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PAIRWISE WEAKLY HAUSDORFF SPACES

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Abstract. In this paper, we introduce and investigate the notion of weakly Hausdorffness in bitopological spaces by using the convergent of nets. Several characterizations of this notion are given. Some relationships between these spaces and other spaces satisfying some separation axioms are studied.

1. Introduction

Kelly [4] introduced and studied the notion of bitopological spaces. A set equipped with two topologies is called a bitopological space. Since then several authors continued investigating such spaces. Concepts of pairwise Hausdorff, pairwise regular and pairwise normal were introduced by Kelly [4], concepts of pairwise T_0 and pairwise T_1 were introduced by Murdeshwar and Naimpally [6] and the concept of pairwise compactness was introduced independently by Fletcher et. al. [2], Kim [5] and Pahk and Choi [7]. Dunham [1] introduced and studied the notion of a new class of topological spaces, namely, weakly Hausdorff spaces, which includes the class of Hausdorff spaces and regular Hausdorff spaces. The purpose of this paper is to introduce and investigate the notion of pairwise weakly Hausdorff spaces. Several characterizations and properties of pairwise weakly Hausdorff spaces have been obtained by using the convergent of nets. Some relationships between these spaces and other spaces satisfying some separation axioms are studied. We prove that a bitopological space is pairwise weakly Hausdorff iff its PT_0–identification is a pairwise Hausdorff. Moreover, we prove that τ_i-closure of τ_i-compact subsets of a pairwise weakly Hausdorff space are τ_j-compact.

For definitions and results of bitopological spaces which are not explained in this paper, we refer to the papers [2, 4, 5, 6], assuming them to be well known. The word "bitopological space", "pairwise" and "pairwise weakly Hausdorff" will be abbreviated as "bts", "P" and "PWT_2" respectively. Also by $\tau_i \cdot \mathrm{cl}\,(A)$ and A' we shall denote respectively the τ_i-closure and the complement of a set A. The set of all τ_i-closed sets and the set of all τ_i-neighbourhoods of a point $x \in X$ denoted

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by τ'_i and $N(x,\tau_i)$, respectively, $i = 1, 2$. Whenever we deal with a statement involving the topologies τ_i and τ_j it will be understood that $i \neq j$ and that i, j take on the values 1 and 2.

2. Pairwise weakly Hausdorffness

In this section we introduce the concept of PWT_2-spaces and study a characterization and relationships with some other spaces. At first we recall the following definitions.

Definition 2.1. A net $S : D \to X$ in a bts (X, τ_1, τ_2) is said to τ_i-converge to a point $x \in X$ (or, x is a τ_i-limit point of S) if for each $U \in N\ (x,\tau_i)$ there exists $n \in D$ such that $S_m \in U$ for each $m \geq n$. The τ_i-limit set of a net S is the set of all x such that the net S τ_i-converges to x. We shall denote this set by $\tau_i \cdot \lim(S)$.

Definition 2.2. A bts (X, τ_1, τ_2) is called PWT_2-space iff $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y)$ whenever there is a net $S : D \to X$ such that $x \in \tau_i \cdot \lim(S)$ and $y \in \tau_j \cdot \lim(S)$.

Theorem 2.3. *A bts (X, τ_1, τ_2) is PWT_2 iff for each $x, y \in X$ one of the following holds:*

(i) $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y)$.

(ii) *There exist $U \in N(x,\tau_i)$ and $V \in N(y,\tau_j)$ such that $U \cap V = \phi$.*

Proof. Let (X, τ_1, τ_2) be PWT_2 and $x, y \in X$. Suppose the condition (ii) does not hold. Then we can define a set $D = \{U \cap V : U \in N(x,\tau_i) \text{ and } V \in N(y,\tau_j)\}$ with ordering by reverse inclusion, and a net $S : D \to X$ by $S(U \cap V) \in U \cap V$, arbitrary. Since $U \cap V \subset U$ and $U \cap V \subset V$, then $x \in \tau_i \cdot \lim(S)$ and $y \in \tau_j \cdot \lim(S)$ and it follows from the assumption that $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y)$.

