)^2 (\delta^{-1} - R)^{2m-4}} \right]^{1/2} .$$

Since $\lim_{\delta \to 0+} \mathscr{J}_2(\delta) = 0$, there is a $\Delta_0 \in (0, (2R)^{-1})$ such that

$$\delta \in (0, \Delta_0) \Rightarrow \mathscr{J}_2(\delta) \leqq (A\beta\|\mu\|)^{1/2} .$$

Consequently,

$$\left[ \int_{R^m} |\operatorname{grad} \varphi_\delta(x)|^2 \, \mathrm{d}H_m(x) \right]^{1/2} \leqq 2(A\beta\|\mu\|)^{1/2} , \tag{37}$$

provided $\delta \in (0, \Delta_0)$.

If $M \subset R^m$ and $\xi = [\xi_1, \ldots, \xi_m]$ is a mapping of $M$ into $R^m$, then $\xi$ is said to be a vector function defined on $M$. In the case that the set $M$ is measurable $(H_m)$ and each $\xi_j$ is measurable $(H_m)$, then $\xi$ will be called $H_m$-measurable vector function. Let us denote by $\mathscr{L}_2$ the linear space of all equivalence classes (with respect to $H_m$) of $H_m$-measurable vector functions $\xi$ defined almost everywhere $(H_m)$ in $R^m$ such that

$$\left( \int_{R^m} \left( \sum_{i=1}^m \xi_i^2(x) \right) \mathrm{d}H_m(x) \right)^{1/2} < \infty .$$

For $\tilde{\xi}, \tilde{\eta} \in \mathscr{L}_2$ the scalar product $(\tilde{\xi}, \tilde{\eta})$ of $\tilde{\xi}$ and $\tilde{\eta}$ is defined by

$$(\tilde{\xi}, \tilde{\eta}) = \int_{R^m} \sum_{i=1}^m \xi_i(x) \cdot \eta_i(x) \, \mathrm{d}H_m(x) , \quad \xi \in \tilde{\xi} , \quad \eta \in \tilde{\eta} .$$

Then $\mathscr{L}_2$ is a Hilbert space and it follows from (37) that the set of vector functions

$$\{\operatorname{grad} \varphi_\delta; \ \delta \in (0, \Delta_0)\} \tag{38}$$

is weakly compact in $\mathscr{L}_2$ (compare the similar proof in [2]). Consequently, there is an $f = [f_1, \ldots, f_m] \in \mathscr{L}_2$ and there exist numbers $\delta^n \in (0, \Delta_0)$ such that $\delta^n \searrow 0$ and the equality

$$\lim_{n \to \infty} \int_{R^m} \operatorname{grad} \varphi_{\delta^n}(x) \cdot g(x) \, \mathrm{d}H_m(x) = \int_{R^m} f(x) \cdot g(x) \, \mathrm{d}H_m(x) \tag{39}$$

holds for each $g \in \mathscr{L}_2$. Write $\varphi_n$ in place of $\varphi_{\delta^n}$. Now we are going to prove that

$$f = \operatorname{grad} U\mu \quad \text{in} \quad \mathscr{L}_2 . \tag{40}$$

For $\varepsilon \in (0, 1)$ denote by

$$G_\varepsilon = \{y \in R^m;\ \varepsilon < \operatorname{dist}(y, B) < \varepsilon^{-1}\}.$$

Fix such an $\varepsilon$ and an $H_m$-measurable set $Q \subset G_\varepsilon$.

Choosing in (39) $g = [\chi_Q, 0, \dots, 0]$ where $\chi_Q$ is the characteristic function of $Q$, we arrive at

$$\lim_{n\to\infty} \int_Q \frac{\partial \varphi_n(x)}{\partial x_1}\, \mathrm{d}H_m(x) = \int_Q f_1(x)\, \mathrm{d}H_m(x).$$

On the other hand, it follows from the definition of $\psi_\delta$, $\varphi_\delta$ and from lemma 22 that

$$\lim_{n\to\infty} \int_Q \frac{\partial \varphi_n(x)}{\partial x_1}\, \mathrm{d}H_m(x) = \int_Q \frac{\partial U\mu(x)}{\partial x_1}\, \mathrm{d}H_m(x).$$

Consequently,

$$f_1 = \frac{\partial U\mu}{\partial x_1} \tag{41}$$

holds for $H_m$-almost all points $x \in G_\varepsilon$. Since $H_m(B) = 0$ and $\varepsilon \in (0, 1)$ was arbitrary, we conclude that (41) holds for $H_m$-almost all points of $R^m$. Corresponding equalities for other components may be verified in a similar way and (40) is established.

