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EXTENSION OF LOCAL AND MEDIAL PROPERTIES TO COMPACTIFICATIONS WITH AN APPLICATION TO ČECH MANIFOLDS *)

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To Eduard Čech: In Memoriam

Conditions are investigated under which certain local and medial connectedness properties of a locally compact space extend to compactifications. Some applications are given for continuous mappings and to generalized manifolds in the sense of E. ČECH.

Introduction. This work was originally inspired by a paper of E.ČECH [2] in which he proposed a definition of generalized closed manifold ("absolute n-manifold") according to the following procedure: (1) One first defines the concept of an *orientable* n-dimensional generalized closed manifold; (2) the n-dimensional generalized closed manifold, orientable or non-orientable, is then defined as a compact space in which each point has a neighborhood whose one point compactification is an orientable n-dimensional generalized closed manifold. In considering this mode of definition, one notes that condition (2) does not state that each point is to have *arbitrarily small* neighborhoods of the type described, so that in the case of the orientable closed manifolds, the entire manifold may be taken as the required neighborhood. This raises the question as to whether one could replace (2) by the following: (2') the n-dimensional generalized closed manifold is a compact space in which each point has arbitrarily small neighborhoods whose one point compactifications are orientable n-dimensional generalized closed manifolds.

Now manifolds are locally connected in all dimensions and simple examples show that the one point compactifications of locally connected, locally compact spaces are not generally locally connected. For example, the subspace of the coordinate plane constituted by the set of points $\{(x, y) \mid x \text{ a positive integer}, y \geqq 0\} \cup \{(x, 0) \mid x \geqq 0\}$ is a connected, 1-lc space, but its one point compactification is not 1-lc. We shall show that the requirement in condition (2') would imply the existence for every point

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of arbitrarily small neighborhoods that are r-acyclic for all r (in terms of homology with compact carriers). Since it is well-known that spaces which are lc^n, $n > 0$, do not generally have such acyclic neighborhoods, the question arises as to whether manifolds must possess them. We give an example of a manifold in which such neighborhoods do not exist for a certain point. Consequently, since the construction given can yield a manifold either orientable or non-orientable, to use condition (2′) would imply an inconsistency (more precisely, the orientable case is defined in (1) without imposition of acyclicity on neighborhoods, while (2′) would impose it). This brings out the fact that the difference noted between (2) and (2′) is quite essential.

We shall begin with an investigation of conditions under which local, and related medial, properties of a locally compact space extend to compactifications thereof. In particular, in § 1 we find conditions on a space which ensure that local connectedness properties extend to certain types of compactifications, such as the one point and Freudenthal "end" compactifications.[1]) In § 2, analogous problems concerning medial properties are treated; such medial properties have been systematically discussed in [10]. And, of importance for the study of manifolds, conditions obtained which ensure that the local Betti numbers $p^r(x)$ shall be $\leqq \omega$ in the compactifications. In § 3 some applications are indicated for continuous mappings and in § 4 applications are made to the matters discussed above.

We shall take $p = 0$ in Čech's definition of "n-manifold of rank p" [2], since it is in this form that the resulting manifolds become a subclass of the generalized manifolds currently employed under a number of equivalent definitions (see for instance [6; VIII]) including Čech's earlier definitions (see references in [2]). For purposes of the present paper only, we designate the former by the term "Čech manifolds"; and as for the latter, we use the symbol "n-gm" to denote "n-dimensional generalized manifold" and "n-gcm" to denote "n-dimensional generalized closed manifold".

1. Extension of local connectedness properties to compactifications. Since the case of the common one point compactification is so simple, we dispose of it separately, saving generalization for subsequent treatment. We employ Čech homology and cohomology with coefficients in an arbitrary algebraic field; the ordinary homology and cohomology groups are indicated by use of the capital H – thus "$H_n(X)$" denotes homology group of X. Since we make such frequent use of the "compact" groups (based on compact carriers of chains and cycles), we indicate these by the lower case "h" – as in "$h_n(X)$". By "$p_n(X)$" and "$p^n(X)$" we denote the dimensions of $h_n(X)$ and $h^n(X)$ respectively.

Lemma. *If X is compact and M a closed subset of X such that both $p_n(X)$ and $p_{n-1}(M)$ are finite, $n > 0$, then $p_n(X; M)$ is finite.*

Proof. Immediate consequence of the exactness of the homology sequence of the compact pair X, M.

[1]) For the classical type of local connectedness, this was treated by L. Zippin [11].

Corollary 1.1. *If X is compact and lc^n, $n > 0$, and M is a closed subset of X such that $p_{n-1}(M)$ is finite, then $p_n(X, M)$ is finite.*

Proof. Immediate consequence of the complex-like character of X (see [6; 180]) and the lemma.

Corollary 1.2. *If X can be imbedded as an open subset of an lc^n compact space S, $n > 0$, so that $S - X$ is complex-like in dimensions 0 to $n - 1$, then X is complex-like in its compact cohomology in dimensions 1 to n.*

Proof. We recall $p_r(X, M) = p^r(X - M)$ for a compact pair X, M.