Conversely, let $S : D \to X$ be a net in X and $x \in \tau_i \cdot \lim(S)$ and $y \in \tau_j \cdot \lim(S)$. Suppose that $\tau_j \cdot \mathrm{cl}\,(x) \neq \tau_i \cdot \mathrm{cl}\,(y)$, i.e. the condition (i) not hold, then the condition (ii) holds and so there exist $U \in N(x,\tau_i)$ and $V \in N(y,\tau_j)$ such that $U \cap V = \phi$. This contradicts that $x \in \tau_i \cdot \lim(S)$ and $y \in \tau_j \cdot \lim(S)$. Thus $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y)$ and so (X, τ_1, τ_2) is PWT_2. □

Theorem 2.4. *A PT_2-space is PWT_2-space.*

Proof. Let (X, τ_1, τ_2) be PT_2-space and let $S : D \to X$ be a net in X such that $x \in \tau_i \cdot \lim(S)$ and $y \in \tau_j \cdot \lim(S)$, for each $x, y \in X$. Since $\tau_j \cdot \lim(S)$ and $\tau_i \cdot \lim(S)$ are unique and equal in a PT_2-space, then $x = y$. Since (X, τ_1, τ_2) is PT_1, then $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i\,\mathrm{cl}\,(y)$. Hence (X, τ_1, τ_2) is PWT_2-space. □

Definition 2.5 [8]**.** A bts (X, τ_1, τ_2) is called:

(i) PR_0-space iff $\tau_j \cdot \mathrm{cl}\,(x) \subset U$; for each $U \in N(x,\tau_i)$.

(ii) PR_1-space iff for each $x, y \in X$ either $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y)$ or there exist $U \in N\big(\tau_j \cdot \mathrm{cl}\,(x), \tau_i\big)$ and $V \in N\big(\tau_i \cdot \mathrm{cl}\,(y), \tau_j\big)$ such that $U \cap V = \phi$.

(iii) PR_2-space iff $(\forall x \in X)(\forall U \in N(x,\tau_i)(\exists V \in N(x,\tau_i)(\tau_j \cdot \mathrm{cl}\,(V) \subseteq U)$.

In the following we shall prove that PR_1 and PWT_2 are identical.

Lemma 2.6. *A PWT_2-space is PR_0-space.*

Proof. Let (X,τ_1,τ_2) be PWT_2-space. Suppose $U\in N(x,\tau_i)$ and $y\in\tau_j\cdot\operatorname{cl}(x)$. Then $\big(\forall V\in N(y,\tau_j)\big)$ $(U\cap V\neq\phi)$. Hence the condition (ii) of Theorem 2.3 is not hold and so, by Theorem 2.3 (i), we have $x\in\tau_i\cdot\operatorname{cl}(y)$. Then $y\in G$ for each $G\in N(x,\tau_i)$. Thus $y\in U$ and so $\tau_j\cdot\operatorname{cl}(x)\subset U$ for each $U\in N(x,\tau_i)$ and hence (X,τ_1,τ_2) is PR_0-space. □

Theorem 2.7. *A bts (X,τ_1,τ_2) is PWT_2-space iff (X,τ_1,τ_2) is PR_1-space.*

Proof. Let (X,τ_1,τ_2) is PWT_2-space. Suppose $\tau_j.cl(x)\neq\tau_i.cl(y)$. By Theorem 2.3 (ii), there exist $U\in N(x,\tau_i)$ and $V\in N(y,\tau_j)$ such that $U\cap V=\phi$. By Lemma 2.6, (X,τ_1,τ_2) is PR_0-space and so $\tau_j\cdot\operatorname{cl}(x)\subset U$ and $\tau_i\cdot\operatorname{cl}(y)\subset V$. Hence (X,τ_1,τ_2) is PR_1-space.

Conversely, if (X,τ_1,τ_2) is PR_1-space and $\tau_j\cdot\operatorname{cl}(x)=\tau_i\cdot\operatorname{cl}(y)$, then Theorem 2.3 (i) holds and so (X,τ_1,τ_2) is PWT_2. Otherwise, if $\tau_j\cdot\operatorname{cl}(x)\neq\tau_i\cdot\operatorname{cl}(y)$, then since (X,τ_1,τ_2) is PR_1, there exist $U\in N\big(\tau_j\cdot\operatorname{cl}(x),\tau_i\big)$ and $V\in N\big(\tau_i\cdot\operatorname{cl}(y),\tau_j\big)$ such that $U\cap V=\phi$. Hence Theorem 2.3 (ii) holds and so (X,τ_1,τ_2) is PWT_2-space. □

