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MUTANTS IN THE SYMMETRIC SEMIGROUPS

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Let S_n be the symmetric (or full transformation) semigroup on n letters. A subset K of S_n is called a mutant if KK is contained in the complement of K in S_n. We shall give an explicit form of a maximal mutant of $T_{n-1} = S_n \backslash D_n$, where D_n denotes the D-class of rank n.

1. INTRODUCTION

KIM [6] has established a generalized Green's Lemma and a generalized Clifford and Miller's Theorem in S_n (and in the multiplicative semigroup $L_n(V)$ of all linear transformations of a finite dimensional vector space $V_n(F)$ over a finite field F [5]). In [3] Kim has proved that if T is a topological semigroup and a in T is not an idempotent, then there exists a maximal open mutant of T containing a. (This does not give any information about the actual form of a maximal mutant of a semigroup.) Using a generalized Clifford and Miller's Theorem for S_n, we shall give an explicit form of a maximal mutant of T_{n-1}. (This is the first time an application of a generalized Clifford and Miller's Theorem of S_n has appeared.) In section 2, we shall establish the rank theorem of S_n by modifying the rank theorem of matrices. In section 3, we shall introduce a generalized Clifford and Miller's Theorem and mention a part of a generalized Green's Lemma in S_n. Section 4 contains some basic results for mutants. From section 5 we shall discuss an explicit form of a mutant of S_n.

2. THE RANK THEOREM OF S_n

Let $S = S_n = S_X$ be the symmetric (or full transformation) semigroup on n letters $\{u_1, u_2, \ldots, u_n\} = X$. The basic results of S can be found in [1, pp. 51–57]. From now on S always denotes the symmetric semigroup on X. We may use the (classical) notation if $v_i \in X$ and $\alpha \in S$,

$$\alpha = \begin{pmatrix} u_1 & u_2 & \ldots & u_n \\ v_1 & v_2 & \ldots & v_n \end{pmatrix}$$

to mean that α is the mapping of X defined by $u_i\alpha = v_i$ $(i = 1, 2, \ldots, n)$. With each element α in S we associate two sets: (1) the range $M(\alpha) = X\alpha$ of α, and (2) the partition $N(\alpha)$ corresponding to α. If $M(\alpha) = \{v_i : i = 1, 2, \ldots, r\}$ and if we define $v_i\alpha^{-1} = V_i = \{u\alpha = v_i\}$, then we may write $N(\alpha) = \{V_i : i = 1, 2, \ldots, r\}$; we can write

$$\alpha = \begin{pmatrix} V_1 & V_2 & \ldots & V_r \\ v_1 & v_2 & \ldots & v_r \end{pmatrix} = (V_1, V_2, \ldots, V_r;\ v_1, v_2, \ldots, v_r) = (V_i, v_i : i = 1, 2, \ldots, r)\,.$$

$|M(\alpha)| = r = \varphi(\alpha)$ is called the rank of α. The rank of the partition $N(\alpha)$ is defined by $r = \varphi(\alpha)$. The following is called the rank theorem of matrices [4].

Theorem A. *Let α and β be two elements of the multiplicative semigroup $L_n(V)$ of all linear transformations of the n-dimensional vector space $V_n(F)$ over a field F. Let $\varphi(\alpha)$ denote the rank of α. Then $\varphi(\alpha\beta) = \varphi(\alpha) - \dim(M(\alpha) \cap N(\beta))$, where $M(\alpha)$ and $N(\beta)$ denote the range space of α and the null space of β, respectively.*

Although we know that Theorem A is applicable in the semigroup S_n in view of Exercise 6-(e) of [1-I, p. 57], the meaning of $\dim(M(\alpha) \cap N(\beta))$ is not so clear when α and β are members of S_n. To get the analogue of Theorem A in S_n we need:

Definition 1. Let α and β be elements of S_n and let $N(\beta) = \{V_1, V_2, \ldots, V_r\}$. Define

$$\|M(\alpha) \cap V_i\| = \begin{cases} |M(\alpha) \cap V_i| - 1 & \text{if } |M(\alpha) \cap V_i| \geqq 1\,, \\ 0 & \text{otherwise}\,; \end{cases}$$

and

$$\|M(\alpha) \cap N(\beta)\| = \sum_{i=1}^{r} \|M(\alpha) \cap V_i\|\,.$$

Now Theorem A takes the following form in S_n.

