Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 17 (1967), No. 4, 535–539

Persistent URL: http://dml.cz/dmlcz/100800

Czechoslovak Mathematical Journal, 17 (92) 1967, Praha

ABELIAN GROUPS WITH IDENTICAL RELATIONS

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(Received May 20, 1966)

Recently there have appeared some papers about the definition of groups by one postulate. Most authors show that classes of groups or the class of all groups can be defined by imposing one postulate on group division (For convenience group division will be denoted by a dot or juxtaposition). The principal aim of this paper is a generalization of results by J. MORGADO [1], HIGMAN and NEUMANN [2] and SHOLANDER [3].

Suppose the groupoid $\langle G, .\rangle$, consisting of a nonempty set $G = \{a, b, c \ldots\}$ and a binary operation . defined everywhere in G, satisfies the identity $b = a(cb \,.\, ca)$. Further let the unary operation $'$ (inversion) be defined by $a' = aa \,.\, a$ and set $a \circ b = ab'$. Then, as Higman and Neumann have shown, the ordered triple $\langle G, \circ, '\rangle$ is an abelian group.Conversely, if $\langle G, \circ, '\rangle$ is an abelian group and . is defined by $a \,.\, b = a \circ b'$ the identity $b = a(cb \,.\, ca)$ is satisfied in $\langle G, .\rangle$.

We remark that the correspondence between $\langle G, \circ, '\rangle$ and $\langle G, .\rangle$ is one-to-one if $\langle G, \circ, '\rangle$ is a group, as can be easily seen from the following equations:

$$a \circ b = a(bb \,.\, b) = a \circ ((b \circ b') \circ a')' = a \circ b$$
$$a' = aa \,.\, a = (a \circ a') \circ a' = a'$$

This permits us to formulate the following theorem:

Theorem 1. *The ordered triple $\langle G, \circ, '\rangle$ is an abelian group if and only if one of the following equivalent conditions is satisfied in $\langle G, .\rangle$:*

MA: $ab \,.\, c = ad \,.\, e$ *implies* $b = d \,.\, ce$

HA: $b = a(cb \,.\, ca)$

SA: $b = a(ac \,.\, bc)$

Condition MA is from J. Morgado [1], HA, as mentioned before, from G. Higman and B. H. Neumann [2] and SA from Sholander [3].

The class of all abelian groups is the class of all groups satisfying the identity $a' \circ b' \circ a \circ b = I$ where I is the unit element. Now suppose $W(x_1, x_2, \ldots, x_m)$ is a word in the group $\langle G, \circ, '\rangle$ and consider the class of all groups satisfying the relation $W = I$. Further let $W'(y_1, \ldots, y_n)$ be another such word. Then the class of all groups satisfying both $W = I$ and $W' = I$ is the class of all groups satisfying

$$(1) \qquad W(x_1, x_2, \ldots, x_m) \circ W'(y_1, \ldots, y_n) = I\,,$$

for if we set $y_1 = y_2 = \ldots = y_n = I$ in (1) we get $W(x_1, \ldots, x_m) = I$. Similarly one deduces $W' = I$ from (1). Analogously the class of all groups satisfying any finite set of identical relations is identical with the class of all groups satisfying a single appropriately chosen relation.

If W is a word in the group $\langle G, \circ, '\rangle$ we can transform W into a word w in $\langle G, .\rangle$ by making use of the relations $a \circ b = a(bb \,.\, b)$ and $a' = aa \,.\, a$. In addition to it we can always retransform w into W by virtue of $ab = a \circ b'$ and the group properties of $\langle G, \circ, '\rangle$.