Theorem 2.8. *A PR_2–space is PWT_2-space.*

Proof. Let $x,y\in X$ and suppose condition (i) of Theorem 2.3 fails. Without loss of generality, we may assume that $\big(\exists U\in N(x,\tau_i)\big)$ $(y\notin U)$. By pairwise regularity, $\big(\exists V\in N(x,\tau_i)\big)$ $\big(V\subseteq\tau_j\cdot\operatorname{cl}(V)\subseteq U\big)$ and so $x\in V$, $y\in\big(\tau_j\cdot\operatorname{cl}(V)\big)'$ and $V\cap\big(\tau_j\cdot\operatorname{cl}(V)\big)'=\emptyset$. This satisfies condition (ii) of Theorem 2.3, and we conclude the space is PWT_2. □

Definition 2.9. A bts (X,τ_1,τ_2) is said to be pairwise zero dimensional iff $(\forall x\in X)\big(\forall G\in N(x,\tau_i)\big)\big(\exists H\in N(x,\tau_i\cap\tau_j')\big)(H\subseteq G)$.

Definition 2.10. A bts (X,τ_1,τ_2) is said to be P–saturated iff arbitrary intersections of members of τ_i is a member of τ_i, $i=1,2$.

Theorem 2.11. *If a bts (X,τ_1,τ_2) is P-saturated, then the following statements are equivalent:*

(i) (X,τ_1,τ_2) *is pairwise zero dimensional.*
(ii) (X,τ_1,τ_2) *is PR_2.*
(iii) (X,τ_1,τ_2) *is PWT_2.*
(iv) (X,τ_1,τ_2) *is PR_0.*
(v) $\tau_i=\tau_j'$.

Proof. (i) $\Longrightarrow$ (ii): Let $x\in X$ and $F\in\tau_i'$ such that $x\notin F$. Then $F'\in N(x,\tau_i)$. Since (X,τ_1,τ_2) is pairwise zero dimensional, then $\big(\exists H\in N(x,\tau_i\cap\tau_j')\big)(H\subseteq F')$. Therefore (X,τ_1,τ_2) is PR_2.

(ii) $\Longrightarrow$ (iii): Follows from Theorem 2.8.

(iii) $\Longrightarrow$ (iv): Follows from Lemma 2.6.

(iv) $\Longrightarrow$ (v): Suppose that (X,τ_1,τ_2) is PR_0-space and let $U\in\tau_i$. Then for each $x\in U$, $\tau_j\cdot\operatorname{cl}(x)\subseteq U$ and so $U=\cup\{\tau_j\cdot\operatorname{cl}(x):x\in U\}$ is τ_j-closed, by the P-saturation property. Thus $\tau_i \subseteq \tau'_j$ and the reverse inclusion follows by complementation.

(v) $\Longrightarrow$ (i): Obvious. $\square$

Let (X, τ_1, τ_2) be any bitopological space and define a relation R on X by xRy iff $\tau_i \cdot \mathrm{cl}\,(x) = \tau_j \cdot \mathrm{cl}\,(y)$. Then $(X/R, \tau_1/R, \tau_2/R)$ is the well-known PT_0-identification with $q : X \to X/R$ the natural quotient mapping.

Proposition 2.12. *For an equivalence relation R on a bts (X, τ_1, τ_2) the following statements are equivalent:*

(i) *The natural mapping $q : (X, \tau_1, \tau_2) \to (X/R, \tau_1/R, \tau_2/R)$ is P-open.*

(ii) *The set $q^{-1}q(U) \subseteq X$ is τ_i-open for every τ_i-open $U \subseteq X$, $i = 1, 2$.*

Proof. (i) $\Longrightarrow$ (ii) Suppose q is P-open and let U be a τ_i-open in X. Then $q(U)$ is τ_i/R-open and hence $q^{-1}q(U)$ is τ_i-open.

(ii) $\Longrightarrow$ (i) Suppose that $q^{-1}q(U)$ is τ_i-open for every τ_i-open subset U in X. Let V be a τ_i-open, then $q^{-1}q(V)$ is τ_i-open. Hence $q(V)$ is τ_i/R-open. Thus q is P-open. $\square$

Proposition 2.13. *If (X, τ_1, τ_2) is PR_0, then $q : (X, \tau_1, \tau_2) \to (X/R, \tau_1/R, \tau_2/R)$ is P-open mapping.*

Proof. It suffices to show that $U = q^{-1}q(U)$ for each U is τ_i-open. Let $x \in q^{-1}q(U)$. Then $q(x) \in q(U)$ and hence there exists $y \in U$ such that $[x] = [y]$. It implies that xRy and hence $\tau_i \cdot \mathrm{cl}\,(x) = \tau_j \cdot \mathrm{cl}\,(y)$ for some $y \in U$. Since (X, τ_1, τ_2) is PR_0, then $x \in \tau_i \cdot \mathrm{cl}\,(x) = \tau_j \cdot \mathrm{cl}\,(y) \subseteq U$. Hence $q^{-1}q(U) \subseteq U$ and then $q^{-1}q(U) = U$. $\square$