Theorem 1. *Let α and β be two elements of the symmetric semigroup S_n. Then $\varphi(\alpha\beta) = \varphi(\alpha) - \|M(\alpha) \cap N(\beta)\|$.*

The proof of Theorem 1 is not hard and shall be omitted.

3. A GENERALIZED CLIFFORD AND MILLER'S THEOREM AND GREEN'S LEMMA

Kim [6] has established the following and Theorem B is considered as a generalized Clifford and Miller's Theorem [1, Theorem 2.17].

Theorem B. (i) *Let $\alpha \in S_n$ and H be an H-class. Then $\alpha H = \bigcup\{H_{\alpha\tau} : \tau \in H\}$.*
(ii) *Let H_i $(i = 1, 2)$ be two H-classes. $H_1H_2 = \bigcup\{H_{\alpha\beta} : \alpha \in H_1$ and $\beta \in H_2\}$.*
(iii) *$\alpha\beta \in D_r$ iff D_r is the D-class of the maximal rank such that $\beta S \cap S\alpha \cap D_i$ contains an idempotent, where $\beta \in S_n$.*

Theorem C. *Let α and β be elements of S_n such that $\alpha \in D_{n-1}$, $\beta \in D_{n-2}$ and $\beta = \gamma\alpha$.*

(i) *Then the mapping $\tau \to \gamma\tau$ $(\tau \in H_\alpha)$ is a one-to-one mapping from H_α onto γH_α.*

(ii) *$\gamma H_\alpha = \bigcup\{H_{\gamma\tau} : \tau \in H_\alpha\}$ and $\{H_{\gamma\tau} : \tau \in H_\alpha\}$ contains $n - 1$ distinct H-classes of rank $n - 2$.*

Theorem C is taken from a generalized Green's Lemma in S_n [6]. To get a Theorem, which we shall need later, we introduce:

Definition 2. (i) $\pi(X)$ denotes the collection of all partitions on X and define $\pi_r(X) = \{N \in \pi(X)$: the rank of N is $r\}$.

(ii) $p(X)$ denotes the collection of all non-empty subsets of X, and $p_r(X) = \{Y \in p(X) : |Y| = r\}$.

(iii) If $N = \{V_1, V_2, \ldots, V_r\} \in \pi_r(X)$, then V_i is called a block of N.

(iv) Let $N_i \in \pi_r(X)$ $(i = 1, 2)$. If for every block V of N_1 there is a block U of N_2 such that V is a subset of U, then we write $N_1 \subset N_2$.

Let $\alpha \in S$. By Lemmas 2.5, 2.6 and 2.7 in [1], if R_α, L_α, H_α, D_α denote, respectively, the R, L, H, D-class containing α, then we can write $N(\alpha) = N(H_\alpha) = N(R_\alpha)$ and $M(\alpha) = M(H_\alpha) = M(L_\alpha)$.

Lemma 1. *$M(\alpha\beta) \subset M(\beta)$ and $N(\alpha) \subset N(\alpha\beta)$.*

Theorem D. *If H_i $(i = 1, 2)$ are two H-classes of rank $n - 1$ and if $H_1 H_2 \subset D_{n-2}$, then $H_1 H_2 = \bigcup\{H_{\alpha\beta} : \alpha \in H_1$ and $\beta \in H_2\} = \bigcup F$ and F contains $(n - 1)^2 \cdot (n - 2)/2$ H-classes such that if $N \in \pi_{n-2}(X)$ and $M \in p_{n-2}(X)$ with $M \subset M(H_2)$ and $N(H_1) \subset N$, then there is $H_{\alpha\beta}$ in F with $N(H_{\alpha\beta}) = N$ and $M(H_{\alpha\beta}) = M$.*

The proof of Theorem D follows from applications of Theorems B, C and Lemma 1.