Using proposition 23 and denoting by $\chi_G$ the characteristic function of $G$ we conclude that $g = \chi_G \cdot \operatorname{grad} U\mu \in \mathscr{L}_2$. The first equality stated in the lemma follows now from (39) and (40).

As for the second equality, let us observe that for each $n$ and each $x \in B$ we have

$$\varphi_n(x) = R_{\delta_n} U\mu(x)$$

and $|\varphi_n| \leqq \beta$ on $B$. By lemma 22,

$$\lim_{n\to\infty} \varphi_n(x) = U\mu(x)$$

holds for $\lambda$-almost all $x \in B$. Now the Lebesgue dominated convergence theorem may be used to complete the proof.

**25. Lemma.** *If $d(y) \neq 0$ for each $y \in B$, then $H_m(B) = 0$.*

Proof. This assertion is an easy consequence of the well-known density theorem. Indeed, suppose that $H_m(B) > 0$. Now the density theorem ([12]; chap. IV.) implies the existence of a $y_0 \in B$ at which $G' = R^m - G$ has $m$-density equal to 1. Consequently, $d(y_0) = 0$, which is a contradiction.

Throughout the rest of the paper *we shall assume that $G$ is connected.*

**26. Theorem.** *Suppose that* $\tilde{a} < 1$ (see (26)), $d(y) \neq 0$ *for each* $y \in B$ *and let* $\nu \in \mathscr{B}^*$ *satisfy*

$$T^* \nu = 0 .$$

*Then* $\nu \in \mathfrak{B}$ *and there exists* $c \in R^1$ *such that* $U\nu = c$ *on* $G$ *and* $c^2 \|\lambda\| = 0$. *If* $c = 0$, *then* $\nu = 0$.

Proof. It is $H_m(B) = 0$ by lemma 25. Using theorem 19 we conclude $\nu \in \mathfrak{B}_* \subset \mathfrak{B}$ and $\mathscr{T}\nu = 0$. By the definition of $\mathscr{T}$,

$$0 = \mathscr{T}\nu(\varphi) = \int_B \varphi(x)\, U\nu(x)\, \mathrm{d}\lambda(x) + \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\nu(x)\, \mathrm{d}H_m(x)$$

for each $\varphi \in \mathscr{D}$.

In view of lemma 24,

$$\int_G |\operatorname{grad} U\nu(x)|^2\, \mathrm{d}H_m(x) + \int_B [U\nu(x)]^2\, \mathrm{d}\lambda(x) = 0 . \tag{42}$$

Since $G$ is connected, there is $c \in R^1$ such that $U\nu = c$ on $G$. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of $\nu$. We have $U\nu^+(x) = c + U\nu^-(x)$ for each $x \in G$. Since $G$ has a positive $m$-dimensional density at any $z \in B$, every fine neighborhood of $z$ (in the Cartan topology) meets $G$ (see [1], chap. VII, §§ 2, 6) and we conclude from the Cartan Theorem ([1], chap. VII, § 6) that $U\nu^+(z) = c + U\nu^-(z)$ (compare with the same reasonings in [4], 4.8). Consequently, $U\nu = c$ holds quasi-everywhere in $B$. Noting that the same is true for $\lambda$-almost all points $x \in B$ we arrive at the equality $c^2 \|\lambda\| = 0$ by (42).

Suppose that $c = 0$, so that $U\nu^+ = U\nu^-$ on $B$. Since $d(y) \neq 0$ for each $y \in B$, the set $G$ is not thin at any $y \in B$ ([1], chap. VII, § 2) and we have $\nu^+ = \nu^-$ (see [5], theorem 5.10 and chap. V, § 1, section 2, 14). In this case $\nu = 0$.

The proof is complete.