Remark. That $p^0(X)$ is not necessarily finite under the hypothesis of Corollary 1.2 is shown by the example in E^2 of $X = \bigcup_{i=1}^{\infty} X_i$ where $X_i = \{(x, y) \mid x = 1/i, 0 \leqq y < < \infty\}$, and S is the one point compactification of X.

Theorem 1.1. *In order that the one point compactification $\hat{X}$ of a connected and lc^n space X should be lc^n, it is necessary and sufficient that X be complex-like in compact cohomology in dimensions 1 to n.*

Proof of necessity. A consequence of Corollary 1.2.

Proof of sufficiency. Since $H_r(\hat{p}) = 0$ for all r (where $\hat{p} = \hat{X} - X$), $H_r(\hat{X}) \approx \approx H_r(\hat{X}, \hat{p}) \approx h^r(X)$ for $r = 1, \ldots, n$; and $p_r(\hat{X})$ is finite since $h^r(X)$ is of finite dimension. Then by Theorem 4 of [5][2]) if $\hat{X}$ were not lc^n, it would fail to be lc^n at a non-degenerate set of points.

If by "lc$^\infty$" we denote possession of the r-lc property for all r, then we can state a similar theorem for lc$^\infty$ spaces.

Theorem 1.2. *In order that the one point compactification $\hat{X}$ of the connected and lc^∞ space X should be lc^∞, it is necessary and sufficient that the compact cohomology groups of X be finitely generated in all dimensions greater than* 0.

Proof of necessity. As above.

Proof of sufficiency. The proof of Theorem 4 of [5] can be applied to show that the property of being lc$^\infty$ is expansive (op. cit.) relative to the class of compact spaces that are complex-like in all dimensions. The proof of Theorem 1.1 is then adaptable to the present theorem.

2. Relations of medial properties of a space to its compactifications and other types of extension. We recall (see [8]) that a subset M of a space X is said to have property $(P, Q)_r$ if for every canonical pair of open sets P, Q (i. e., $\bar{Q}$ is compact and $P \supset \bar{Q}$), the group $h_r(M \cap Q \mid M \cap P)$ is finitely generated. (By $h_r(U \mid V)$ we denote the image of $h_r(U)$ under the inclusion mapping $U \to V$). Property $(P, Q)^r$ is similarly defined in terms of cohomology.

2) Although the results of [5] were stated only for metric spaces, their extension to the non-metric cases presents no difficulty.

Of equal interest are medial properties defined in terms of bounding (or cobounding): Thus a subset M of X has property $(P, Q, \sim)_r$ if for every canonical pair P, Q of open subsets of X, the image of $i\partial$ in the sequence of homomorphisms

$$h_{r+1}(M, M \cap Q) \xrightarrow{\partial} h_r(M \cap Q) \xrightarrow{i} h_r(M \cap P)$$

where i is induced by inclusion and ∂ by the boundary operator, is finitely generated. The corresponding cohomology property is denoted by $(P, Q, \sim)^r$.

It is clear from their definitions that these medial properties, as applied to *subsets* of a space, are positional or relative in character, inasmuch as the sets P and Q are taken as open in the "parent space". However, as applied to a space X and its topological images, they are topological invariants, since here the sets P and Q are open relative to X (or its images). Consequently in discussions where the medial properties of a space X and those of its compactifications are concerned, it becomes necessary to distinguish between those which are relative to X itself (and hence topological) and those properties of X which are relative to the compactifications (and hence positional); we shall call the former *intrinsic* and the latter *extrinsic*. The following example will make this distinction clearer:

Example. In E^2, let $A = \{(x, y) \mid 0 < x \leqq 1/\pi, y = \sin(1/x)\}$, $B = \{(x, y) \mid x = 0, -1 \leqq y \leqq 1\}$, and let C be an arc joining $(1/\pi, 0)$ and $(0, -1)$ in the fourth quadrant of E^2 but not meeting $A \cup B$ otherwise. Let X be the bounded domain having $A \cup B \cup C$ as boundary, and $\hat{X} = \bar{X}$ (each with the subspace topology induced by the topology of E^2). Then X, as a subspace of $\hat{X}$, does not have property $(P, Q)_0$ extrinsically; however, it does have property $(P, Q)_0$ intrinsically, since X is homeomorphic with the open circular disk bounded by $x^2 + y^2 = 1$.

Remark. Clearly if X, as a subset of a space $\hat{X}$, has one of the medial properties defined above extrinsically, then it has it intrinsically.

To indicate these medial properties over a range of dimensions k to n inclusive, $k < n$, we use pairs of indices; thus "${}_k(P, Q)_n$" indicates property $(P, Q)_r$ for $r = k, k+1, \ldots, n$.

Since many of our conclusions hinge upon certain groups being finitely generated, we shall use the abbreviation "f. g." to denote "finitely generated".