Theorem 2.14. *A bts (X, τ_1, τ_2) is PWT_2 iff $(X/R, \tau_1/R, \tau_2/R)$ is PT_2.*

Proof. Suppose that (X, τ_1, τ_2) is PWT_2 and $q(x) \neq q(y)$ in X/R. Then $\tau_i \cdot \mathrm{cl}\,(x) \neq \tau_j \cdot \mathrm{cl}\,(y)$, and by Theorem 2.3 there exist $U \in N(x, \tau_j)$ and $V \in N(y, \tau_i)$ such that $U \cap V = \emptyset$. Thus $q(x) \in q(U)$ and $q(y) \in q(V)$. By Lemma 2.6 and Proposition 2.13, $q(U) \in \tau_j/R$ and $q(V) \in \tau_i/R$. It remains only to show $q(U) \cap q(V) = \emptyset$. But if $q(z) \in q(U) \cap q(V)$, then there exist $x^* \in U$ and $y^* \in V$ such that $\tau_i \cdot \mathrm{cl}\,(z) = \tau_j \cdot \mathrm{cl}\,(x^*)$ and $\tau_i \cdot \mathrm{cl}\,(z) = \tau_j \cdot \mathrm{cl}\,(y^*)$. Applying Lemma 2.6, $\tau_j \cdot \mathrm{cl}\,(y^*) \subseteq V$ and hence $x^* \in U \cap \tau_i \cdot \mathrm{cl}\,(z) = U \cap \tau_j \cdot \mathrm{cl}\,(y^*) \subseteq U \cap V$, a contradiction.

Conversely, suppose that $S : D \to X$ is a net with $x \in \tau_i \cdot \lim(S)$ and $y \in \tau_j \cdot \lim(S)$. Then $q(x) \in \tau_i \cdot \lim(q \circ S)$ and $q(y) \in \tau_j \cdot \lim(q \circ S)$. Since $(X/R, \tau_1/R, \tau_2/R)$ is PT_2, then $q(x) = q(y)$. Thus $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y)$ and so (X, τ_1, τ_2) is PWT_2. $\square$

3. Some basic properties

Theorem 3.1. *A PWT_2 is hereditary property.*

Proof. Let $(Y, \tau_{1_Y}, \tau_{2_Y})$ be a bitopological subspace of a PWT_2-space (X, τ_1, τ_2). Let $S : D \to Y$ be a net in Y with $y_1 \in \tau_{i_Y} \cdot \lim(S)$ and $y_2 \in \tau_{j_Y} \cdot \lim(S)$ for some $y_1, y_2 \in Y$. It is clear that $y_1 \in \tau_i \cdot \lim(S)$ and $y_2 \in \tau_j \cdot \lim(S)$. Since

(X,τ_1,τ_2) is PWT_2-space and $y_1,y_2 \in X$, then $\tau_j \cdot \mathrm{cl}\,(y_1) = \tau_i \cdot \mathrm{cl}\,(y_2)$. Thus $\tau_{j_Y} \cdot \mathrm{cl}\,(y_1) = Y \cap \tau_j \cdot \mathrm{cl}\,(y_1) = Y \cap \tau_i \cdot \mathrm{cl}\,(y_2) = \tau_{i_Y} \cdot \mathrm{cl}\,(y_2)$. Thus $(Y,\tau_{i_Y},\tau_{2_Y})$ is PWT_2-space and hence PWT_2 is hereditary property. $\square$