4. PRELIMINARY RESULTS

We rewrite Theorem 2.10-(i) of [1] in the following:

Lemma 2. *Let H be an H-class of rank r with $H = (V_i, v_i)$. Then H contains an idempotent iff $|V_i \cap M(H)| = 1$ for every V_i.*

Definition 3. (i) Y in $p_r(X)$ is said to be a cross section of $N \in \pi_r(X)$, denoting by $Y \# N$, if every block of N contains just one element of Y.

(ii) Let $N \in \pi(X)$ and $M \in p(X)$. A pair $[N, M]$ is called a partitionrange. $\pi_r x p_s(X) = \{[N, M] : N \in \pi_r(X)$ and $M \in p_s(X)\}$.

(iii) Let $[N, M] \in \pi_t xp_s(X)$. $[N, M](D_r) = \{\beta \in D_r : M(\beta) \subset M$ and $N \subset N(\beta)\}$.

(iv) A non-empty subset A of $\pi_r xp_r(X)$ is called a section (of $\pi_r xp_r(X)$) if there are two sets I and J such that $A = \{[N_j, M_i] : j \in J$ and $i \in I\}$.

(v) A section A as in the above (iv) is said to be idempotent free if non $N_j(j \in J)$ is a cross section of $M_i(i \in I)$.

(vi) Two sections A_1 and A_2 in $\pi_r xp_r(X)$ are said to be orthogonal if $N_j \neq N_t$ and $M_i \neq M_s$ for all elements $[N_j, M_i]$ of A_1 and all elements $[N_t, M_s]$ of A_2.

(vii) Two sections A_1 and A_2 are said to be parallel if for each $[N_j, M_i]$ in A_1 there is $[N_t, M_s]$ in A_2 such that $M_i = M_s$ and vice versa.

(viii) A collection $F = \{A_i \in \pi_r xp_r(X)\}$ of sections A_i is said to be orthoparallel if any two distinct elements A_i and A_j in F are either orthogonal or parallel and if there are no partitionranges $[N, M]$ in A_i and $[N', M']$ in A_j with $N = N'$ for $i \neq j$.

Definition 4. A subset K of a semigroup S is called a mutant if $KK \subset S \setminus K$.

Lemma 3. *Let $F = \{A_i \in \pi_r xp_r(X)\}$ be a collection of orthoparallel and idempotent free sections. Then $F(D_r) = \{\alpha \in D_r : [N(\alpha), M(\alpha)] \in \bigcup\{A_i \in F\}\}$ is a mutant.*

Proof. (1) Let $A_i \in F$ and let α and β be elements of $A_i(D_r)$. Since A_i is idempotent free, there is a block V of $N(\beta)$ such that V contains at least two elements of $M(\alpha)$ and hence $\|M(\alpha) \cap N(\beta)\| \geqq 1$. By Theorem 1, $\varphi(\alpha\beta) = \varphi(\alpha) - \|M(\alpha) \cap N(\beta)\| < r$, and hence $\alpha\beta \notin F(D_r)$.

(2) Let A_i and A_j be elements of F. Let $\alpha \in A_i(D_r)$ and $\beta \in A_j(D_r)$. If A_i and A_j are parallel, then $A_i \cup A_j$ is a section. By (1), $\alpha\beta \notin F(D_r)$. If A_i and A_j are orthogonal, then we have that either $\varphi(\alpha\beta) = r$ or $\varphi(\alpha\beta) < r$; the latter case we have $\alpha\beta \notin F(D_r)$. If $\varphi(\alpha\beta) = r$, then, by Theorem 2.17 of [1], $H_\alpha H_\beta = H_{\alpha\beta}$. From $N(\alpha) = N(H_{\alpha\beta})$ and $M(\beta) = M(H_{\alpha\beta})$, it follows that $[N(\alpha), M(\beta)] \notin (\bigcup\{A_i \in F\})$, whence $\alpha\beta \notin F(D_r)$. Thus $F(D_r)$ is a mutant of S_n.