Theorem 2.1. *Let $\hat{X}$ be a locally compact space, T a closed, totally disconnected subset of $\hat{X}$, and $X = \hat{X} - T$. If X has property $(P, Q, \sim)_n$ intrinsically, then $\hat{X}$ has property $(P, Q, \sim)_n$. Conversely, if $\hat{X}$ has property $(P, Q, \sim)_n$, then X has property $(P, Q, \sim)_n$ both extrinsically and intrinsically.*

Proof. Let P, Q be a canonical pair of open subsets of $\hat{X}$; we may assume that $\bar{P}$ is compact. Let $\hat{T} = (\hat{X} - P) \cup \bar{Q} \cup T$. Then $\hat{T}$ is closed, and $\hat{X} - P$ and $\bar{Q}$ are disjoint closed subsets of $\hat{T}$. We assert that there exists an open subset R of $\hat{X}$ such that (1) $P \supset\supset R \supset\supset Q$ and (2) $\hat{T} \cap F(R) = \emptyset$.

To see this, we note that since $\hat{T}$ is a locally compact subspace of $\hat{X}$ and $\bar{Q}$ is a compact subset of $\hat{T}$ there exists a decomposition $\hat{T} = \hat{T}_1 \cup \hat{T}_2$ separate, where $\hat{T}_1 \supset \bar{Q}$ and $\hat{T}_2 \supset \hat{X} - P$. (See [6; 100, Th. 1.3].) For each $x \in \hat{T}_1$ there exists an open subset U_x of $\hat{X}$ such that $\bar{U}_x \subset P$ and $\bar{U}_x \cap \hat{T}_2 = \emptyset$. As $\hat{T}_1$ is compact, a finite number of such sets U_x covers $\hat{T}_1$ and their union, R, is a set of the type desired.

Since $\hat{T}$ is closed, there exist open sets P_1 and Q_1 such that (1) $P \supset\supset P_1 \supset\supset R \supset\supset Q_1 \supset\supset Q$ and (2) $\hat{T} \cap (\bar{P}_1 - Q_1) = \emptyset$; and open sets U_1 and V_1 such that (1) $U_1 \supset F(P_1)$, $V_1 \supset F(Q_1)$, (2) $\hat{T} \cap (\bar{U}_1 \cup \bar{V}_1) = \emptyset$, (3) $\bar{U}_1 \subset P - \bar{R}$, $\bar{V}_1 \subset R - \bar{Q}$.

Now suppose Q contains an infinite collection $\{Z_n^i\}$ of compact n-cycles that are lirh in P and bound on $\hat{X}$. Then there exists, for each i, a cycle Z_{n+1}^i mod $\bar{Q} \cup F(P_1)$ on $\bar{P}_1$ such that

$$\partial Z_{n+1}^i = Z_n^i - \gamma_n^i, \text{ where } \gamma_n^i \text{ is a cycle on } F(P_1)\,.$$

The portion of Z_{n+1}^i on $\bar{P}_1 - Q_1$ is a relative cycle $\bar{Z}_{n+1}^i$ such that

$$\partial \bar{Z}_{n+1}^i = \gamma_n^i - w_n^i, \text{ where } w_n^i \text{ is on } F(Q_1)\,;$$

and since X has property $(P, Q, \sim)_n$ intrinsically, there exists a homology

$$\sum a^i \gamma_n^i - \sum a_i w_n^i \sim 0$$

in $U_1 \cup V_1$. But the sets U_1, V_1 are disjoint, so that this implies a homology $\sum a_i \gamma_n^i \sim 0$ in U_1. But the γ_n^i must be lirh in P since the Z_n^i are, so the existence of the cycles Z_n^i must be impossible.

To prove the converse, let P, Q be a canonical pair as before, and select R as above. Since X is open, there exists an open set U such that $P - \bar{Q} \supset U \supset F(R)$ and $U \subset X$. This time we suppose the Z_n^i lie in $X \cap Q$ and bound in X. They are therefore homologous in $P \cap X$ to cycles γ_n^i on $F(R)$, and using the $(P, Q, \sim)_n$ property of $\hat{X}$, we find that the γ_n^i are not lirh in U. But as $U \subset X$, this implies the Z_n^i not lirh in $P \cap X$. We conclude that X has property $(P, Q, \sim)_n$ extrinsically and hence intrinsically.

Corollary 2.1. *With X and $\hat{X}$ as in Theorem 2.1, if X has property $(P, Q, \sim)_n$ intrinsically, then it has property $(P, Q, \sim)_n$ extrinsically.*

Remark. That Theorem 2.1 fails if "$(P, Q)_n$" is substituted for "$(P, Q, \sim)_n$" is shown by the following example: Let $\hat{X}$ be the space of [6; 341, 5.19], consisting of a denumerable set of circles C_n successively tangent and converging to a point p. Let T consist of p together with a point x_n of each C_n, which may as well be distinct from the points of tangency with C_{n-1} and C_{n+1}. Here the set $X = \hat{X} - T$ has property $(P, Q)_1$ even extrinsically, yet $\hat{X}$ does not. This example also shows, incidentally, that property "${}_0(P, Q)_n$" cannot be substituted for "$(P, Q, \sim)_n$".