Theorem 3.2. *A PWT_2 is a topological property.*

Proof. Let $f : (X,\tau_1,\tau_2) \to (Y,\Delta_1,\Delta_2)$ be a P-homeomorphism and let (X,τ_1,τ_2) be PWT_2-space. Let $S : D \to Y$ be a net in Y with $y_1 \in \Delta_i \cdot \lim(S)$ and $y_2 \in \Delta_{j}. \cdot \lim(S)$. Since f is $1-1$, then there exist $x_1,x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus $x_1 = f^{-1}(y_1) \in \tau_i \cdot \lim(f^{-1} \circ S)$ and $x_2 = f^{-1}(y_2) \in \tau_j \cdot \lim(f^{-1} \circ S)$, where $f^{-1} \circ S$ is a net in X. Since (X,τ_1,τ_2) is PWT_2-space, then $\tau_j \cdot \mathrm{cl}\,(x_1) = \tau_i \cdot \mathrm{cl}\,(x_2)$. Since f is P-homeomorphism and $\tau_j \cdot \mathrm{cl}\,\big(f^{-1}(y_1)\big) = \tau_i \cdot \mathrm{cl}\,\big(f^{-1}(y_2)\big)$, then $\Delta_j \cdot \mathrm{cl}\,(y_1) = \Delta_i \cdot \mathrm{cl}\,(y_2)$. Thus (Y,Δ_1,Δ_2) is PWT_2-space. $\square$

Theorem 3.3. *A PWT_2 is productive property.*

Proof. Necessity. Let $X = \prod\limits_{\alpha} X_\alpha$ and $\tau_1 = \prod\limits_{\alpha} \tau_\alpha^1$ and $\tau_2 = \prod\limits_{\alpha} \tau_\alpha^2$. For each $\alpha \in \Gamma$, there exists a subspace of $\Big(\prod\limits_{\alpha} X_\alpha, \prod\limits_{\alpha} \tau_\alpha^1, \prod\limits_{\alpha} \tau_\alpha^2\Big)$ which is P-homeomorphic to $(X_\alpha,\tau_\alpha^1,\tau_\alpha^2)$. Then $(X_\alpha,\tau_\alpha^1,\tau_\alpha^2)$ is PWT_2, by Theorems 3.1 and 3.2.

Sufficiency. Let $S : D \to X$ be a net in $X = \prod\limits_{\alpha} X_\alpha$ and $x \in \tau_i \cdot \lim(S)$, $y \in \tau_j \cdot \lim(S)$ in $\prod\limits_{\alpha} X_\alpha$, where $\tau_i = \prod\limits_{\alpha} \tau_\alpha^i$, $i = 1,2$. Then for each $\alpha \in \Gamma, P_\alpha \circ S$ is a net in X_α and $P_\alpha(x) \in \tau_i \cdot \lim(P_\alpha \circ S), P_\alpha(y) \in \tau_j \cdot \lim(P_\alpha \circ S)$, where $P_\alpha : \prod\limits_{\alpha} X_\alpha \to X_\alpha$ is the projection. Since $(X_\alpha,\tau_\alpha^1,\tau_\alpha^2)$ is PWT_2 for each $\alpha \in \Gamma$, we have $\tau_\alpha^j \cdot \mathrm{cl}\,\big(P_\alpha(x)\big) = \tau_\alpha^i \cdot \mathrm{cl}\,\big(P_\alpha(y)\big)$, for each $\alpha \in \Gamma$. Then $\tau_j \cdot \mathrm{cl}\,(x) = \Big(\prod\limits_{\alpha} \tau_\alpha^j\Big) \cdot \mathrm{cl}\,(x) = \prod\limits_{\alpha} \big(\tau_\alpha^j \cdot \mathrm{cl}\,(P_\alpha(x))\big) = \prod\limits_{\alpha} \big(\tau_\alpha^i \cdot \mathrm{cl}\,(P_\alpha(y))\big) = \Big(\prod\limits_{\alpha} \tau_\alpha^i\Big) \cdot \mathrm{cl}\,(y) = \tau_i \cdot \mathrm{cl}\,(y)$. Thus $\Big(\prod\limits_{\alpha} X_\alpha, \prod\limits_{\alpha} \tau_\alpha^1, \prod\limits_{\alpha} \tau_\alpha^2\Big)$ is PWT_2. $\square$

Theorem 3.4. *A bts (X,τ_1,τ_2) is PT_2 iff it is PT_0 and PWT_2.*

Proof. The necessity follows immediately from Theorem 2.4. To prove sufficiency, let $x,y \in X$ with $x \neq y$. Since (X,τ_1,τ_2) is PT_0-space, then $\tau_j \cdot \mathrm{cl}\,(x) \neq \tau_i \cdot \mathrm{cl}\,(y)$ and the result follows from condition (ii) of Theorem 2.3. $\square$

4. Pairwise Compactness

Definition 4.1. A bts X is called P-paracompact if it is PT_2 and if every P-open cover of X has a locally finite P-open refinement cover.