Theorem 2. *Let $w(x_1, \ldots, x_m)$ be a word in the groupoid $\langle G, .\rangle$. Then $\langle G, \circ, '\rangle$ is an abelian group satisfying the identical relation corresponding to $w = I$ if and only if one of the following equivalent conditions is satisfied in $\langle G, .\rangle$:*

A1: $ab \,.\, c = ad \,.\, e$ *implies* $b = dw \,.\, ce$

A2: $b = aw \,.\, (cb \,.\, ca)$

A3: $b = aw \,.\, (ac \,.\, bc)$

Proof. It is easily seen that A1, A2 and A3 are satisfied if $\langle G, \circ, '\rangle$ is an abelian group satisfying the identical relation corresponding to $w = I$. To show the converse it suffices by Theorem 1 to show that A1 implies MA, A2 implies HA, A3 implies SA and that A1, A2 and A3 each imply $w = I$.

1. A1 implies MA. Since $ab \,.\, c = ab \,.\, c$ we have by A1

$$(1) \qquad b = bw \,.\, cc \quad \text{for all} \quad b, c \in G\,.$$

Now suppose $ab = ac$. Then $ab \,.\, d = ac \,.\, d$ and this implies by A1 $b = cw \,.\, dd$, which gives by (1) $b = c$. So we have the left cancellation.

From $bw \,.\, cc = bw \,.\, dd$, which holds by (1) for any $c, d \in G$, it follows by left cancellation that $cc = dd$. Thus, cc does not depend on c and we set $cc = i$. Now clearly, by (1), $w = ww \,.\, cc = ii = i$. Thus

$$(2) \qquad b = bi \,.\, i$$

By the foregoing $ai \,.\, ai = aa \,.\, i$, which implies by A1 $i = aw \,.\, (ai \,.\, i)$. By (2) it follows $i = aw \,.\, a$. However, also $i = aw \,.\, aw$, so that by left cancellation $a = aw$, and this means A1 implies MA. Therefore $\langle G, \circ, '\rangle$ is a group and $w = i = cc = c \circ c' = I$.

2. A2 implies HA. Let $R(a)$ be a mapping of G into G defined by $x\,R(a) = xa$ ($R(a)$ applied to x gives xa), let $L(a)$ be defined by $x\,L(a) = ax$ and let J denote the identity mapping. Then A2 can be written in the form:

$$L(c)\,R(ca)\,L(aw) = J\,. \tag{3}$$

Now we make use of the fact that if S and T are two single-valued mappings of G with $ST = P$, where P is a permutation of G, then T is onto G and S is one-to-one. Thus $L(aw)$ is onto G and $L(c)$ is one-to-one. Since c can be any element of G the application $L(aw)$ is one-to-one. Now $L(aw)$ is a one-to-one mapping of G onto G, that is a permutation, and has an inverse. Hence (3) becomes:

$$L(c)\,R(ca) = L(aw)^{-1} \tag{4}$$

and $R(ca)$ is onto G. Because the right side of (4) is independent of c we have

$$L(c)\,R(ca) = L(b)\,R(ba)\,, \tag{5}$$

or, by applying both sides to a, $ca\,.\,ca = ba\,.\,ba$ for all a, b, c. By substituting ca for a this gives $(c\,.\,ca)\,(c\,.\,ca) = (b\,.\,ca)\,(b\,.\,ca)$. Now let a and c be fixed elements of G. Then the right side of this equation is also a fixed element of G, say i. Since $R(ca)$ is onto G there is a b for any d such that $b\,.\,ca = d$. Thus we have

$$i = dd \quad \text{for any} \quad d\,. \tag{6}$$

By A2 $w = ww\,.\,(ww\,.\,ww)$, which gives by (6) $w = i$. This, A2 and (6) yield $i = ai\,.\,(i\,.\,ia)$. On the other hand, $L(aw) = L(ai)$ is a permutation and $i = ai\,.\,ai$. Thus,

$$ai = i\,.\,ia \tag{7}$$

By applying both sides of (5) to i we get $ci\,.\,ca = bi\,.\,ba$, which implies $ci\,.\,i = i\,.\,ic$, wherefrom it follows by (7),

$$ai = ai\,.\,i \tag{8}$$

Now, since $R(ca)$ is onto G and $i = ii$, the application $R(i)$ is onto G, too. Therefore every element b of G can be represented in the form ai. So we have by (8) $b = bi$. Hence A2 implies HA and $\langle G, \circ, ' \rangle$ is a group with $w = i = cc = c \circ c' = I$.