Corollary 2.2. *If X is a locally compact space having property $(P, Q, \sim)_n$, and $\hat{X}$ is a compactification of X such that $\hat{X}-X$ is a closed, totally disconnected subset of $\hat{X}$, then $\hat{X}$ has property $(P, Q, \sim)_n$.*

Corollary 2.2a. *If X is a locally connected, locally compact space having property $(P, Q, \sim)_n$, then the Freudenthal compactification [3] of X has property $(P, Q, \sim)_n$.*

Theorem 2.2. *Let $\hat{X}$ be a compact space and T a closed, totally disconnected subset of $\hat{X}$ such that $X = \hat{X} - T$ has property $(P, Q, \sim)_n$ intrinsically, is $(n + 1)$-lc, and has finite $p^{n+1}(X)$. Then $\hat{X}$ is $(n + 1)$-lc and, moreover, has property $(P, Q)_{n+1}$.*

Proof. Since $H_{n+1}(\hat{X}, T) \approx h^{n+1}(X)$ and $p^{n+1}(X)$ is finite, the group $H_{n+1}(\hat{X}, T)$ is f. g. And since T is closed and totally disconnected, $H_{n+1}(T) = 0$. It follows, from the homology sequence of the pair $\hat{X}$, T, that $H_{n+1}(\hat{X})$ is f. g. And since $\hat{X}$ is compact, $\hat{X}$ is semi-$(n + 1)$-connected.

Since X has property $(P, Q, \sim)_n$ and is $(n + 1)$-lc, X has property $(P, Q, \sim)_{n+1}$ [10; Th. III 1]. By Theorem 2.1, $\hat{X}$ has property $(P, Q, \sim)_{n+1}$. Hence by [10; Lemma II 1], $\hat{X}$ has property $(P, Q)_{n+1}$ and is a fortiori $(n + 1)$-lc.

Corollary 2.3. *If X is a locally compact, $(n + 1)$-lc space having property $(P, Q, \sim)_n$ and $\hat{X}$ is a compactification of X such that the set $T = \hat{X} - X$ is a closed, totally disconnected subset of $\hat{X}$, then a sufficient condition that $\hat{X}$ be $(n + 1)$-lc is that $p^{n+1}(X)$ be finite. And if either (1) $n > 0$ or (2) T is finite, then this condition is necessary.*

Proof. For the necessity, $\hat{X}$ has property $(P, Q, \sim)_n$ by Theorem 2.1 and together with the fact that $\hat{X}$ is $(n + 1)$-lc this implies that $p_{n+1}(\hat{X})$ is finite [10; Cor. III 2]. Hence if $n > 0$ or T is finite it will follow from the homology sequence of the pair $\hat{X}$, T that $p_{n+1}(\hat{X}, T) = p^{n+1}(X)$ is finite.

That the necessity fails when $n = 0$ and T is not finite is shown by the familiar examples of dendrites having a closed, infinite set of endpoints; denoting such a dendrite by $\hat{X}$ and the set of endpoints by T, $p^1(\hat{X} - T)$ is infinite.

Corollary 2.4. *If X is a locally compact, lc^m_{n+1} space, $n < m \leqq \infty$, having property $(P, Q, \sim)_n$ and $\hat{X}$ is a compactification of X such that the set $T = \hat{X} - X$ is a closed, totally disconnected subset of $\hat{X}$, then a sufficient condition that $\hat{X}$ be lc^m_{n+1} is that the numbers $p^r(X)$, $r = n + 1, \ldots, m$, all be finite. And if either (1) $n > 0$ or (2) T is finite, then this condition is necessary.*

Proof of sufficiency. By [10; Th. III 2], X has property ${}_{n+1}(P, Q)_m$ and a fortiori property ${}_{n+1}(P, Q, \sim)_m$. Hence by Corollary 2.3, $\hat{X}$ is lc^m_{n+1}.

Corollary 2.5. *In order that the one point compactification of a locally compact, lc^m_{n+1} space X having property $(P, Q, \sim)_n$ should be lc^m_{n+1}, $n < m \leqq \infty$, it is necessary and sufficient that the numbers $p^r(X)$, $r = n + 1, \ldots, m$, all be finite. In particular, if X is lc^m, then for $\hat{X}$ to be lc^m it is sufficient that the numbers $p^r(X)$, $r = 0, 1, \ldots, n$, all be finite.*

Proof. For the lc^m case, we recall that the 0-lc and $(P, Q)_0$ condition are equivalent. And if $p^0(X)$ is finite, X has only finitely many components so that the proof of sufficiency for 0-lc reduces to an appeal to the fact that no continuum can fail to be 0-lc at one point. That $\hat{X}$ is also lc^m_1 follows from Corollary 2.3.

That $p^0(X)$ is not of necessity finite, in general, is shown by such an example as that in the Remark following Corollary 1.2.

