Malcolm T. McGregor
Short proofs of some inequalities of Horst Alzer

Archivum Mathematicum, Vol. 29 (1993), No. 3-4, 167--168

Persistent URL: http://dml.cz/dmlcz/107479

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 29 (1993), 167 – 168

SHORT PROOFS OF SOME INEQUALITIES OF HORST ALZER

Malcolm T. McGregor

Abstract. We provide elementary proofs of some inequalities of Horst Alzer.

Let the real numbers a_i be such that $0 < a_i \le \frac{1}{2}$ for all $i = 1, \dots, n$, and let

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i, \qquad G_n = \prod_{i=1}^{n} a_i^{1/n}, \qquad H_n = n/\sum_{i=1}^{n} 1/a_i$$
$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1-a_i), \quad G'_n = \prod_{i=1}^{n}(1-a_i)^{1/n}, \quad H'_n = n/\sum_{i=1}^{n} 1/(1-a_i)\,.$$

In [1] Horst Alzer establishes the inequalities

$$1/H'_n - 1/H_n \le 1/G'_n - 1/G_n \le 1/A'_n - 1/A_n\,, \tag{1}$$

where the sign of equality holds if and only if $a_1 = \dots = a_n$. Furthermore, in [1] he refers to the inequalities

$$H_n/H'_n \le G_n/G'_n \le A_n/A'_n \tag{2}$$

discovered by W.-L. Wang and P.-F. Wang, and Ky Fan. (See [3] and [2, p. 5].)

In this note we use the classical arithmetic, geometric, harmonic mean inequalities

$$A_n \ge G_n \ge H_n \tag{3}$$

together with (2) to establish the inequalities (1). We begin by showing that

$$\frac{1}{H_n}\left(1 - \frac{H_n}{H'_n}\right) \ge \frac{1}{G_n}\left(1 - \frac{G_n}{G'_n}\right) \tag{4}$$

1991 *Mathematics Subject Classification*: 26D15.
Key words and phrases: arithmetic and geometric means.
Received December 3, 1992.

which is equivalent to the left hand inequality in (1). Clearly, since $0 < a_i \leq \frac{1}{2}$ for all $i = 1, \ldots, n$ we have $H'_n \geq H_n > 0$ and $G'_n \geq G_n > 0$, and using (2) we deduce that

$$1 - \frac{H_n}{H'_n} \geq 1 - \frac{G_n}{G'_n} \geq 0\,. \tag{5}$$

In addition, the right hand inequality in (3) yields

$$\frac{1}{H_n} \geq \frac{1}{G_n} > 0\,, \tag{6}$$

and forming the products of corresponding sides in the inequalities (5) and (6) readily produces (4). The right hand inequality in (1) is proved in an analogous way.

Remark. It is worth noting that the above argument may be used to show that if $p > 0$ then

$$(1/H'_n)^p - (1/H_n)^p \leq (1/G'_n)^p - (1/G_n)^p \leq (1/A'_n)^p - (1/A_n)^p\,.$$

References

[1] Alzer, H., *Inequalities for arithmetic, geometric and harmonic means*, Bull. London Math. Soc. **22** (1990), 362-366.

[2] Beckenbach, E. F., Bellman, R., *Inequalities*, Springer, Berlin, 1961.

[3] Wang, W.-L., Wang, P.-F., *A class of inequalities for the symmetric functions (Chinese)*, Acta Math. Sinica **27** (1984), 485-497.

Malcolm T. McGregor
Department of Mathematics and Computer Science
University College of Swansea
Singleton Park
Swansea, SA2 8PP
WALES, U.K.