Lemma 4. *Let $A_1 = \{[N_j, M] \in \pi_r xp_r(X) : j \in J\}$, $A_2 = \{[N, M_i] \in \pi_r xp_r(X): : i \in I\}$ and $A_3 = \{[N_j, M_i] \in \pi_r xp_r(X) : j \in J$ and $i \in I\}$. If M is a cross section of N, then $A_1(D_r)\, A_2(D_r)$ contains $A_3(D_r)$.*

Proof. Let $H = (N_j, M_i) \in A_3(D_r)$ be an H-class determined by N_j and M_i. Choosing two H-classes $H_1 = (N_j, M) \in A_1(D_r)$ and $H_2 = (N, M_i) \in A_2(D_r)$, we have that $H_1 H_2 = H$ by Theorem 2.17 of [1]. This completes the proof.

Definition 5. (i) Let $A_1 = \{[N_j, M_i] \in \pi_r xp_r(X) : j \in J_1$ and $i \in I_1\}$ and $A_2 = \{[N_j, M_i] \in \pi_r xp_r(X) : j \in J_2$ and $i \in I_2\}$ be two orthogonal sections. Then $A_3 = \{[N_j, M_i] : j \in J_1$ and $i \in I_2\}$ and $A_4 = \{[N_j, M_i] : j \in J_2$ and $i \in I_1\}$ are called the right and left complementary sections of A_3 and A_4, respectively.

(ii) If $A_3(A_4)$ has an element $[N, M]$ with $M \# N$, then we shall say that the right (left) complementary section of A_1 and A_2 has a cross section. If A_3 and A_4 have cross sections then we shall say that the complementary sections of A_1 and A_2 have cross sections.

Lemma 5. *If A_i $(i = 1, 2)$ are orthogonal in $\pi_r xp_r(X)$ and if the left (right) complementary section $A_4(A_3)$ of A_1 and A_2 has a cross section, then $A_1(D_r)\,A_2(D_r)$ $(A_2(D_r)\,A_1(D_r))$ contains $A_3(D_r)\,(A_4(D_r))$.*

The proof of the lemma follows from Lemma 4.

5. MUTANTS IN S_n

To get an explicit form of a mutant of T_{n-1}, we introduce:

Definition 6. (i) Define $M_i = X \backslash u_i$. N_{ij} denotes a partition of rank $n - 1$ having one block consisting of two elements u_i and u_j $(i < j)$. $\bar{m}$ denotes the set $\{1, 2, \ldots, m\}$.

(ii) (ij) denotes a sequence from the set $\bar{n}$ with $i < j$. Let (ij) and (st) be two distinct sequences from the set $\bar{n}$. $(ij) < (st)$ if either $j < t$ or $j = t$ and $i < s$. Letting $(n_1 n_2) < (m_1 m_2)$, define $[n_1 n_2, m_1 m_2] = \{N_{ij} \in \pi_{n-1}(X) : (n_1 n_2) \leqq (ij) \leqq (m_1 m_2)\}$.

(iii) Let $t_1, t_2 \in \bar{n}$ with $t_1 < t_2$. Define $[t_1 t_2] = \{M_i \in p_{n-1}(X) : i = t_1, t_1 + 1, \ldots \ldots, t_2\}$.