Corollary 2.6. *In order that the Freudenthal compactification $\hat{X}$ of a connected, locally connected, lc^m_{n+1}, $n < m \leqq \infty$, locally compact space X having property $(P, Q, \sim)_n$ should be lc^m_{n+1}, it is necessary and sufficient that the numbers $p^r(X)$, $r = n + 1, \ldots, m$, all be finite. In particular, if X is lc^m, then for $\hat{X}$ to be lc^m it is sufficient that the numbers $p^r(X)$, $n = 1, \ldots, m$ all be finite.*

Let us turn now to the cohomology case. Here we can expect substantial differences, inasmuch as r-colc at a point x is equivalent to $p^r(x) = 0$, while the range of possible values of $p^r(x)$ is infinite. (See [6; 190, § 6.6].) On the other hand, in the case of homology the corresponding numbers $g_r(x)$ have only two possible values, 0 and ∞, the former corresponding to r-lc at x. (See [6; 192].) However, corresponding to Theorem 2.1 we have:

Theorem 2.3. *Let $\hat{X}$ be a compact space, T a closed, totally disconnected subset of $\hat{X}$, and $X = \hat{X} - T$. If X has property $(P, Q, \sim)^n$ intrinsically, then $\hat{X}$ has property $(P, Q, \sim)^n$. Conversely, if $\hat{X}$ has property $(P, Q, \sim)^n$, then X has property $(P, Q, \sim)^n$ both extrinsically and intrinsically.*

Proof. By the fundamental duality between homology and cohomology of "$(P, Q, \sim)$" properties [10; Th. II 1], if $n > 0$, X has property $(P, Q, \sim)_{n-1}$ intrinsically, so that by Theorem 2.1, $\hat{X}$ has property $(P, Q, \sim)_{n-1}$ and, by duality, property $(P, Q, \sim)^n$. When $n = 0$, and P, Q form a canonical pair, every cobounding 0-cocycle of $Q \cap \hat{X}$ is in the same cohomology class of $Q \cap \hat{X}$ as a 0-cocycle of $Q \cap X$, so that the $(P, Q, \sim)^0$ property of X yields the desired result immediately. Conversely, if $\hat{X}$ has property $(P, Q, \sim)^n$, X has $(P, Q, \sim)^n$ intrinsically since X is open. However, to show the property is extrinsic we proceed as in the converse of Theorem 2.1, this time letting U_1 and U_2 be open sets such that

$$P - \bar{Q} \supset\supset U_1 \supset\supset U_2 \supset F(R), \ X \supset U_1 \ \text{and} \ \bar{U}_2 \ \text{is compact}.$$

Then U_1, U_2 form a canonical pair in $\hat{X}$; and if z^n_i are cobounding cocycles of X in Q, they are cohomologous in $X \cap P$ to cocycles γ_i^n in U_2 which are related (because of the properties of $\hat{X}$) by cohomologies in U_1, hence in $P \cap X$.

Theorem 2.4. *Let $\hat{X}$ be a compact space, T a closed, totally disconnected subset of $\hat{X}$, and $X = \hat{X} - T$. If X has property $(P, Q, \sim)^{n+1}$ and $p^n(x) \leqq \omega$ for all $x \in X$, $n \geqq 1$, then in order that $p^n(x) \leqq \omega$ for $x \in \hat{X}$ it is sufficient that $p^n(X)$ be finite. And if $n > 1$ or T is finite, this condition is also necessary.*

Proof of sufficiency. By [10; Th. VI 2], X has property $(P, Q, \sim)^n$ and hence by Theorem 2.3, $\hat{X}$ has property $(P, Q, \sim)^n$. And if $p^n(X)$ is finite, it follows from the exactness of the sequence

$$h^n(\hat{X}, X) \xleftarrow{j} h^n(\hat{X}) \xleftarrow{i} h^n(X) \xleftarrow{\delta} h^{n-1}(\hat{X}, X)$$

and the fact that $h^n(\hat{X}, X) = 0$ for $n \geqq 1$, that $p^n(\hat{X})$ is finite. Hence by [10; Lemma II 2] $\hat{X}$ has property $(P, Q)^n$ and a fortiori that $p^n(x) \leqq \omega$ for all $x \in \hat{X}$.

Proof of necessity for case $n > 1$ or T finite: We are given that X has property $(P, Q, \sim)^{n+1}$, $p^n(x) \leqq \omega$ for all $x \in \hat{X}$, and must show that $p^n(X)$ is finite. Since, by Theorem 2.3, $\hat{X}$ has property $(P, Q, \sim)^{n+1}$, it follows that $\hat{X}$ has property $(P, Q)^n$ [10; Th. VI 2]. Hence $p^n(\hat{X})$ is finite. It then follows from the above exact sequence that if $n > 1$ (in which case

$$h^{n-1}(\hat{X}, X) = h^{n-1}(T) = 0 = h^n(\hat{X}, X)$$

is finite) or T is finite then $p^n(X)$ is finite.

Remark. In the sufficiency proof of Theorem 2.4, we also proved:

Theorem 2.4a. *If $\hat{X}$, X and T are as in the hypothesis of Theorem 2.4, then $\hat{X}$ has property $(P, Q)^n$.*

The extensions of the preceding two theorems by induction are obvious. In particular, we have:

Theorem 2.5. *Let $\hat{X}$ be a compact space, T a closed, totally disconnected subset of $\hat{X}$ and $X = \hat{X} - T$. If k and n are positive integers such that $k \leqq n$ and X has property $(P, Q, \sim)^{n+1}$, $p^r(x) \leqq \omega$ for all $x \in X$ and $p^r(X)$ finite for $r = k, k+1, \ldots, n$, then $\hat{X}$ has property ${}^k(P, Q)^n$. Moreover, $\hat{X}$ is lc_k^n.*

That $\hat{X}$ is lc_k^n follows from [10; Th. VI 10].