**27. Lemma.** *Suppose that* $G$ *is bounded. If* $f(x) = 1$ *for any* $x \in B$, *then*

$$\tilde{W} f = 0 .$$

Proof. Let us construct $\varphi \in \mathscr{D}$ such that $\varphi = 1$ on $\operatorname{cl} G$. Using (5) we have for any $y \in B$

$$\tilde{W} f(y) = A d(y) f(y) + \langle f, \nu_y \rangle = A d(y)\, \varphi(y) + \langle \varphi, \nu_y \rangle =$$
$$= \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\delta_y(x)\, \mathrm{d}H_m(x) = 0 .$$

**28. Theorem.** *Suppose that* $d(y) \neq 0$ *for each* $y \in B$ *and*

$$a' < 1 .$$

*Then*

$$\mathscr{T}(\mathfrak{B}) = \mathfrak{B} \tag{43}$$

*with the only exception which occurs if G is bounded and* $\lambda = 0$. *In this case*

$$\mathscr{T}(\mathfrak{B}) = \{v \in \mathfrak{B};\ v(B) = 0\}\,.$$

Proof. Suppose that $\mathscr{T}v = 0$ holds for a $v \in \mathfrak{B}$. Noting that $\tilde{a} \leqq a'$ we may apply theorem 26 to assert that there is a $c \in R^1$ such that $Uv = c$ on $G$ and $c^2\|\lambda\| = 0$. If either $G$ is not bounded or $\lambda \neq 0$ we conclude that $c = 0$ and theorem 26 implies $v = 0$. In this case (43) follows by the Riesz-Schauder theory.

It remains only to consider the case that $G$ is bounded and $\lambda = 0$. In this case we have $T = \tilde{W}$ and we know that $\tilde{W}\mathscr{C} \subset \mathscr{C}$ (see (16) in [7]). Denote $^\wedge\tilde{W}$ the restriction of $\tilde{W}$ to $\mathscr{C}$. Then $\mathscr{T}$ is a dual operator to $^\wedge\tilde{W}$ (see remark 32 in [8]). Referring to the remark 32 in [8] (the equality (92)), and to the lemma 33 in [8] we see that the assumption $a' < 1$ guarantees the applicability of the Riesz-Schauder theory to the pair of operators $^\wedge\tilde{W}, \mathscr{T}$.

Using theorem 26 we conclude that the space $\mathscr{N}^*$ of all solutions of the equation

$$\mathscr{T}\mu = 0 \quad \text{on} \quad \mathfrak{B}$$

has dimension at most one. By the Riesz-Schauder theory, $\mathscr{N}^*$ has same dimension as the space $\mathscr{N}$ of all solutions of the equation

$$^\wedge\tilde{W}g = 0 \quad \text{on} \quad \mathscr{C}\,.$$

Consequently, lemma 27 implies that $\mathscr{N}$ consists precisely of functions constant on $B$. Finally, the Riesz-Schauder theory implies that $v \in \mathscr{T}(\mathfrak{B})$ if and only if $\langle f, v\rangle = 0$ for any $f \in \mathscr{N}$, or, which is the same, if and only if $v(B) = 0$.

The proof is complete.

**29. Remark.** Using the notation introduced in [8] we can state a corollary of the preceding theorem here:

Suppose that the potential $U(\lambda - \hat{\lambda})$ is continuous at each point of $\operatorname{cl}[B - (B_1 \cup B_2)]$. If

$$k_1 < A\,, \quad k_2 < \tfrac{1}{2}A\,, \tag{44}$$

then the assertion of theorem 28 is true.

Indeed, the inequalities in (44) secure $a' < 1$ by theorem 31 and lemma 33 in [8] and the last inequality implies $d(y) \neq 0$ for any $y \in B$ by theorem 20 and lemma 33 in [8].

In particular, if $\lambda = 0$ and (44) holds, theorem 28 contains an assertion connected with the Neumann problem for the case of a domain. The last result slightly generalizes the result of 4.11 in [4] for the case of connected $G$. The above mentioned corollary generalizes essentially the corresponding result of [13].

Let us recall here the definition of the space $\mathfrak{B}_H$ introduced in [7]. $\mathfrak{B}_H$ is the space of all elements of $\mathfrak{B}$ which are absolutely continuous with respect to $H$. Roughly speaking, $\mathfrak{B}_H$ consists of all elements having a density with respect to an area measure.