(iv) $[n_1 n_2, m_1 m_2]\,[t_1] = \{[N_{ij}, M_{t_1}] : N_{ij} \in [n_1 n_2, m_1 m_2]\}$, $[n_1 n_2]\,[t_1, t_2] = \{[N_{n_1 n_2}, M_t] : M_t \in [t_1, t_2]\}$, and $[n_1 n_2, m_1 m_2]\,[t_1, t_2] = \{[N_{ij}, M_t] : N_{ij} \in [n_1 n_2, m_1 m_2]$ and $M_t \in [t_1, t_2]\}$.

(v) $K_3 = \{[N_{12}, M_3], [N_{13}. M_2], [N_{23}, M_1]\}$, $K_4 = \{[12, 23]\,[4] \cup [14, 24]\,.\,[3] \cup [34]\,[1, 2]$.

(vi) $K_5 = \bigcup_{i=1}^{4} A_{5i}$, $A_{51} = [12, 24]\,[4, 5]$, $A_{52} = [14]\,[2, 3] \cup [15]\,[2, 3]$, $A_{53} = [24, 34]\,[1]$, and $A_{54} = [25, 45]\,[1]$. $K_t = \bigcup_{i=1}^{4} A_{ti}\,(t = 6, 7, 8)$, $A_{61} = [12, 34]\,.\,[5, 6]$, $A_{62} = [15, 25]\,[3, 4]$, $A_{63} = [35, 16]\,[2]$, $A_{64} = [26, 56]\,[1]$, $A_{71} = [12, 45]\,[6, 7]$, $A_{72} = [16, 36]\,[4, 5]$, $A_{73} = [46, 17]\,[2, 3]$, $A_{74} = [27, 67]\,[1]$, $A_{81} = [12, 56]\,[7, 8]$, $A_{82} = [17, 47]\,[5, 6]$, $A_{83} = [57, 28]\,[3, 4]$ and $A_{84} = [38, 78]\,[1, 2]$.

(vii) For $9 \geqq n$, $A(n) = \bigcup_{i=1}^{4} A_{ni}$, $A_{n1} = [1\ n-3,\ n-3\ n-2]\,[n-1, n]$, $A_{n2} = [1\ n-1, n-4\ n-1]\,[n-3, n-2]$, $A_{n3} = [n-3\ n-1, n-6\ n]\,[n-5, n-4]$, and $A_{n4} = [n-5\ n, n-1\ n]\,[n-7, n-6]$.

(viii) K_n $(n = 5, 6, 7, 8)$ and $A(n)$ are called ladders. A_{ni} is called the ith step of the ladder $A(n)$ or K_n.

(ix) If n takes the form $n = 4m + 4 + i$ for $1 \leqq i \leqq 4$, then we define $K_n = A(n) \cup A(n-4) \cup \ldots \cup A(n - (m-1)\,4) \cup K_{4+i}$.

Lemma 6. *Let $n = 4m + 4 + i$ $(1 \leqq i \leqq 4)$. K_n $(n \geqq 5)$ is a union of orthoparallel and idempotent free sections A_{tj} $(j = 1, 2, 3, 4)$ for $t \in \{n, n - 4, \ldots, n - (m - 1) 4\}$.*

Proof. We can see that K_{4+i} $(i = 1, 2, 3, 4)$ is a union of orthoparallel and idempotent free sections. We can also see that each $A(n - 4k)$ $(k = 0, 1, \ldots, m - 1)$ is a union of orthoparallel and idempotent free sections by Definition 6-(vi). Now consider $A(t)$ and $A(t - 4)$. The third step of the ladder $A(t)$ is parallel to the first step of $A(t - 4)$ and the end of the first step of the ladder $A(t)$ is touched at the top of the fourth step of the ladder $A(t - 4)$. This is true for $t = n - 4k$. Let us consider $A(n - (m - 1) 4)$ and K_{4+i}. In the above statement, we can replace $A(t)$ by $A(n - (m - 1) 4)$ and $A(t - 4)$ by K_{4+i}. It is clear that A_{tj} is idempotent free by construction. This proves the lemma.