Theorem 2.6. *If $\hat{X}$ is the compactification of a locally compact space X by the addition of a point set T which is a closed and totally disconnected subset of $\hat{X}$ and X has property $(P, Q, \sim)^{n+1}$ and $p^r(x) \leqq \omega$ at all $x \in X$, $r = k, k+1, \ldots, n$ where*

$1 \leqq k \leqq n$, then for $p^r(x)$ to be $\leqq \omega$ at all points of $\hat{X}$ (for the same range of r), it is sufficient that $p^r(X)$ be finite. And if $k > 1$ or T is finite, this condition is also necessary.

3. Applications to continuous mappings. Generally, the image of an lc^n space, $n > 0$, under a continuous mapping is not lc^n. For instance, if on the circle $C = \{(x, y) \mid x^2 + y^2 = 1\}$, the points p_n, q_n obtained by intersection of C with the line $x = (n-1)/n$ are identified for $n = 1, 2, 3, \ldots$, the resulting configuration C' is not lc^1, although C is. It is therefore of importance to know under what conditions a mapping preserves the lc^n property (see [8; VIII], for instance). From the theorems of § 2 we can obtain conditions of this nature.

Theorem 3.1. *Let X be a locally compact, lc^n space ($n \leqq \infty$), and T a closed, totally disconnected subset of X such that the groups $h^r(X - T)$, $r \leqq n$, are all finitely generated. Then the space Y formed by identifying all points of T is lc^n.*

Proof. Denoting by y the point of Y formed by identification of the points of T, we have

$$h^r(Y - y) \approx h^r(X - T)\,, \quad r \leqq n\,,$$

so that the groups $h^r(Y - y)$ are finitely generated. But Y can be considered as the one point compactification of $Y - y$, so that Y is lc^n by Corollary 2.5.

Theorem 3.2. *Let U be an lc^m_{n+1} open subset, $n < m$, of a compact space X such that U has property $(P, Q, \sim)_n$ intrinsically and $p^r(U)$ is finite for $r = n + 1, \ldots, m$. Then if $f : X \to Y$ is a continuous mapping of X onto a locally compact space Y such that $f \mid U$ is a homeomorphism, $f(U) \cap f(X - U) = \emptyset$, and $f(X - U)$ is a closed, totally disconnected subset of Y, then Y is lc^m_{n+1}.*

Proof. The set $V = f(U)$ has property $(P, Q, \sim)_n$ intrinsically, is lc^m_{n+1}, and the numbers $p^r(V)$, $r = n + 1, \ldots, m$, are all finite. Accordingly, by Corollary 2.4, Y is lc^m_{n+1}.

4. Applications to the Čech manifolds. We return now to the discussion of the Introduction concerning Čech manifolds.

Theorem 4.1. *Let M be an n-gm and x a point of M having arbitrarily small neighborhoods U_x whose one point compactifications are orientable n-gcms. Then x has arbitrarily small neighborhoods V_x which are orientable n-gms for which $h_r(V_x) = h^{n-r}(V_x) = 0$, $1 \leqq r \leqq n - 1$.*

Proof. Since an n-gm is r-lc for all r, we may limit our attention to U_x's such that $h_r(U_x \mid M) = 0$. Let $\hat{U}_x$ denote a one point compactification of such a U_x, forming an orientable n-gcm by the addition of an ideal point $\hat{p}$. We assert that U_x is a V_x of the desired type.

Let Z_r be any compact r-cycle of U_x, $r \leqq n - 1$; it is carried by a compact subset K of $\hat{U}_x - \hat{p}$. Let N be a neighborhood of $\hat{p}$ in $\hat{U}_x$ such that $N \cap K = \emptyset$. Since $\hat{U}_x$ is an orientable n-gcm, it is completely r-avoidable [6; 229] at $\hat{p}$ for $r \leqq n-2$, and locally $(n-1)$-avoidable [6; 218] at $\hat{p}$. Hence there exist neighborhoods P and Q of $\hat{p}$ in $\hat{U}_x$ such that $N \supset \bar{P} \supset P \supset \bar{Q} \supset Q$, and such that every r-cycle on $F(P)$ bounds on $\hat{U}_x - Q$, $r \leqq n-1$.

Since $h_r(U_x \mid M) = 0$, $Z_r \sim 0$ on M. Hence $Z_r \sim 0 \bmod M - U_x$, and accordingly $Z_r \sim 0 \bmod \bar{P}$ on $\hat{U}_x$ (we continue to use the same symbols for subsets and cycles of U_x, whether considered as a subset of M or of $\hat{U}_x$). It follows that there exists a cycle C_r on $F(P)$ such that $Z_r \sim C_r$ on $\hat{U}_x - P$. And since $C_r \sim 0$ on $\hat{U} - Q = U - Q$, so must $Z_r \sim 0$ on a compact subset of U. Hence $h_r(U) = 0$; and since U is an orientable n-gm, $h^{n-r}(U_x) = 0$ by duality [6; 260, Lemma 5.16].