An easy consequence of theorem 28 and of proposition 12 in [7] is the following assertion.

**30. Theorem.** *Suppose that $d(y) \neq 0$ for any $y \in B$, $a' < 1$ and $\lambda \in \mathfrak{B}_H$. Then*

$$\mathcal{T}(\mathfrak{B}_H) = \mathfrak{B}_H \tag{45}$$

*with the only exception which occurs if $G$ is bounded and $\lambda = 0$. In this case*

$$\mathcal{T}(\mathfrak{B}_H) = \{\nu \in \mathfrak{B}_H;\ \nu(B) = 0\}\,. \tag{46}$$

Proof. It is known from proposition 12 in [7] that $\mathcal{T}(\mathfrak{B}_H) \subset \mathfrak{B}_H$ and $\mathcal{T}\nu \in \mathfrak{B}_H$ for a $\nu \in \mathfrak{B}$ implies $\nu \in \mathfrak{B}_H$.

If the exceptional case does not occur, then $\mathcal{T}(\mathfrak{B}_H) = \mathfrak{B}_H$ follows from theorem 28 and (45) is verified.

If $G$ is bounded and $\lambda = 0$, then clearly

$$\mathcal{T}(\mathfrak{B}_H) \subset \{\nu \in \mathfrak{B}_H;\ \nu(B) = 0\}\,.$$

On the other hand, if $\nu \in \mathfrak{B}_H$ and $\nu(B) = 0$, then there is a $\mu \in \mathfrak{B}$ such that $\mathcal{T}\mu = \nu$ by theorem 28. Consequently, $\mu \in \mathfrak{B}_H$. Thus (46) is established and the proof is complete.

## *References*


[1] *M. Brelot:* Eléments de la théorie classique du potentiel, Les cours de Sorbonne, Paris, 1959.

[2] *Ju. D. Burago* and *V. G. Mazja:* Some questions in potential theory and function theory for regions with irregular boundaries (Russian), Zapiski nauč. sem. Leningrad. otd. MIAN 3 (1967).

[3] *Ju. D. Burago, V. G. Mazja* and *V. D. Sapožnikova:* On the theory of potentials of a double and a simple layer for regions with irregular boundaries (Russian), Problems Math. Anal. Boundary Value Problems Integr. Equations (Russian), 3—34, Izdat. Leningrad. Univ., Leningrad, 1966.

[4] *J. Král:* The Fredholm method in potential theory, Trans. Amer. Math. Soc. *125* (1966), 511—547.

[5] *N. L. Landkof:* Fundamentals of modern potential theory (Russian), Izdat. Nauka, Moscow, 1966.

[6] *I. Netuka:* The Robin problem in potential theory, Comment. Math. Univ. Carolinae *12* (1971), 205—211.
[7] *I. Netuka:* Generalized Robin problem in potential theory, Czechoslovak Math. J. *22* (*97*) (1972), 312—324.
[8] *I. Netuka:* An operator connected with the third boundary value problem in potential theory, Czechoslovak Math. J. *22* (*97*) (1972), 462—489.
[9] *J. Plemelj:* Potentialtheoretische Untersuchungen, Leipzig, 1911.
[10] *J. Radon:* Über die Randwertaufgaben beim logaritmischen Potential, Sitzungsber. Akad. Wiss. Wien (2a) 128 (1919), 1123—1167.
[11] *F. Riesz* and *B. Sz. Nagy*: Leçons d'analyse fonctionelle, Budapest, 1952.
[12] *S. Saks:* Theory of the integral, Hafner Publishing Comp., New York, 1937.
[13] *V. D. Sapožnikova:* Solution of the third boundary value problem by the method of potential theory for regions with irregular boundaries (Russian), Problems Math. Anal. Boundary Value Problems Integr. Equations (Russian), 35—44, Izdat. Leningrad. Univ., Leningrad, 1966.
[14] *L. Schwartz:* Théorie des distributions, Hermann, Paris, 1950.
[15] *K. Yosida:* Functional analysis, Springer Verlag, Berlin, 1965.

*Author's address:* Praha 1, Malostranské nám. 25, ČSSR (Matematicko-fyzikální fakulta UK).