Lemma 7. (i) $K_n(D_{n-1})$ $(n = 3, 4, 5, 6, 7, 8)$ *is a mutant of* S_n.

(ii) $K_n(D_{n-1})$ $(n \geqq 9)$ *is a mutant of* S_n.

The proof follows from Lemmas 3 and 6.

6. $K_3(D_{3-1})$ IS A MAXIMAL MUTANT OF T_{3-1}

We see that

$$K_3(D_{3-1}) = \left\{\begin{pmatrix} \{u_1, u_2\} & \{u_3\} \\ u_1 & u_2 \end{pmatrix}, \begin{pmatrix} \{u_1, u_2\} & \{u_3\} \\ u_2 & u_1 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \{u_1, u_3\} & \{u_2\} \\ u_1 & u_3 \end{pmatrix}, \begin{pmatrix} \{u_1, u_3\} & \{u_2\} \\ u_3 & u_1 \end{pmatrix}, \begin{pmatrix} \{u_1\} & \{u_2, u_3\} \\ u_2 & u_3 \end{pmatrix}, \begin{pmatrix} \{u_1\} & \{u_2, u_3\} \\ u_3 & u_2 \end{pmatrix}\right\}$$

is a mutant of S_3 by Lemma 7-(i). Since D_{3-2} is the set of idempotents and $D_{3-1} \cdot K_3(D_{3-1})$ is a union of H-classes each of which is a group, $K_3(D_{3-1})$ is a maximal mutant in $T_{3-1} = D_{3-1} \cup D_{3-2}$.

7. $K_8(D_{8-1})$ IS A MAXIMAL MUTANT OF T_{8-1}

(i) $K_8(D_{8-1}) = \bigcup_{i=1}^{4} A_{8i}(D_{8-1})$, and A_{8i} are orthogonal and idempotent free sections.

(ii) Since any two distinct sections A_{8i} and A_{8j} have cross sections, $\bigcup\{A_{8i}(D_{8-1}) A_{8j}(D_{8-1})\}$ contains $D_{8-1} \setminus \bigcup_{i=1} A_{8i}(D_{8-1})$ by Lemma 5.

(iii) In K_8, for each N_{ij}, there are two partitionranges $[N_{ij}, M_k]$ and $[N_{ij}, M_{k-1}]$. Thus if $H_1 = (N_{ij}, M_k)$ and $H_2 = (N_{ij}, M_{k-1})$, then by Theorem D, $H_1 H_2 \subset D_{8-2}$

and $H_1H_2 \cup H_2H_1$ contains $(n-1)(n-2)/2$ R-classes R_i such that if N is a partition of rank 6 with $N_{ij} \subset N$ then there is R_i such that $N(R_i) = N$ for some i. Hence we can infer that $\bigcup_{i,j=1}^{4} \{A_{8i}(D_{8-1})\, A_{8j}(D_{8-1})\}$ contains D_{8-2}.

Definition 7. (i) Let $M \in p_{n-3}(X)$ and let $[N_{ij}, M_t] \in \pi_{n-1} x p_{n-1}(X)$. If $M \subset M_t$ and $\{u_i, u_j\} \not\subset M$, then we say that M is passable to $[N_{ij}, M_t]$.

(ii) Let $[N_{ij}, M_t] \in A$. If Y is passable to $[N_{ij}, M_t]$, then we shall say that Y is pasaable to A.

(iv) We can check that for any element M in $p_{8-3}(X)$, there is $[N_{ij}, M_t]$ in K_8 to which M is passable.

For example, if $M = \{u_1, u_2, u_3, u_4, u_5\} \in p_{8-3}(X)$, then taking $[N_{56}, M_8]$ in A_{81}, M is passable to $[N_{56}, M_8]$. Hence every element Y in $p_{8-3}(X)$ is passable to K_8.