Theorem 4.2. *Let (X, τ_1, τ_2) be P-paracompact. Then (X, τ_1, τ_2) is PR_2 iff (X, τ_1, τ_2) is PWT_2.*

Proof. Sufficiency follows from Theorem 2.8. To prove necessity, suppose (X, τ_1, τ_2) is PWT_2 and $x \notin F \in \tau_i'$. Then for each $y \in F$, $x \notin \tau_i \cdot \mathrm{cl}\,(y)$ and so $\tau_j \cdot \mathrm{cl}\,(x) \neq \tau_i \cdot \mathrm{cl}\,(y)$. By Theorem 2.3 (ii), there are $U_y \in N(x, \tau_i)$ and $V_y \in N(y, \tau_j)$ such that $U_y \cap V_y = \phi$. The family $\mathcal{U} = F' \cup \{V_y : y \in F\}$ is a P-open cover of X. Since X is P-paracompact, then there is a locally finite P-open refinement cover $\{W_\alpha : \alpha \in \Gamma\}$ of X. Let $W = \cup\{W_\alpha : W_\alpha \cap F \neq \phi\}$. Then we have $F \subset W$ and will show that $x \in \big(\tau_i \cdot \mathrm{cl}\,(W)\big)'$. For otherwise, $x \in \tau_i \cdot \mathrm{cl}\,(W) = \cup\{\tau_i \cdot \mathrm{cl}\,(W_\alpha) : W_\alpha \cap F \neq \phi\}$ by locally finiteness, and so $x \in \tau_i \cdot \mathrm{cl}\,(W_{\alpha^*})$ where $W_{\alpha^*} \cap F \neq \phi$ for some $\alpha^* \in \Gamma$. Thus, $W_{\alpha^*} \not\subset F'$ and so $W_{\alpha^*} \subset V_{y^*}$ for some $y^* \in F$. But then $x \in \tau_j \cdot \mathrm{cl}\,(W_{\alpha^*}) \subset \tau_j \cdot \mathrm{cl}\,(V_{y^*})$, a contradiction since $x \in U_{y^*}$ and $U_{y^*} \cap V_{y^*} = \phi$. Thus $F \subset W \in \tau_j$ and $x \in \big(\tau_i \cdot \mathrm{cl}\,(W)\big)' \in \tau_i$ with $W \cap \big(\tau_i \cdot \mathrm{cl}\,(W)\big)' = \phi$. Hence (X, τ_1, τ_2) is PR_2-space. □

Corollary 4.3. *A P-compact space is PR_2 if it is PWT_2.*

Proof. Since every finite subcover is locally finite refinement, it is clear that a P-compactness implies P-paracompactness and the result follows from Theorem 4.2. □

Corollary 4.4. *A P-paracompact, WPT_2-space is P-normal space (PR_3-space, for short).*

Proof. A P-paracompact, WPT_2-space is P-paracompact and PR_2-space and thus PR_3-space. □

Corollary 4.5. *A P-compact, WPT_2-space is $PR_{2\frac{1}{2}}$ and PR_3.*

Proof. PR_2 follows from Corollary 4.3 and PR_3 follows from Corollary 4.4. Thus the space is $PR_{2\frac{1}{2}}$ as well. □

Theorem 4.6. *In a WPT_2-space, τ_i-closure of τ_j-compact sets are τ_j-compact.*

Proof. Let (X, τ_1, τ_2) be WPT_2 and $A \subset X$ be τ_j-compact set. Then if $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ is an τ_j-open cover of $\tau_i \cdot \mathrm{cl}\,(A)$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $A \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$. Now if $x \in \tau_i \cdot \mathrm{cl}\,(A)$, there is a net $S : D \to A$ such that $x \in \tau_i \cdot \lim(S)$. Hence by τ_j-compactness of A, there is a subnet $T : D \to A$ such that $y \in \tau_j \cdot \lim(T)$ for some $y \in A$. Since $x \in \tau_i \cdot \lim(T)$ and $y \in \tau_j \cdot \lim(T)$ and (X, τ_1, τ_2) is WPT_2, then $\tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y)$ and thus by Lemma 2.6, we have $x \in \tau_j \cdot \mathrm{cl}\,(x) = \tau_i \cdot \mathrm{cl}\,(y) \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}$. It follows that $\tau_i \cdot \mathrm{cl}\,(A) \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. Hence $\tau_i \cdot \mathrm{cl}\,(A)$ is τ_j-compact. □

Theorem 4.7. *A P-locally compact, WPT_2-space is $PR_{2\frac{1}{2}}$.*

Proof. It follows from the fact that every P-compact is P-locally compact and by Corollary 4.5. □

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