It is interesting to note that Theorem 4.1 has a converse:

Theorem 4.2. *Let M be an n-gm and x a point of M having a neighborhood U which is an r-acyclic, orientable n-gm, $r \leqq n-1$. Then the one point compactification of U is a spherelike n-gcm.*[3])

Proof. Let $\hat{U}$ denote the compactification of U by an ideal point $\hat{p}$. Consider the exact sequence

$$(1) \qquad \ldots \to h^r(U) \to h^r(\hat{U}) \to h^r(\hat{U}, U) \to \ldots$$

Since

$$h_r(U) \approx h^{n-r}(U) \quad [6;\, 260,\ \text{Lemma } 5.16]\,,$$

and

$$h^r(\hat{U}, U) \approx h^r(\hat{p}) = 0\,, \quad 1 \leqq r \leqq n-1\,,$$

it follows from (1) that $h^r(\hat{U}) = 0$. Hence by duality, $h_r(\hat{U}) = 0$, so that $h_r^a(\hat{U}) = 0$ for all $r \leqq n-1$.

Let P be any neighborhood of $\hat{p}$ in $\hat{U}$, and let Z_r, $1 \leqq r \leqq n-1$, be a cycle of $\hat{U} \bmod \hat{U} - P$. Since $h_r^a(U) = 0$, there is a cycle C_r of $\hat{U}$ such that, for some neighborhood Q of $\hat{p}$, $C_r \sim Z_r \bmod \hat{U} - Q$. And since $h_r(\hat{U}) = 0$, we have $C_r \sim 0$ on $\hat{U}$ and accordingly $Z_r \sim 0 \bmod \hat{U} - Q$. It follows that $p_p^r(\hat{U}) = 0$.[4]) By Theorem 2.6,

$$p^r(\hat{p}) \leqq \omega \quad \text{for} \quad r = 1, \ldots, n-1\,,$$

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[3]) When this paper was completed, we noticed that what is really proved here is that the one point compactification of an r-acyclic, orientable n-gm, $r \leqq n - 1$, is a spherelike n-gcm; and that the latter result has recently been established by F. Raymond in a paper to appear in the Pacific Journ. Math. Since our proof is evidently quite different from Raymond's, as well as for reasons of completeness, we include it here.

[4]) This is the dimension of the Alexandroff group $H_p^r(\hat{U})$; [7: § 2].

and by [7; Th. 4],

$$p^r(\hat{p}) = p^r_{\hat{p}}(\hat{U}) = 0\,.$$

It remains to show that $p^n(\hat{p}) = 1$.

Since U is orientable, it carries a non-bounding infinite cycle C_n. Let P be a neighborhood of $\hat{p}$ such that $\hat{U} - \bar{P} \neq \emptyset$. Since by Theorem 1.1, $\hat{U}$ is lc^{n-1}, there is a neighborhood Q of $\hat{p}$ such that $\bar{Q} \subset P$ and $h_{n-1}(\bar{Q} \mid P) = 0$. As C_n is cycle mod $\bar{Q}$, its boundary ∂C_n is a cycle of $\bar{Q}$, and since $\partial C_n \sim 0$ in P, there is an absolute cycle Z_n of $\hat{U}$ such that $C_n \sim Z_n$ mod $\bar{P}$. Now $Z_n \sim 0$ mod $\hat{U} - Q$ else (since U is n-dimensional) $Z_n = C_n = 0$ on $Q - \hat{p}$, implying C_n is carried by the closed proper subset $U - Q$ of U and hence is ~ 0 on U. We conclude that $Z_n \nsim 0$ mod $\hat{U} - Q$ and that $p_n(\hat{p}) \geqq 1$. Finally, suppose Z^1_n, Z^2_n are cycles of $\hat{U}$ mod $\hat{U} - P$ for some neighborhood P of $\hat{p}$. Since $h_{n-1}(U) = 0$, they are extendible (as was Z_r) to cycles C^1_n, C^2_n, respectively, of $\hat{U}$ in such a way that $C^i_n \sim Z^i_n$ mod $\hat{U} - Q$ for some neighborhood $Q \subset P$, $i = 1, 2$. But U is an orientable n-gm, so there must be a relation $aC^1_n \sim bC^2_n$ mod $\hat{p}$, implying that $aC^1_n = bC^2_n$ mod $\hat{U} - Q$. We conclude that $p_n(\hat{p}) \leqq 1$, and, with the above relation, that $p_n(\hat{p}) = 1$.

That $\hat{U}$ is orientable follows from the sequence (1), which gives $h^r(U) \approx h^r(\hat{U})$.