Lemma 8. *If $Y \in p_{n-3}(X)$ is passable to $[N_{ij}, M_t]$, then*

(i) *a subset M of Y is passable to $[N_{ij}, M_t]$;*

(ii) *letting $H = (N_{ij}, M_t)$ and $\alpha = (N_1, Y) \in D_{n-3}$ for $N_1 \in \pi_{n-3}(X)$, αH contains α;*

(iii) *if $\beta = (N_2, M) \in D_{n-3-i}$, then βH contains β.*

Proof. (i) follows from Definition 7. (ii) By Definition 7, $\|M(\alpha) \cap N_{ij}\| = 0$; $\varphi(\alpha\beta) = \varphi(\alpha)$ for $\beta \in H$, by Theorem 1. It follows from $N(\alpha\beta) \subset N(\alpha)$ and $M(\alpha\beta) \subset \subset M(\beta) \supset Y = M(\alpha)$ that $\alpha \in \alpha H$. The proof of (iii) is analogous as the above.

Finally we have that if $\alpha \in D_{8-3-i}$ then $\alpha K_8(D_{8-1})$ contains α, whence $K_8(D_{8-1})$ is a maximal mutant of T_{8-1} in view of (ii), (iii) and Lemma 8.

8. $K_n(D_{n-1})$ $(9 \geqq n)$ AND D_{n-i} $(i \geqq 2)$

We begin with

Lemma 9. *$K_n(D_{n-1})\, K_n(D_{n-1})$ contains D_{n-2}.*

Proof. Let $H = (H_1, Y)$ be an H-class in D_{n-2}. Then there exist two partitionranges $[N_{ij}, M_t]$ and $[N_{ab}, M_s]$ in K_n with $N_{ij} \subset N_1$ and $Y \subset M_s$. Let $H_1 = (N_{ij}, M_t)$ and $H_2 = (N_{ab}, M_s)$ be H-classes determined by the partitionranges. Then H_1H_2 contains H by Theorem D. This completes the proof.

Lemma 10. *Every element Y in $p_{n-3}(X)$ is passable to K_n for $n = 4, 5, 6, 7, 8$.*

Lemma 11. *Let $n \geqq 9$.* (i) $p_{n-3}(X) = \bigcup_{i=1}^{n-2} Q_i$, *where* $Q_1 = \{Y \in p_{n-3}(X) : n \notin Y\}$, $Q_k = \{Y \in p_{n-3}(X) \setminus \bigcup_{i=1}^{k-1} Q_i : n-k+1 \notin Y\}$ $(k = 1, 2, \ldots, n-2)$.

(ii) *Every Y in* $\bigcup_{i=1}^{4} Q_i$ *is passable to* $A(n)$.

(iii) *Let* $W = \{a_1, a_2, \ldots, a_s\}$ *be a set of s elements and let* $Z = \{b_1, b_2, \ldots, b_{4t}\}$ *be a set of 4t elements with* $W \cap Z = \emptyset$, *the empty set. Let* $Y_1 \in p_{s-3}(W)$ *and* $[N_1, M_1] \in \pi_{s-1} x p_{s-1}(W)$ *such that* Y_1 *is passable to* $[N_1, M_1]$. *Then setting* $N_2 = \{N_1, \{b_1\}, \{b_2\}, \ldots, \{b_{4t}\}\}$, $M_2 = M_1 \cup Z$ *and* $Y_2 = Y_1 \cup Z$, Y_2 *is passable to* $[N_2, M_2]$.

(iv) *Every Y in* $p_{n-3}(X)$ *is passable to* K_n.

Proof. (i) and (iii) are clear.