3. A3 implies SA. With the same definitions as before we can write A3 in the form

$$R(c)\,L(ac)\,L(aw) = J\,.$$

It follows that $R(c)$ is one-to-one and that $L(aw)$ is an application onto G. That is, to every d there is a c with $d = aw\,.\,c$. Substituting aw for a and b in A3 we have

$$aw = (aw\,.\,w)\,.\,dd \tag{9}$$

For $d = w$ and by left multiplication with $aw \,.\, w$ we get $(aw \,.\, w) \,.\, aw = (aw \,.\, w) \,.$ $.\, (aw \,.\, w)(ww)$. By A3 the right side of this equation is w, so we have

$$(10) \qquad (aw \,.\, w) \,.\, aw = w \,.$$

By another left multiplication with $aw \,.\, w$ we have $(aw \,.\, w) \,.\, (aw \,.\, w)(aw) =$ $= (aw \,.\, w)\, w$; hence, by application of A3 again, it follows for every a,

$$(11) \qquad a = (aw \,.\, w)\, w \,.$$

By setting $a = d = w$ in (9) we get $ww = (ww \,.\, w) \,.\, ww$. On the other hand, it follows from (10), $(ww \,.\, w) \,.\, ww = w$. Thus $w = ww$. For $d = w$ in (9) we have again $aw = (aw \,.\, w)\, w$, but the right side is a by (11). Hence $aw = a$ and A3 implies SA. Thus, $\langle G, \circ, ' \rangle$ is an abelian group and from $a \circ w' = aw = a$ it follows that $w = I$, because the unit element is unique.

For non-abelian groups the following theorem holds:

Theorem 3. *Let $w(x_1, \ldots, x_m)$ be a word in the groupoid $\langle G, . \rangle$. Then $\langle G, \circ, ' \rangle$ is a group satisfying the identical relation corresponding to $w = I$ if and only if one of the following equivalent conditions is satisfied in $\langle G, . \rangle$:*

H: $$a \,.\, ((aa \,.\, w)\, b \,.\, c)\,((aa \,.\, a)\, c) = b$$

BI: $$ab \,.\, c = ad \,.\, e \quad \textit{implies} \quad c = ew \,.\, bd$$

Condition H is from Higman and Neumann [2], BI from G. Baron and the author [7]. The proof of Theorem 3 under condition BI can easily be led in such a way to prove also:

Theorem 4. *The ordered triple $\langle G, \circ, ' \rangle$ is a group if and only if the following condition is satisfied in $\langle G, . \rangle$:*

I: $$ab \,.\, c = ad \,.\, e \quad \textit{implies} \quad c = e \,.\, bd$$

Condition I looks very much like MA for abelian groups and is simpler than the equivalent condition

BM: $$a(bb \,.\, b) \,.\, (cc \,.\, c) = a(dd \,.\, d) \,.\, (ee \,.\, e) \quad \text{implies} \quad b = d \,.\, ce \,,$$

which has been found independently by G. Baron [4] and J. Morgado [5]. The above mentioned close relationship between axioms for abelian and non-abelian groups (conditions I and MA) has an analogoue. Slater has shown [6] that the ordered triple $\langle G, \circ, ' \rangle$ is a group if and only if the following condition is fulfilled:

S: $$(a \circ b) \circ c = (a \circ d) \circ e \quad \text{implies} \quad b = d \circ (e \circ c')$$

By the same method it can be shown that $\langle G, \circ, ' \rangle$ is an abelian group if and only if

SA: $$(a \circ b) \circ c = (a \circ d) \circ e \quad \text{implies} \quad c = e \circ (d \circ b') \,.$$

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