Example *of a 3-gcm having a point p which does not have arbitrarily small 1-acyclic neighborhoods.* Let A denote the solid Alexander horned sphere in S^3; i. e., the "wild" sphere of [1] together with its (tame) interior. Let S denote the quotient space resulting from identifying all points of A, and p the point of S corresponding to A. Then S is an orientable 3-gcm of the same homology type as S^3 (see [8]). We shall show that p does not have arbitrarily small 1-acyclic neighborhoods, or, which is equivalent, that A does not have arbitrarily small 1-acyclic neighborhoods in S^3.

Referring to the Alexander construction [1], let E denote the totally disconnected, closed set of "endpoints" needed to complete the "horns", and suppose U is a 1-acyclic neighborhood of A. Define stages of construction of the horned sphere such that: (1) At stage 1, there are just two "interlocked" horns; (2) at stage 2, there are just 4 new "interlocked" horns, emanating in pairs from the horns of stage 1; ...; (n) at stage n, there are 2^n new "interlocked" horns, etc.

Clearly there exists n such that all 2^n horns of the n'th stage lie in U; moreover, we may assume (see Figure 1) that the connecting cylinders $C^1_n, C^2_n, \ldots, C^{2^n}_n$ (which do not form part of the horned sphere, of course) all lie in U. Let C be the connecting cylinder C^1_{n-1} of stage $n - 1$, containing C^1_n and C^2_n (see Figure 1). The curves J_1 and J_2, lying as shown on the n'th stage horns and running through C^1_n and C^2_n lie in U.

Consider J_1; let Z_1 denote its fundamental 1-cycle. Since U is 1-acyclic, $Z_1 \sim 0$ in U. Then Z_1 is homologous in $C \cap \bar{U}$ to a cycle Z_1' on $F(C)$ – see [6; 203, 1.13]. We may assume that there exists a chain C^2 in $C \cap \bar{U}$ such that $Z_1' \sim Z_1$ on $\|C^2\|$ irreducibly,[5]) so that

$$K = \|C^2 - Z_1' - Z_1\|$$

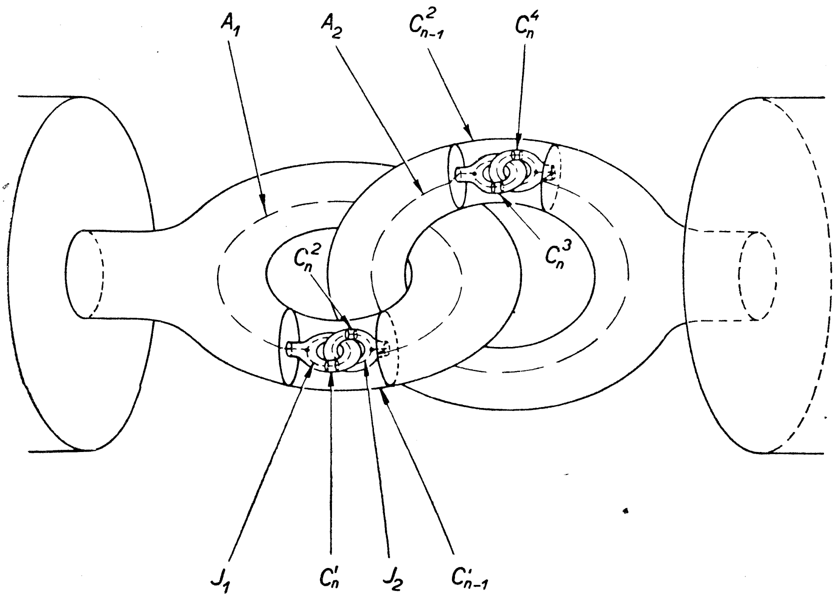

Fig. 1.

is connected (see [4; 299, Lemma 5]). However, since $Z_1' \sim 0$ on $F(C)$, it follows that K must meet J_2, inasmuch as J_1 and J_2 are linked. This implies that the arc A_1 on the parent horn (see the Figure) can be extended through C over to a simple closed curve A_1' in U; and similar situations prevail in regard to each of the 2^{n-1} parent horns of the other horn-pairs of stage n.

Now consider the pair A_1', A_2' of closed curves obtained by extensions of A_1 and A_2 as described above. As these lie in U, the fundamental 1-cycle on A_1' bounds in U, and we can proceed as before to show that corresponding to the associated parent horn of the 2^{n-2}-th stage there exists a simple closed curve analogous to J_1 and A_1'. And this process can be continued back to the 1st stage.

But U, when taken as a sufficiently close approximation to A, will not permit bounding of the curve indicated at the 1st stage.

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[5]) If D is a chain, then by $\|D\|$ we denote a carrier of D.

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Резюме

РАСПРОСТРАНЕНИЕ ЛОКАЛЬНЫХ И МЕДИАЛЬНЫХ СВОЙСТВ НА КОМПАКТНЫЕ РАСШИРЕНИЯ С ПРИМЕНЕНИЕМ К МНОГООБРАЗИЯМ В СМЫСЛЕ ЧЕХА

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Рассматриваются условия, при которых некоторые локальные, а также так наз. медиальные свойства, касающиеся связности, переносятся с пространства на компактные расширения. Даются приложения полученных результатов, в частности, в теории обобщенных многообразий в смысле Э. Чеха.