(ii) Let $Y \in Q_1$. We note that $|Y| = n - 3 < |[1\, n-3, n-3\, n-2]\,[n]| = (n-4) + (n-3)$, and $Y \subset M_n$. If Y contains u_{n-2}, then $Y \backslash u_{n-2}$ contains $n-4$ elements. But in $[1\, n-3, n-3\, n-2]\,[n]$ there are $n-3$ distinct partitionranges of the form $[N_{in-2}, M_n]$ $(i = 1, 2, \ldots, n-3)$. Thus there must exist a partitionrange $[N_{in-2}, M_n]$ such that Y is passable to $[N_{in-2}, M_n]$ for some $i \in \{1, 2, \ldots, n-3\}$. If Y does not contain u_{n-2}, then Y is passable to $[N_{in-2}, M_n]$ for all $i \in \{1, 2, \ldots, n-3\}$. Similarly, we can show that for any Y in $\bigcup_{i=1}^{4} Q_i$ there is a partitionrange $[N_{ij}, M_t]$ in $A(n)$ to which Y is passable.

(iv) We shall prove the part (iv) by induction on $n = 4m + 4 + i$. If $n = 4 + i$ $(i \leqq 4)$ then it follows from Lemma 10. We assume that we have been proved that (iv) holds for $n \leqq t - 1$. Now let $n = t$. From (iii) and the inductional assumption, it follows that any Y in $\bigcup_{n-2}^{4(k+1)} \bigcup_{4k+1} Q_i$ is passable to $A(n - 4k)$ $(0 \leqq k \leqq m - 1)$; similarly, every Y in $\bigcup_{4m+1} Q_i$ is passable to K_{4+i}. This proves Lemma 11. By Lemmas 7, 8, 10 and 11, we have that if $\alpha \in D_{n-i}$ $(i \geqq 3)$, then $\alpha K_n(D_{n-1})$ contains α.

9. $K_n(D_{n-1})$ AND D_{n-1} $(9 \geqq n)$

Let $n = 4m + 4 + i$ $(1 \leqq i \leqq 4)$. Consider $A(s)$. We abbreviate $[N_{ij}, M_t]$ as $[ij, t]$.

Definition 8. (i) $(-A(s)) = \{[ij, t] : (1\, s-3) \leqq (ij) \leqq (s-1\, s) \text{ and } t \geqq s+1\} \cup \{[ij, t] : (1\, s-1) \leqq (ij) \leqq (s-1\, s) \text{ and } t = s, s-1\} \cup \{[ij, t] : (s-3\, s-1) \leqq (ij) \leqq (s-1\, s);\ t = s-2, s-3\} \cup \{[ij, t] : (s-5\, s) \leqq (ij) \leqq (s-1\, s);\ t = s-4, s-5\}$.

(ii) $(-K_n) = (-A(n)) \cup (-A(n-4)) \cup \ldots \cup (-A(n-(m-1)4)) \cup (-K_{4+i})$ is called the left complementary set of K_n.

(iii) $(K_n -) = \pi_{n-1} x p_{n-1}(X) \backslash K_n \cup (-K_n)$ is called the right complementary set of K_n.

Lemma 12. (i) $K_n(D_{n-1})\,K_n(D_{n-1})$ *contains* $(-K_n)\,(D_{n-1})$.

(ii) If $\alpha \in (K_n-)\,(D_{n-1})$, *then* $\alpha K_n(D_{n-1})$ *contains* α.

Proof. (i) follows from Lemma 4 (and Lemma 5 if necessary).

(ii) For any partitionrange $[N_{ij}, M_t]$ in (K_n-), there are $[N_{ij}, M_s]$ and $[N_{ab}, M_s]$ in K_n, and $[N_{ab}, M_t]$ in (K_n-) such that M_t is a cross section of N_{ab}. Therefore, taking $H_1 = (N_{ij}, M_t)$ and $H_2 = (N_{ab}, M_s)$ we have $H_1 H_2 = (N_{ij}, M_s)$. This proves the part (ii).

Finally we have:

Theorem 2. *Let* S_n *be the symmetric semigroup on n letters.*

(i) Let $n \geqq 3$. $K_n(D_{n-1})$ is a mutant of S_n.

(ii) $K_n(D_{n-1})$ is a maximal mutant of T_{n-1}.

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